NUMERICAL ALGORITHMS AND ISSUES CONCERNING THE DISCRETE-TIME OPTIMAL PROJECTION EQUATIONS FOR SYSTEMS WITH WHITE PARAMETERS

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ABSTRACT

The discrete-time optimal projection equations for systems with white parameters are *strengthened.* For the class of minimal ms (mean square) stabilizing compensators the strengthened discrete-time optimal projection equations are proved to be equivalent to first-order necessary optimality conditions for optimal reduced-order dynamic compensation of systems with white parameters. The conventional discrete-time optimal projection equations are proved to be *weaker*. As a result solutions of the conventional discrete-time optimal projection equations may *not* correspond to optimal reduced-order compensators. To compute optimal reduced-order compensators two numerical algorithms are proposed. One is a homotopy algorithm and one is based on *iteration* of the strengthened discrete-time optimal projection equations. The latter algorithm is a generalization of the algorithm that solves the full-order problem, which in turn is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration. Therefore the efficiency of these three types of algorithms is comparable. It is demonstrated that, despite the strengthening of the optimal projection equations, the optimal reduced-order compensation problem, in general, *may* posses multiple extrema

Keywords: Multiplicative White Noise, White Stochastic Parameters, Reduced-order Control, Numerical Algorithms, Optimal Projection Equations

1. INTRODUCTION

There are mainly two reasons why discrete-time systems with white parameters are important. Firstly, these systems arise naturally in the field of digital control where some of the parameters may be white such as the sampling period [1], the controller parameters [2], or the parameters of the plant [3]. In all these cases it is possible to convert such a digital control system to an equivalent discrete-time system with white parameters [4],[5]. Also inherent discrete-time systems, such as economic systems, may have white parameters. Secondly, the parameters of an inherent or equivalent discrete-time system may be assumed to be white for the purpose of a robust control system design. It is well known that the standard LQG design does not lead in general to a robust control system with respect to parameter deviations [6]. The advantage of a model with white parameters is that it fits naturally in the LQ design context. Therefore the use of white parameters to model the system uncertainty allows for non-conservative robust control system design with respect to structured parameter variations [7]-[9].

In this paper optimal reduced-order dynamic output feedback, called optimal reduced-order compensation, of linear discrete-time systems with stationary white stochastic parameters is considered. The criteria are quadratic. The synthesis of the *unique* optimal full-order compensator was presented in [10]. The result is partially based on the optimal projection approach to reduced-order dynamic compensation [11]. The conventional discrete-time optimal projection equations (CDOPE) for systems with deterministic parameters were presented in [14], and modified in [13],[22].

Recently *strengthened* discrete-time optimal projection equations (SDOPE) for systems with deterministic parameters, were presented [12]. In [12] the CDOPE are proved to be *weaker* and having solutions that do not correspond to (locally) optimal reduced order compensators. To compute (locally) optimal reduced order compensators two numerical algorithms were presented in [12]. One is a homotopy algorithm and one is based on *iteration* of the SDOPE. Here the SDOPE and algorithms in [12] are generalized to systems with white stochastic parameters, using results from [10]. As in [12], despite the strengthening of the optimal projection equations and opposite to the full-order case, in the reduced-order case the optimal compensation problem, in general, *may* have multiple extrema. This suggests that some of the results presented in [13], [23] and [24] should be reconsidered.

2. THE OPTIMAL REDUCED-ORDER COMPENSATION PROBLEM AND REDUCED-ORDER MEAN-SQUARE COMPENSATABILITY

For the properties of ms-stability, ms-stabilizability and ms-detectability of linear discrete-time systems with white parameters the reader is referred to [10],[15],[16]. Consider the system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i \qquad (1.1)$$

$$y_i = C_i x_i + w_i,\ i = 0,1,... \qquad (1.2)$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ the control, $y_i \in R^l$ the observation, $v_i \in R^n$ the system noise, $w_i \in R^l$ the observation noise. The processes $\{\Phi_i\}, \{\Gamma_i\}, \{C_i\}$ are sequences of independent random matrices with appropriate dimensions and constant statistics and $\{v_i\}, \{w_i\}$ are mutually independent sequences of independent stochastic vectors with constant statistics. The initial condition x_0 is stochastic with mean $\bar{x}_0$ and covariance P_0 and is independent of $\{\Phi_i, \Gamma_i, C_i, v_i, w_i\}$. Moreover, Φ_i, Γ_i and C_i are independent of $v_j, w_j, i \neq j$ and uncorrelated with v_i, w_i. The processes $\{v_i\}$ and $\{w_i\}$ are zero-mean with covariance's $V \geq 0$ and $W > 0$. As a controller we choose the reduced-order compensator

$$\hat{x}_{i+1} = F x_i + K y_i \tag{2.1}$$

$$u_i = -L x_i \tag{2.2}$$

where $\hat{x}_i \in R^{n_c}$, $n_c \leq n$ is the compensator state and F, K, L are real matrices of appropriate dimensions. The initial condition $\hat{x}_0$ is deterministic. Compensator *(2)* is characterized by the triple (F, K, L). Let an overbar denote expectation. The closed-loop system *(1)*,*(2)* is called ms-stable if $\overline{\|x_i\|^2}$ and $\overline{\|\hat{x}_i\|^2}$ converge as $i \to \infty$ to values which do not depend on x_0 and $\hat{x}_0$. Let (Φ_i, Γ_i, C_i) denote system *(1)* without the additive noise terms v_i, w_i and with x_0 deterministic. Now we define reduced-order ms-compensatability.

Definition 1
(Φ_i, Γ_i, C_i) is called n_c-ms-compensatable if there exists an F, K and L such that the closed-loop system is ms-stable. ●

A number of properties concerning reduced-order ms-compensatability is stated in the following theorem.

Theorem 1
a) (Φ_i) ms-stable $\Rightarrow (\Phi_i, \Gamma_i, C_i)$ n_c-ms-compensatable $\forall n_c$
b) (Φ_i, Γ_i, C_i) n_c-ms-compensatable $\Leftrightarrow (\Phi_i^T, C_i^T, \Gamma_i^T)$ n_c-ms-compensatable
c) If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$ and $C_i = C$ with Φ, Γ, C deterministic and constant then (Φ_i, Γ_i, C_i) n_c-ms-compensatable $\Leftrightarrow (\Phi, \Gamma, C)$ - n_c-compensatable [12].
d) $n_1 > n_2 \Rightarrow$ (n_2-ms-compensatability $\Rightarrow n_1$-ms-compensatability) ●

Proof
The proof follows from theorem 1 in [10] and theorem 1 in [13] and their consecutive proofs. ●

Compensator *(2)* is called minimal if (F, K) is reachable and (F, L) is observable. The *optimal reduced-order compensation problem* is to find the minimal ms-stabilizing compensator (F^*, K^*, L^*) for the system *(1)* which minimizes the criterion

$$s_\infty = \lim_{i \to \infty} \frac{1}{N} E\left\{ \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) \right\} \tag{3}$$

and to find the minimum $s_\infty^*(F^*, K^*, L^*)$ where $Q \geq 0$ and $R > 0$ are real symmetric matrices of appropriate dimensions.

3. THE STRENGTHENED OPTIMAL PROJECTION EQUATIONS

Define $\tilde{\Phi} = \Phi_i - \overline{\Phi}$, $\tilde{\Gamma} = \Gamma_i - \overline{\Gamma}$ and $\tilde{C} = C_i - \overline{C}$ and assume Γ_i and C_i are independent. Then as in [10],[12] first-order necessary optimality conditions for the optimal reduced-order compensation problem can be obtained using the matrix minimum principle [17]. The following theorem is obtained after rearranging these first-order necessary optimality conditions as in [10],[12].

Theorem 2
A ms-stabilizing compensator (F, K, L) satisfies the first-order necessary optimality conditions and is minimal *if and only if* there exist nonnegative symmetric $n \times n$ matrices $P, S, \hat{P}, \hat{S}$ such that,

$$F = H\left(\overline{\Phi} - K_{P,\hat{P}} \overline{C} - \overline{\Gamma} L_{S,\hat{S}}\right) G^T \tag{4.1}$$

$$K = H K_{P,\hat{P}} \tag{4.2}$$

$$L = L_{S,\hat{S}} G^T \tag{4.3}$$

and such that $P, S, \hat{P}, \hat{S}, t$ satisfy,

$$P = \overline{\left(\Phi - K_{P,\hat{P}} C\right) P \left(\Phi - K_{P,\hat{P}} C\right)^T} + V + \overline{\left(\tilde{\Phi} - \tilde{\Gamma} L_{S,\hat{S}}\right) \hat{P} \left(\tilde{\Phi} - \tilde{\Gamma} L_{S,\hat{S}}\right)^T}$$
$$+ K_{P,\hat{P}} \left(W + \overline{\tilde{C} \hat{P} \tilde{C}^T} \right) K_{P,\hat{P}}^T + t_\perp \Psi^1_{P,S,\hat{P},\hat{S}} t_\perp^T \tag{4.4}$$

$$S = \overline{\left(\Phi - \Gamma L_{S,\hat{S}}\right)^T S \left(\Phi - \Gamma L_{S,\hat{S}}\right)} + Q + \overline{\left(\tilde{\Phi} - K_{P,\hat{P}} \tilde{C}\right)^T \hat{S} \left(\tilde{\Phi} - K_{P,\hat{P}} \tilde{C}\right)}$$
$$+ L_{S,\hat{S}}^T \left(R + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Gamma}} \right) L_{S,\hat{S}} + t_\perp^T \Psi^2_{P,S,\hat{P},\hat{S}} t_\perp \tag{4.5}$$

$$\hat{P} = \tfrac{1}{2}\left[t \Psi^1_{P,S,\hat{P},\hat{S}} + \Psi^1_{P,S,\hat{P},\hat{S}} t^T \right] \tag{4.6}$$

$$\hat{S} = \tfrac{1}{2}\left[t^T \Psi^2_{P,S,\hat{P},\hat{S}} + \Psi^2_{P,S,\hat{P},\hat{S}} t \right] \tag{4.7}$$

$$rank\left(\hat{P}\right) = rank\left(\hat{S}\right) = rank\left(\hat{P}\hat{S}\right) = n_c \tag{4.8}$$

$$t = \hat{P}\hat{S}\left(\hat{P}\hat{S}\right)^{\#} \tag{4.9}$$

where,

$$\Psi^1_{P,S,\hat{P},\hat{S}} = \left(\overline{\Phi} - \overline{\Gamma} L_{S,\hat{S}}\right) \hat{P} \left(\overline{\Phi} - \overline{\Gamma} L_{S,\hat{S}}\right)^T$$
$$+ K_{P,\hat{P}} \left(\overline{C P C^T} + \overline{\tilde{C} \hat{P} \tilde{C}^T} + W \right) K_{P,\hat{P}}^T \tag{5.1}$$

$$\Psi^2_{P,S,\hat{P},\hat{S}} = \left(\overline{\Phi} - K_{P,\hat{P}} \overline{C}\right)^T \hat{S} \left(\overline{\Phi} - K_{P,\hat{P}} \overline{C}\right)$$
$$+ L_{S,\hat{S}}^T \left(\overline{\Gamma^T S \Gamma} + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Gamma}} + R \right) L_{S,\hat{S}} \tag{5.2}$$

$$K_{P,\hat{P}} = \left(\overline{\Phi P C^T} + \overline{\tilde{\Phi} \hat{P} \tilde{C}^T} \right) \left(\overline{C P C^T} + \overline{\tilde{C} \hat{P} \tilde{C}^T} + W \right)^{-1} \tag{5.3}$$

$$L_{S,\hat{S}} = \left(\overline{\Gamma^T S \Gamma} + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Gamma}} + R \right)^{-1} \left(\overline{\Gamma^T S \Phi} + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Phi}} \right) \tag{5.4}$$

$$t_\perp = I_n - t, \tag{5.5}$$

In *(4.9)* # denotes the group generalized inverse which is unique [18]. In equations *(4.1)-(4.3)* $G, H \in R^{n_c \times n}$ are two matrices that satisfy,

$$GH^T = I_{n_c},\ \ G^T H = t \qquad (5.6)$$

The matrices G and H are unique up to a basis transformation within R^{n_c} [14]. This reflects the independence with respect to the internal realization of the (optimal) compensator. For the costs of the compensator (F, K, L) we have,

$$s_{\infty} = s_{Q,R} = s_{V,W} \qquad (6.1)$$

$$s_{Q,R} = tr\left[QP + \left(Q + L_{S,\hat{S}}^T R L_{S,\hat{S}}\right)\hat{P}\right] \qquad (6.2)$$

$$s_{V,W} = tr\left[VS + \left(V + K_{P,\hat{P}} W K_{P,\hat{P}}^T\right)\hat{S}\right] \qquad (6.3)$$

●

Proof
The proof follows from the proofs of comparable theorems in [11] theorem 3.1, see also [10], and [12] theorem 2. ●

Equations *(4.4)-(4.9)* are the strengthened discrete-time optimal projection equations (SDOPE). The difference between the SDOPE and CDOPE relates to the following equalities which must hold if the first-order necessary conditions are to be satisfied [12],

$$\hat{P} = t\Psi_{P,S,\hat{P},\hat{S}}^1 = \Psi_{P,S,\hat{P},\hat{S}}^1 t^T = t\Psi_{P,S,\hat{P},\hat{S}}^1 t^T \qquad (7.1)$$

$$\hat{S} = t^T\Psi_{P,S,\hat{P},\hat{S}}^2 = \Psi_{P,S,\hat{P},\hat{S}}^2 t = t^T\Psi_{P,S,\hat{P},\hat{S}}^2 t \qquad (7.2)$$

Now equations *(4.6)*, *(4.7)* ensure that *(7.1)*, *(7.2)* are satisfied [12]. The final expressions in *(7.1)*, *(7.2)* which, instead of *(4.6)*, *(4.7)*, determine $\hat{P}, \hat{S}$ in the CDOPE *not* necessarily imply the second and third equality in *(7.1)*, *(7.2)*. From *(7.1)*, *(7.2)* observe that *(4.6)*, *(4.7)* may be replaced with,

$$\hat{P} = \Psi_{P,S,\hat{P},\hat{S}}^1 - \tau_{\perp}\Psi_{P,S,\hat{P},\hat{S}}^1 \tau_{\perp}^T \qquad (7.3)$$

$$\hat{S} = \Psi_{P,S,\hat{P},\hat{S}}^2 - \tau_{\perp}^T\Psi_{P,S,\hat{P},\hat{S}}^2 \tau_{\perp} \qquad (7.4)$$

Finally note that if $n_c = n$, $t = I_n$ and G and H may be chosen equal to I_n [11],[12]. Then the result in [10] is obtained.

Definition 2
The triple (Φ_i, Γ_i, C_i) is called ms-detectable if (Φ_i, C_i) and (Φ_i^T, Γ_i^T) are ms-detectable. ●

Theorem 3
Assume $(\Phi_i, V^{1/2}, Q^{1/2})$ ms-detectable. Then all nonnegative solutions of the SDOPE correspond to all compensators $(F, K, L) \in C_{stab}^m$, given by *(4.1)-(4.3)*, that satisfy the first-order necessary optimality conditions ●

Proof
This can be seen from [10], theorem 3. The order of the compensator plays no role. ●

4. NUMERICAL ALGORITHMS

Assuming Φ_i is independent of Γ_i and C_i the strengthened optimal projection equations *(4.4)-(4.9)* can be written slightly different to simplify the presentation [10]. Based on this simplification define the following nonlinear transformation where S^n denotes the space of real symmetric $n \times n$ dimensional matrices

$$\Re_a X : S^n \times S^n \times S^n \times S^n \to S^n \times S^n \times S^n \times S^n,\ a \in [0,1]$$

$$\begin{aligned}\Re_a X = \Big(& \overline{\Phi X_1 \Phi^T} - K_{X_1,X_3}\left(\overline{CX_1C^T} + \overline{\tilde{C}X_3\tilde{C}^T} + W\right)K_{X_1,X_3}^T + V \\ & + \overline{\tilde{\Phi}X_3\tilde{\Phi}^T} + \overline{\tilde{\Gamma}L_{X_2,X_4}X_3L_{X_2,X_4}^T\tilde{\Gamma}^T} + \tau_{\alpha\perp}\Psi_{X_1,X_2,X_3,X_4}^1\tau_{\alpha\perp}^T, \\ & \overline{\Phi^T X_2 \Phi} - L_{X_2,X_4}^T\left(\overline{\Gamma^T X_2 \Gamma} + \overline{\tilde{\Gamma}^T X_4 \tilde{\Gamma}} + R\right)L_{X_2,X_4} + Q \\ & + \overline{\tilde{\Phi}^T X_4 \tilde{\Phi}} + \overline{\tilde{C}^T K_{X_1,X_3}^T X_4 K_{X_1,X_3}\tilde{C}} + \tau_{\alpha\perp}^T\Psi_{X_1,X_2,X_3,X_4}^2\tau_{\alpha\perp}, \\ & \tfrac{1}{2}\left[\tau_\alpha\Psi_{X_1,X_2,X_3,X_4}^1 + \Psi_{X_1,X_2,X_3,X_4}^1\tau_\alpha^T\right], \\ & \tfrac{1}{2}\left[t_a^T\Psi_{X_1,X_2,X_3,X_4}^2 + \Psi_{X_1,X_2,X_3,X_4}^2 t_a\right]\Big)\end{aligned} \qquad (8.1)$$

where,

$$X = (X_1, X_2, X_3, X_4),\ \ X_1, X_2, X_3, X_4 \in S^n \qquad (8.2)$$

$$K_{X_1,X_3} = \overline{\Phi}X_1\overline{C}\left(\overline{CX_1C^T} + \overline{\tilde{C}X_3\tilde{C}^T} + W\right)^{-1} \qquad (8.3)$$

$$L_{X_2,X_4} = \left(\overline{\Gamma^T X_2 \Gamma} + \overline{\tilde{\Gamma}^T X_4 \tilde{\Gamma}} + R\right)^{-1}\overline{\Gamma}^T X_2 \overline{\Phi} \qquad (8.4)$$

$$\tau_a = U_{X_3X_4}\begin{bmatrix} I_{n_c^{'}} & 0 \\ 0 & (1-\alpha)I_{n-n_c^{'}} \end{bmatrix}U_{X_3X_4}^{-1} \qquad (8.5)$$

$$n_c^{'} = \min\left(n_c, rank(X_3X_4)\right) \qquad (8.6)$$

In equation *(8.5)* the columns of $U_{X_3X_4}$ are eigenvectors of X_3X_4 obtained from an eigenvalue decomposition. The columns associated with the $n_c^{'}$ *largest positive* eigenvalues of X_3X_4 appear first. If during the iteration, for some reason, $rank(X_3X_4) < n_c$, equation *(8.5)* ensures that τ_α is still properly computed, i.e. according to equation *(4.9)*. On the other hand, during the iteration, $rank(X_3X_4)$ may increase, or in other words, "recover" [12]. Call $X = (X_1, X_2, X_3, X_4)$ nonnegative if $X_1, X_2, X_3, X_4 \geq 0$. From *(4),(5)* note that for $a = 0$ a nonnegative solution of $X = \Re_a X$ satisfies the first-order necessary optimality conditions for full-order compensation, while for $a = 1$ it satisfies these conditions for reduced-order compensation. Denote the parameterized equation $Y^a = \Re_a Y^a$ by $H(Y^a, a) = 0$, where Y^a denotes the nonnegative solution of the equation $X = \Re_a X$. The function $H(Y^a, a)$ is called a homotopy. We may follow the solution path as a changes from 0 to 1 and the full-order compensation problem is deformed into the reduced-order compensation problem [19]. Observe that when $a = 1$, $(X_{1i}, X_{2i}, X_{3i}, X_{4i}) = \Re_a^i(0, 0, I_n, I_n)$ are iterations of equations *(4.4-4.8)* when $(X_1, X_2, X_3, X_4) = (P, S, \hat{P}, \hat{S})$. When $a = 0$

these iterations converge to the unique nonnegative solution of the optimal full-order compensation problem [10]. Based on these iterations and $H\left(Y^a, a\right)$ the following discrete homotopy algorithm is proposed

Algorithm 1
Initialization:
$X_1^0 = 0$, $X_2^0 = 0$, $X_3^0 = I_n$, $X_4^0 = I_n$, $a = 0$, $\Delta a = 1/N$, $N \geq 1$ and integer.
compute $Y^a = \lim_{i\to\infty} \Re_a^i(X^0)$ through iteration
Loop:
$a := a + \Delta a$
determine, through iteration, whether $Y^a = \lim_{i\to\infty} \Re_a^i(Y^{a-\Delta a})$ exists
stop when $a = 1$ ●

Consider again the homotopy $H\left(Y^a, a\right)$. If the number of solutions of the equation $H\left(Y^a, a\right) = 0$ from $a = 0$ to $a = 1$ remains constant, then, given the uniqueness of the full-order compensator, theorem 3 would give us necessary and sufficient conditions for the existence of a unique optimal reduced-order compensator. Similarly the algorithm that computes the unique solution of the full-order order problem through iteration [10], could be carried over to an algorithm that computes the unique nonnegative solution of the SDOPE, through iteration [13]. In the continuous-time and discrete-time deterministic parameter case results have been published concerning conditions under which the number of solutions along the solution path remains constant [13],[23],[24]. On the other hand numerical examples in [12] and convex analysis of systems controlled by output feedback [20] indicate that, in general, multiple nonnegative solutions satisfying the SDOPE may exist. In section 6 example 2 demonstrates this phenomena in the stochastic parameter case. Despite this result we could still pursue the idea of iterating the SDOPE.

Algorithm 2
Initialization:
$X_1^1 = 0$, $X_2^1 = 0$, $X_3^1 = \Lambda_1$, $X_4^1 = \Lambda_2$
with $\Lambda_1, \Lambda_2 \geq 0$, symmetric, random, and with rank n_c
Computation:
Determine, through iteration, whether $Y^1 = \lim_{i\to\infty} \Re_1^i(X^1)$ exists ●

Theorem 4
If $\left(\Phi, V^{1/2}, Q^{1/2}\right)$ ms-detectable then algorithms 1 and 2, if they converge to $Y^1 \geq 0$, generate compensators $(F, K, L) \in C_{stab}^m$, given by *(4.1)-(4.3)*, with a minimal dimension equal to $n_c' = rank\left(Y_3^1 Y_4^1\right) \leq n_c$ and costs s_∞, given by *(6)*, where $\left(P, S, \hat{P}, \hat{S}\right) = \left(Y_1^1, Y_2^1, Y_3^1, Y_4^1\right)$. These compensators are local or global *minima* of the optimal reduced-order dynamic compensation problem when the prescribed compensator order is n_c'. ●

Proof
If the algorithms converge to $Y^1 \geq 0$ from *(8.6)*, $rank\left(Y_3^1 Y_4^1\right) \leq n_c$. Then from *(8)* and theorem 3, Y^1 corresponds to a minimal ms-stabilizing compensator with dimension $n_c' = rank\left(Y_3^1 Y_4^1\right) \leq n_c$ which satisfies the first-order necessary optimality conditions *(4),(5)* when the prescribed compensator order equals n_c'. Because both algorithm 1 and 2 are *generalizations* of the algorithm that solves the four equations of the full-order compensation problem *through iteration*, they converge to local (global) minima, *not* to local (global) maxima, which also satisfy the first-order necessary optimality conditions. ●

The examples in section 5 and a huge number of examples in [12] *suggest* that the algorithm has the following two important properties. 1) $n_c' = n_c$, unless a solution with $n_c' = n_c$ does not exist. 2) if the algorithms converge $Y^1 \geq 0$. From definition 1, theorem 1 and theorem 4 the following numerical test, representing *sufficient* conditions for n_c-ms-compensatability, is obtained.

Ms-compensatability test
Check if Φ_i is ms-stable. If so $\left(\Phi_i, \Gamma_i, C_i\right)$ is n_c-ms-compensatable $\forall n_c$. If not choose $R = I, W = I, Q = I$, $V = I$. Then $\left(\Phi_i, \Gamma_i, C_i\right)$ is n_c-ms-compensatable if algorithm 1 or algorithm 2, for some Λ_1, Λ_2, converges to $Y^1 \geq 0$. If not, nothing can be concluded with respect to the n_c-ms-compensatability of $\left(\Phi_i, \Gamma_i, C_i\right)$ ●

5. NUMERICAL ISSUES AND EXAMPLES

In *(8.6)* $rank\left(X_3 X_4\right)$ is computed as the number of eigenvalues of $X_3 X_4$ with a magnitude larger than 10^{-6} times the largest. In $\Re_a$ terms like $\overline{\Phi^T A \Phi}$ for some matrix A occur. They may be written as $st^{-1}\left[\left(\overline{\Phi \otimes \Phi}\right)^T st\left(A\right)\right]$ where st denotes the stack operator and $\otimes$ the Kronecker product [21]. Therefore $\overline{\Phi \otimes \Phi}$ needs to be calculated only once. Also $\overline{\Phi \otimes \Phi} = \overline{\Phi} \otimes \overline{\Phi} + \overline{\tilde{\Phi} \otimes \tilde{\Phi}}$ and similarly for Γ_i and C_i. In view of this it is convenient to specify the needed statistics of the parameters by $\overline{\tilde{\Phi} \otimes \tilde{\Phi}}$, $\overline{\tilde{\Gamma} \otimes \tilde{\Gamma}}$ and $\overline{\tilde{C} \otimes \tilde{C}}$. Furthermore in equation *(5)* $G = \left[A^T \quad 0\right] U_{X_3 X_4}^T$ and $H = \left[A^{-1} \quad 0\right] U_{X_3 X_4}^{-1}$ where $A \in R^{n_c \times n_c}$ is an arbitrary invertible matrix. The iteration in theorem 3 is numerically stable in general. In critical situations the symmetry of X_1 and X_2 must be enforced further by performing $X_1 = \frac{1}{2}\left(X_1 + X_1^T\right)$, $X_2 = \frac{1}{2}\left(X_2 + X_2^T\right)$ at the end of each iteration. To prevent ill conditioning of $U_{X_3 X_4}$ it is computed from the eigenvalue decomposition of $X_3 X_4 + 10^{-12} I_n$. In case of numerically difficult examples, where the system is only *just* reduced-order ms-compensatable the *numerical* stability is greatly enhanced if we apply the following computation, which implements a numerical damping, at the end of each iteration $X_{j_i} = \left(1 - a\right) X_{j_i} + a X_{j_{i-1}}$, $j = 1,2,3,4$, $0 \leq a < 1$. In the following examples we applied the computations mentioned above with $a = 0.25$.

Example 1

$$\overline{\Phi} = \begin{bmatrix} 0.3884 & 1.6578 & 0.0613 & 0.0137 & 0 \\ 0.0834 & 0.6802 & 0.0948 & 0.6800 & 0 \\ 1.2041 & 0.9213 & 0.9395 & 0.1186 & 0 \\ 1.2048 & 1.4738 & 1.1904 & 0.7405 & 0 \\ 0 & 0 & 0 & 0 & 0.95 \end{bmatrix},$$

$$\overline{\Gamma}^T = \begin{bmatrix} 0.5890 & 0.9304 & 0.8462 & 0.5269 & 0 \\ 0.0920 & 0.6539 & 0.4160 & 0.7012 & 0 \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} 0.9103 & 0.2625 & 0.7361 & 0.6326 & 0.9910 \\ 0.7622 & 0.0475 & 0.3282 & 0.7564 & 0.3653 \end{bmatrix},$$

$$\overline{\tilde{\Phi} \otimes \tilde{\Phi}} = b\left(\overline{\Phi} \otimes \overline{\Phi}\right),\ \overline{\tilde{\Gamma} \otimes \tilde{\Gamma}} = b\left(\overline{\Gamma} \otimes \overline{\Gamma}\right),\ \overline{\tilde{C} \otimes \tilde{C}} = b\left(\overline{C} \otimes \overline{C}\right)$$

$$V = diag\left[0.2470 \quad 0.9826 \quad 0.7227 \quad 0.7534 \quad 0.6515\right],$$

$$Q = diag\left[0.8847 \quad 0.2727 \quad 0.4364 \quad 0.7665 \quad 0.4777\right],$$

$$W = diag\left(0.0727 \quad 0.6316\right),\ R = diag\left(0.2378 \quad 0.2749\right) \quad \bullet$$

Note that b may be conceived as an uncertainty measure of $\left(\Phi_i, \Gamma_i, C_i\right)$. If $b = 0$ the system has deterministic parameters. The parameter uncertainty increases as b increases. Observe that $r\left(\overline{\Phi} \otimes \overline{\Phi}\right) = r\left(\overline{\Phi}\right)^2 = 6.25$ and $r\left(\overline{\Phi \otimes \Phi}\right) = (1+b)\, r\left(\overline{\Phi} \otimes \overline{\Phi}\right)$ so $\left(\overline{\Phi}\right)$ is highly unstable in the usual sense and $\left(\Phi_i\right)$ is highly ms-unstable. Also $\left(\overline{\Phi}, \overline{\Gamma}, \overline{C}\right)$ has a single uncontrollable mode with an associated spectral radius of 0.95 so $\left(\Phi_i, \Gamma_i, C_i\right)$ is only *just* full-order ms-compensatable. These properties make the example numerically difficult. Convergence of the iteration is assumed when the relative difference of consecutive values of $tr\left(X_1 + X_2\right)$ is less than 10^{-6} [10]. For $n_c = 3$, $b = 5*10^{-4}$, table 1 lists solutions of the CDOPE obtained through iteration with different initial values $\hat{P}_0, \hat{S}_0$. If the first-order necessary optimality conditions are satisfied $s_{Q,R}$ equals $s_{V,W}$ [12]. Therefore the solutions in table 1 do *not* correspond to (local) optimal reduced-order compensators. Iterations of the CDOPE leave t unchanged [12]. So only if τ_0, which depends on $\hat{P}_0, \hat{S}_0$, is optimal, do iterations of the CDOPE converge to solutions that correspond to (local) optimal reduced-order compensators. Table 2 and 3 list the results obtained from iteration of the SDOPE, i.e. algorithm 2, for $b = 5*10^{-4}$ and $b = 5*10^{-3}$ respectively. In the latter case for $n_c = 1$ the algorithm does not converge and the system *may* not be n_c-compensatable. Algorithm 1 produced the same answers.

Table 1: Solutions of the CDOPE, $n_c = 3$, $b = 5*10^{-4}$

1: $\quad \hat{P}_0 = \hat{S}_0 = diag\left(1 \quad 1 \quad 1 \quad 0 \quad 0\right)$

2: $\quad \hat{P}_0 = \hat{S}_0 = diag\left(1 \quad 0 \quad 1 \quad 1 \quad 0\right)$

3: $\quad \hat{P}_0 = \hat{S}_0 = diag\left(1 \quad 1 \quad 0 \quad 1 \quad 0\right)$

	1	2	3
$s_{Q,R}$:	1361.0	14985	1375.9
$s_{V,W}$:	2101.9	30207	1182.4

Table 2: Solutions of the SDOPE, $b = 5*10^{-4}$

n_c: Prescribed compensator order

$n_c^{'}$: Minimal order of the compensator

i : Number of iterations necessary to reach convergence.

CT: Computation time in seconds on a pentium 90 MHz PC using MATLAB version 4.2c2.

n_c:	5	4	3	2	1
$n_c^{'}$:	5	4	3	2	1
$s_{Q,R}$:	215.44	219.91	242.71	407.84	15549
$s_{V,W}$:	215.44	219.91	242.71	407.84	15553
i :	128	178	192	199	586
CT:	4.06	5.60	5.99	6.21	18.46

Table 3: Solutions of the SDOPE , $\beta = 5*10^{-3}$

n_c:	5	4	3	2	1
$n_c^{'}$:	5	4	3	2	1
$s_{Q,R}$:	1662.2	1671.6	1970.2	6314.1	∞
$s_{V,W}$:	1662.3	1672.2	1968.8	6301.0	∞
i :	250	248	186	354	∞
CT:	7.80	7.74	5.83	10.98	∞

The next example demonstrates the non-uniqueness of optimal reduced-order compensators.

Example 2

$$\overline{\Phi} = \begin{bmatrix} -0.7336 & 0.6036 \\ -0.6036 & -0.7336 \end{bmatrix},\ \overline{\Gamma} = \begin{bmatrix} 0.4492 \\ 0.1784 \end{bmatrix},\ \overline{C}^T = \begin{bmatrix} 0.6171 \\ 0.3187 \end{bmatrix}$$

$$\overline{\tilde{\Phi} \otimes \tilde{\Phi}} = b\left(\overline{\Phi} \otimes \overline{\Phi}\right),\ \overline{\tilde{\Gamma} \otimes \tilde{\Gamma}} = b\left(\overline{\Gamma} \otimes \overline{\Gamma}\right),\ \overline{\tilde{C} \otimes \tilde{C}} = b\left(\overline{C} \otimes \overline{C}\right)$$

$$V = diag\left(0.7327 \quad 0.8612\right),\ W = 0.9334,$$

$$Q = diag\left(0.0437 \quad 0.1108\right),\ R = 0.3311 \quad \bullet$$

For $n_c = 1$ and $\beta = 0$, i.e. the deterministic parameter case, two optimal compensators may be found using algorithm 2. The associated costs are 0.9957 and 1.1315 respectively. For $n_c = 1$ and $\beta = 0.05$ also two optimal compensators are found using algorithm 2. The associated costs are 1.4415 and 1.7963 respectively.

6. CONCLUSIONS

Strengthened discrete-time optimal projection equations (SDOPE) and the property of *reduced-order* ms-compensatability for linear systems with white parameters have been introduced. For the class of minimal ms-stabilizing compensators the SDOPE were proved to be *equivalent* to first-order necessary optimality conditions. The conventional discrete-time optimal projection equations (CDOPE) were shown to be *weaker* and having solutions which do not correspond to optimal reduced-order compensators.

Based on the SDOPE two algorithms were proposed to compute optimal reduced-order compensators. One is a homotopy algorithm. The other algorithm iterates the SDOPE and is a generalization of the algorithm that solves the full-order problem for systems with white parameters [10]. Using these algorithms numerical examples were presented which illustrate the following important properties of reduced-order compensation problems and the algorithms. 1) In general the optimal reduced-order compensation problem *may* have multiple solutions which correspond to (locally) optimal minimal ms-stabilizing compensators. This suggests that some of the results presented in [13], [23], [24] must be reconsidered. 2) The algorithms, if they converge to Y^1, seem to have the property that $Y^1 \geq 0$ and also $n_c^{'} = n_c$, unless a solution with $n_c^{'} = n_c$ does not exist [12]. *During* the iterations negative matrices may sometimes occur. 3) Although not explicitly shown in one of the examples, as in [12], the homotopy algorithm often, but not always, converges to the global minimum. Clearly these properties require further investigation and proof and so do the convergence properties of the algorithms.

With respect to the possible non-uniqueness of the optimal reduced-order compensator the following practical approach is suggested. Apply algorithm 2 several times with different random initial values. Pick the best solution. Of course one can never be sure that better solutions do not exist. However compared to the performance of the optimal full-order compensator, which represents the global minimum obtainable with *any* compensator, the loss of performance may serve as a criterion for acceptance of a (locally) optimal reduced-order compensator.

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