FINITE AND INFINITE HORIZON FIXED-ORDER LQG COMPENSATION USING THE DELTA OPERATOR

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ABSTRACT

The strengthened discrete-time optimal projection equations (SDOPE) are presented in a form based on the delta operator. This form unifies discrete-time and continuous-time results. Based on this unification, recently established results and algorithms for finite and infinite-horizon fixed-order LQG compensation of discrete-time systems are carried over to the continuous-time case. The results concern the *equivalence* of the strengthened optimal projection equations to first-order necessary optimality conditions together with the condition that the compensator is minimal. Furthermore in the finite-horizon continuous-time case the problem of stating the optimal projection equations explicitly in the LQG problem parameters is explained and resolved. The algorithms exploit the resemblance between the strengthened optimal projection equations and the Riccati equations of full-order LQG control. They allow for efficient numerical computation of fixed-order LQG compensators through repeated forward and backward iteration (integration) of the SDOPE. They are illustrated with four numerical examples.

1. INTRODUCTION

For finite and infinite-horizon discrete-time fixed-order LQG compensation so called strengthened discrete-time optimal projection equations (SDOPE) have been developed recently [3], [4], [10]. As opposed to the conventional version [1], [6], which turned out to be weaker, the SDOPE are equivalent to first-order necessary optimality conditions together with the condition that the compensator is minimal. Based on the SDOPE in the discrete-time case efficient numerical algorithms have been presented to solve the SDOPE and compute the associated optimal fixed-order LQG compensator [3], [4], [10].

In this paper, for the first time, the SDOPE are presented in a form that is based on the delta operator. This form unifies continuous and discrete-time results [8]. Furthermore it circumvents numerical difficulties encountered in discrete-time, when the sampling time becomes very small and discrete-time results tend to continuous-time results. Both these advantages will be exploited in this paper. Using the unification property it is established that the optimal projection equations that have been presented for the finite and infinite horizon continuous-time case [5], [7] coincide with the strengthened optimal projection equations based on the delta operator. Note that in [5], [7] the equivalence of the optimal projection equations to first-order necessary optimality conditions together with the condition that the compensator is minimal was suggested, but not proved. In the finite-horizon continuous-time case the inability to specify the optimal projection equations explicitly in the LQG problem parameters [5] is explained and resolved.

The efficient numerical algorithms recently presented in [3], [4], [10] are carried over to the continuous-time case, and are illustrated with numerical examples. To the best knowledge of the authors for the first time a numerical solution of a finite-horizon continuous-time fixed-order LQG problem is presented. The SDOPE for finite-horizon fixed-order LQG compensation will be the starting point since the SDOPE for infinite-horizon fixed-order LQG compensation are a special case of the former.

2. THE FINITE-HORIZON DISCRETE-TIME OPTIMAL FIXED-ORDER LQG COMPENSATION PROBLEM

Consider the time-varying discrete-time system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i \qquad (1.1)$$

$$y_i = C_i x_i + w_i, \; i = 0,1,..,N-1 \qquad (1.2)$$

where $x_i \in R^{n_i}$ is the state, $u_i \in R^{m_i}$ is the control, $y_i \in R^{l_i}$ is the observation, $v_i \in R^{n_{i+1}\times n_i}$ the system noise, $w_i \in R^{l_i}$ the observation noise and $\Phi_i \in R^{n_{i+1}\times n_i}$, $\Gamma_i \in R^{n_{i+1}\times m_i}$, and $C_i \in R^{l_i \times n_i}$ are real matrices. The processes $\{v_i\}$ and $\{w_i\}$ are mutually independent zero-mean white noise sequences with covariance $V_i \geq 0, V_i \in R^{n_{i+1}\times n_{i+1}}$ and $W_i > 0, W_i \in R^{l_i \times l_i}$ respectively. The initial condition $x_0 \in R^{n_0}$ is a stochastic variable with mean $\bar{x}_0 \in R^{n_0}$ and covariance $X \in R^{n_0 \times n_0}$ and is independent of $\{v_i\}$ and $\{w_i\}$. Note that the system (1) has time-varying dimensions so Φ_i may not be square. As controller the following time-varying dynamic compensator is chosen,

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_i y_i, \quad i = 0,1,..,N-1 \qquad (2.1)$$

$$u_i = -L_i \hat{x}_i, \quad i = 0,1,\ldots,N-1 \qquad (2.2)$$

where $\hat{x}_i \in R^{n_i^c}$ is the compensator state. $F_i \in R^{n_{i+1}^c \times n_i^c}$, $K_i \in R^{n_{i+1}^c \times l_i}$ and $L_i \in R^{m_i \times n_i^c}$ are real matrices. Note that the compensator has time-varying dimensions so F_i may not be square. The initial condition $\hat{x}_0 \in R^{n_0^c}$ is deterministic. Compensator *(2)* is denoted by $(\hat{x}_0, F^N, K^N, L^N)$ where $F^N = \{F_i,\ i = 0,1,..,N-1\}$, $K^N = \{K_i,\ i = 0,1,..,N-1\}$, $L^N = \{L_i, i = 0,1,..,N-1\}$. As to the notation I_k denotes the $k \times k$ identity matrix and Θ_k the $k \times k$ zero matrix.

Problem formulation

Given the system *(1)* the optimal fixed-order dynamic compensation problem is to find a compensator *(2)*, with prescribed dimensions n_i^c, $i = 0,1,..,N$ which minimises the criterion,

$$J_N(\hat{x}_0, F^N, K^N, L^N) = E\left\{x_N^T Z x_N + \sum_{i=0}^{N-1}\left(x_i^T Q_i x_i + u_i^T R_i u_i\right)\right\} \quad (3.1)$$

$$Q_i \geq 0,\quad R_i > 0,\quad i = 0,1,..,N-1,\quad Z \geq 0 \quad (3.2)$$

and to find the minimum value of J_N. ●

In [10], [11] the above problem is solved. Furthermore in [11] the minimality property of a finite-horizon discrete-time compensator was defined as follows.

Definition 1

$(\hat{x}_0, F^N, K^N, L^N)$ is called minimal if $\forall i \in [0, N-1]$, $M_{i,N}$ full rank and if $\forall i \in [1, N]$, $W_{0,i}$ full rank and if in addition $n_0^c = 1$ and $n_N^c = 0$ where,

$$W_{0,i+1} = F_i W_{0,i} F_i^T + K_i K_i^T,\ i = 0,..,N-1, \qquad W_{0,0} = \hat{x}_0^T \hat{x}_0 \in R^{n_0^c \times n_0^c} \quad (4.1)$$

$$M_{i,N} = F_i^T M_{i+1,N} F_i + L_i^T L_i,\ i = 0,..,N-1, \qquad M_{N,N} = \Theta_{n_N^c} \in R^{n_N^c \times n_N^c} \quad (4.2)$$

●

In definition 1 $M_{i,N}$ is the observability grammian and $W_{0,i}$ a modified reachability grammian because its initial condition is non-zero. Due to the boundary conditions $n_0^c = 1$, $W_{0,0} = \hat{x}_0^T \hat{x}_0$ and $n_N^c = 0$, $M_{N,N} = \Theta_{n_N^c}$ at the boundaries the dimensions of the state of a minimal discrete-time compensator *change in one or several time steps*. Furthermore application of the optimal projection approach in this case requires the use of the Moore-Penrose instead of the standard inverse, which constitutes a major generalisation [11].

Because time-varying dimensions are incompatible with the continuous-time case and the description based on the delta operator from now on the system *(1)* and the compensator *(2)* are assumed to have a constant dimension, i.e. $n_i = n$, $n_i^c = n^c$, $i = 0,1,..,N$. As a result in our analysis non-minimal discrete-time compensators will be considered. This again requires the use of the Moore-Penrose instead of the standard inverse in applying the optimal projection approach [11].

3. PROBLEM REFORMULATION BASED ON THE DELTA OPERATOR

The delta operator δ is defined as follows [8],

$$\delta a_i = (a_{i+1} - a_i)/T,\ T > 0 \quad (5)$$

where a_i are vectors or matrices of equal dimension appearing in *(1)-(3)* and where $T > 0$ is arbitrarily fixed. Then the discrete-time LQG problem *(1)-(3)* may be written in terms of the delta operator,

$$\delta x_i = \Phi_i^\delta x_i + \Gamma_i^\delta u_i + v_i^\delta,\quad i = 0,1,..,N-1 \quad (6.1)$$

$$y_i = C_i^\delta x_i + w_i^\delta,\ i = 0,1,...,N \quad (6.2)$$

$$\hat{x}_{i+1} = F_i^\delta \hat{x}_i + K_i^\delta y_i,\quad i = 0,1,..,N-1 \quad (6.3)$$

$$u_i = -L_i^\delta \hat{x}_i,\quad i = 0,1,...,N-1 \quad (6.4)$$

$$J_N(\hat{x}_0, F^{\delta N}, K^{\delta N}, L^{\delta N}) = E\left\{x_N^T Z x_N + \sum_{i=0}^{N-1}\left(x_i^T Q_i^\delta x_i + u_i^T R_i^\delta u_i\right)\right\} \quad (6.5)$$

where the following correspond,

$$\Phi_i \leftrightarrow T\Phi_i^\delta + I_n,\ \Gamma_i \leftrightarrow T\Gamma_i^\delta,\ C_i \leftrightarrow C_i^\delta,$$
$$V_i \leftrightarrow T^2 V_i^\delta, W_i \leftrightarrow W_i^\delta \quad (6.6)$$

$$Q_i \leftrightarrow Q_i^\delta,\ R_i \leftrightarrow R_i^\delta \quad (6.7)$$

$$F_i \leftrightarrow TF_i^\delta + I_{n^c},\ K_i \leftrightarrow TK_i^\delta,\ L_i \leftrightarrow L_i^\delta \quad (6.8)$$

Because our aim is to unify discrete-time and continuous-time results, the discrete-time LQG problem description *(6)* is now linked to that of a continuous-time LQG problem. To do this associate the discrete-time instants i with continuous-time instants $t = iT$. In the limit $T \to 0$, $t = iT$ fixed, the problem *(6)* equals the following continuous-time LQG problem

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v^c(t) \quad (7.1)$$

$$y(t) = C^c(t)x(t) + w^c(t),\ t \in [0, t_f] \quad (7.2)$$

$$\dot{\hat{x}}(t) = F^c(t)\hat{x}(t) + K^c(t)y(t) \quad (7.3)$$

$$u(t) = -L^c(t)\hat{x}(t),\ t \in [0, t_f] \quad (7.4)$$

$$J_{t_f}(\hat{x}(0), F^c(\cdot), K^c(\cdot), L^c(\cdot)) = E\{x(t_f)Zx(t_f)\} + E\left\{\int_0^{t_f} x^T(t)Q^c(t)x(t) + u^T(t)R^c(t)u(t)dt\right\} \quad (7.5)$$

where $v^c(\cdot)$ and $w^c(\cdot)$ are zero-mean white noise processes with intensity matrices $V^c(\cdot)$ and $W^c(\cdot)$ respectively and where the following correspond,

$$\Phi_t^\delta \leftrightarrow A(t),\ \Gamma_i^\delta \leftrightarrow B(t),\ C_i^\delta \leftrightarrow C^c(t),$$

$$V_i^\delta \leftrightarrow \frac{1}{T}V^c(t), W_i^\delta \leftrightarrow \frac{1}{T}W^c(t),\ NT \leftrightarrow t_f \tag{7.6}$$

$$Q_i^\delta \leftrightarrow TQ^c(t),\ R_i^\delta \leftrightarrow TR^c(t) \tag{7.7}$$

$$F_i^\delta \leftrightarrow F^c(t) \leftrightarrow,\ K_i^\delta \leftrightarrow K^c(t),\ L_i^\delta \leftrightarrow L^c(t) \tag{7.8}$$

Remark 1
Equation *(6)* is an Euler approximation with time step T of *(7)*. The so called equivalent discrete-time LQG problem concerns the optimal LQG control of *(7.1)*, *(7.2)* based on the integral criterion *(7.5)* using a *digital* compensator with a sampling period T [9]. Equation *(6)* is also an approximation of this equivalent discrete-time LQG problem. In the limit $T \to 0$, $t = iT$ fixed, both the Euler approximation *(6)* and the equivalent discrete-time LQG problem become identical to *(7)*. ●

4. STRENGHTENED OPTIMAL PROJECTION EQUATIONS BASED ON THE DELTA OPERATOR

In [11], theorem 2, strengthened discrete-time optimal projection equations (SDOPE) for finite-horizon optimal fixed-order LQG compensation of time-varying discrete-time systems were presented. The following theorem is identical but expressed in terms of the delta operator matrices Φ_i^δ, Γ_i^δ, C_i^δ, V_i^δ, W_i^δ, F_i^δ, K_i^δ, L_i^δ, Q_i^δ, R_i^δ using *(6.6)-(6.8)*.

Theorem 1
The compensator $\left(\hat{x}_0, F^{\delta N}, K^{\delta N}, L^{\delta N}\right)$ satisfies the first-order necessary optimality conditions and is pseudo minimal, as defined in [11], *if and only if* there exist nonnegative symmetric $n \times n$ matrices $P_i, \hat{P}_i,\ i = 1,2,..,N$ and $S_i, \hat{S}_i,\ i = 0,1,..N-1$ that satisfy,

$$\delta P_i = T\Phi_i^\delta P_i \Phi_i^{\delta T} - K_i^{0\delta}\left(TC_i^\delta P_i C_i^{\delta T} + TW_i^\delta\right)K_i^{0\delta T} + TV_i^\delta + \Phi_i^\delta P_i + P_i\Phi_i^{\delta T} + \tau_{\perp_{i+1}}\Psi_i^{1\delta}\tau_{\perp_{i+1}}^T + \tau_{\perp_{i+1}}\frac{1}{T}\hat{P}_i\tau_{\perp_{i+1}}^T, \tag{8.1}$$
$$i = 0,1,2,..,N-1,\ P_0 = X$$

$$-\delta S_i = T\Phi_i^{\delta T} S_{i+1}\Phi_i^\delta - L_i^{0\delta T}\left(T\Gamma_i^{\delta T} S_{i+1}\Gamma_i^\delta + \frac{1}{T}R_i^\delta\right)L_i^{0\delta} + \frac{1}{T}Q_i^\delta + \Phi_i^{\delta T}S_{i+1} + S_{i+1}\Phi_i^\delta + \tau_{\perp_i}^T\Psi_{i+1}^{2\ \delta}\tau_{\perp_i} + \tau_{\perp_i}^T\frac{1}{T}\hat{S}_{i+1}\tau_{\perp_i}, \tag{8.2}$$
$$i = 0,1,2,..,N-1, S_N = Z$$

$$\delta\hat{P}_i = \Psi_i^{1\delta} - \tau_{\perp i+1}\Psi_i^{1\delta}\tau_{\perp i+1}^T - \tau_{\perp i+1}\frac{1}{T}\hat{P}_i\tau_{\perp i+1}^T, \tag{8.3}$$
$$i = 0,1,2,..,N-1,\ \hat{P}_0 = \bar{x}_0\bar{x}_0^T$$

$$-\delta\hat{S}_i = \Psi_{i+1}^{2\ \delta} - \tau_{\perp i}^T\Psi_{i+1}^{2\ \delta}\tau_{\perp i} - \tau_{\perp i}^T\frac{1}{T}\hat{S}_{i+1}\tau_{\perp i}, \tag{8.4}$$
$$i = 0,1,2,..,N-1,\ \hat{S}_N = \Theta_n$$
$$rank\left(\hat{P}_i\right) = rank\left(\hat{S}_i\right) = rank\left(\hat{P}_i\hat{S}_i\right) = r_i \le n^c,$$

$$i = 1,..,N-1 \tag{8.5}$$

$$\tau_i = \hat{P}_i\hat{S}_i\left(\hat{P}_i\hat{S}_i\right)^\# \tag{8.6}$$

such that

$$F_i^\delta = H_{i+1}\left[\Phi_i^\delta - K_i^{0\delta}C_i^\delta - \Gamma_i^\delta L_i^{0\delta}\right]G_i^T + \left(H_{i+1}G_i^T - I_{n^c}\right)/T,\ i = 0,1,2,..,N-1 \tag{9.1}$$

$$K_i^\delta = H_{i+1}K_i^{0\delta},\ i = 0,1,2,..,N-1 \tag{9.2}$$

$$L_i^\delta = -L_i^{0\delta}G_i^T,\ i = 0,1,2,..,N-1 \tag{9.3}$$

$$\hat{x}_0 = H_0\bar{x}_0 \tag{9.4}$$

with,

$$K_i^{0\delta} = \left(T\Phi_i^\delta + I_n\right)P_iC_i^{\delta T}(TC_i^\delta P_iC_i^{\delta T} + TW_i^\delta)^{-1}$$
$$i = 0,1,2,..,N-1 \tag{10.1}$$

$$L_i^{0\delta} = (T\Gamma_i^{\delta T}S_{i+1}\Gamma_i^\delta + \frac{1}{T}R_i^\delta)^{-1}\Gamma_i^{\delta T}S_{i+1}\left(T\Phi_i^\delta + I_n\right)$$
$$i = 0,1,2,..,N-1 \tag{10.2}$$

$$\Psi_i^{1\delta} = T\left(\Phi_i^\delta - \Gamma_i^\delta L_i^{0\delta}\right)\hat{P}_i\left(\Phi_i^\delta - \Gamma_i^\delta L_i^{0\delta}\right)^T + K_i^{0\delta}\left(TC_i^\delta P_iC_i^{\delta T} + TW_i^\delta\right)K_i^{0\delta T} + \left(\Phi_i^\delta - \Gamma_i^\delta L_i^{0\delta}\right)\hat{P}_i + \hat{P}_i\left(\Phi_i^\delta - \Gamma_i^\delta L_i^{0\delta}\right)^T \tag{10.3}$$

$$\Psi_{i+1}^{2\ \delta} = T\left(\Phi_i^\delta - K_i^{0\delta}C_i^\delta\right)^T\hat{S}_{i+1}\left(\Phi_i^\delta - K_i^{0\delta}C_i^\delta\right) + L_i^{0\delta T}\left(T\Gamma_i^{\delta T}S_{i+1}\Gamma_i^\delta + \frac{1}{T}R_i^\delta\right)L_i^{0\delta} + \left(\Phi_i^\delta - K_i^{0\delta}C_i^\delta\right)^T\hat{S}_{i+1} + \hat{S}_{i+1}\left(\Phi_i^\delta - K_i^{0\delta}C_i^\delta\right) \tag{10.4}$$

$$\tau_{\perp i} = I_n - \tau_i \tag{10.5}$$

where # in *(8.6)* denotes the group generalised inverse which is unique. $G_i, H_i \in R^{n_i^c \times n_i}$ are two matrices that satisfy,

$$G_iH_i^T = \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n^c \times n^c},\quad G_i^TH_i = \tau_i \tag{10.6}$$

The costs of the compensator $\left(\hat{x}_0, F^{\delta N}, K^{\delta N}, L^{\delta N}\right)$ are,

$$J_N = J_{N_1} = J_{N_2} \tag{11.1}$$

$$J_{N_1} = tr\left[Z\left(P_N + \hat{P}_N\right)\right] + T\sum_{i=0}^{N-1} tr\left[\frac{1}{T}Q_i^\delta P_i + \left(\frac{1}{T}Q_i^\delta + L_i^{0\delta T}\frac{1}{T}R_i^\delta L_i^{0\delta}\right)\hat{P}_i\right] \tag{11.2}$$

$$J_{N_2} = tr\left[X\left(S_0 + \hat{S}_0\right) + \bar{x}_0\bar{x}_0^TS_0\right] + T\sum_{i=0}^{N-1} tr\left[TV_i^\delta S_{i+1} + \left(TV_i^\delta + K_i^{0\delta}TW_i^\delta K_i^{0\delta T}\right)\hat{S}_{i+1}\right] \tag{11.3}$$

●

Remark 2
Observe that equations *(8.1)-(8.4)* with their associated boundary conditions constitute a two point boundary value problem ●

Remark 3
In the limit $T \rightarrow 0$, $t = iT$ fixed, theorem 1 applies to the continuous-time LQG problem *(7)* where the correspondence between the matrices of the continuous-time problem and those in theorem 1 are given by equations *(7.6)-(7.8).* ●

As $T \rightarrow 0$, $t = iT$ fixed, the *changes* of the dimension of the state of a minimal discrete-time compensator at the boundaries, which occur in one or several time steps, occur *instantaneously* in the continuous-time case.

Definition 2
A finite-horizon continuous-time compensator (i.e. with a constant dimension n^c) defined over $[0, t_f]$ is called minimal if its observability grammian is full rank at all times $t \in [0, t_f)$ and if its modified reachability grammian is full rank at all times $t \in (0, t_f]$ ●

Remark 4
In the finite-horizon time-varying case using *(10.5),*

$$\tau_{\perp i+1} \hat{P}_i \tau^T_{\perp i+1} = (\tau_{i+1} - \tau_i) \hat{P}_i (\tau_{i+1} - \tau_i)^T \qquad (12.1)$$

$$\tau^T_{\perp i} \hat{S}_{i+1} \tau_{\perp i+1} = (\tau_i - \tau_{i+1})^T \hat{S}_{i+1} (\tau_i - \tau_{i+1}) \qquad (12.2)$$

Since *(12.1)*, *(12.2)* are quadratic in $(\tau_{i+1} - \tau_i)$, in the continuous-time case, i.e. in the limit $T \rightarrow 0$, iT fixed, the final terms in *(8.1)-(8.4)* tend to zero. Then from definition 2 and [11] it follows that if the compensator in theorem 1 is minimal, $r_i = n^c$ in theorem 1 is satisfied, except at the initial and final time. Using *(10.6)* and presuming $r_i = n^c$, in *(9.1)* the final term,

$$\left(H_{i+1} G^T_i - I_{n^c}\right) / T = (H_{i+1} - H_i) G^T_i / T \rightarrow \dot{H} G^T \qquad (13)$$

Then the finite-horizon continuous-time fixed-order LQG result [5] is obtained. The violation of the condition $r_i = n^c$ at the initial and final time causes the inability in [5] to specify the optimal projection equations explicitly in the LQG problem parameters. To overcome this problem the use of the Moore-Penrose instead of the standard inverse is required [11] ●

Remark 5
When all matrices in the problem formulation are time-invariant and if J_N is replaced with $\frac{1}{N} J_N$, when $N \rightarrow \infty$, the time-invariant infinite-horizon case is obtained. After removal of the boundary conditions and time-indices equations *(8)-(11)* characterise the solution of this problem when the left hand sides of *(8.1)-(8.4)*, i.e. δP, δS, $\delta \hat{P}$, $\delta \hat{S}$ are zero. Using [3],

$$\hat{P} = \tau \hat{P} = \hat{P} \tau^T = \tau \hat{P} \tau^T \qquad (14.1)$$

$$\hat{S} = \tau^T \hat{S} = \hat{S} \tau = \tau^T \hat{S} \tau \qquad (14.2)$$

and *(8.6),(10.5)*, the final terms in *(8.1)-(8.4)* and *(9.1)* are equal to zero. Then the infinite-horizon time-invariant fixed-order LQG results in discrete-time [3] and, given remarks 1 and 3, in continuous-time [7], are obtained, if the inequality in equation *(8.5)* is replaced by an equality ●

Remark 6
If the horizon $t_f = NT$ is fixed as $T \rightarrow 0$, $N \rightarrow \infty$. To perform numerical computations, the number of (integration) time steps must always be finite so in practice T will only be small and N will be finite. In this case *(8.1)-(8.4)* without the final terms and multiplied by T and *(11.2)*, *(11.3)* describe exactly the *Euler integration* with time step T of $P, S, \hat{P}, \hat{S}$ and the cost expressions respectively ●

From remarks 1,3,5 and theorem 1 the following theorem is obtained.

Theorem 3
The optimal projection equations for the infinite-horizon time-invariant continuous-time case, presented in [7], are equivalent to first-order necessary optimality conditions together with the condition that the compensator is minimal ●

Although the equivalence stated in theorem 3 was suggested in the paper [7], the associated main theorem only proved that the optimal projection equations were *implied* by the first-order necessary optimality conditions together with the condition that the compensator is minimal. In the discrete-time case the optimal projection equations were modified twice [1], [3], [6] before equivalence was obtained.

5. NUMERICAL ALGORITHMS FOR FINITE AND INFINITE-HORIZON FIXED-ORDER LQG COMPENSATION

Based on the delta operator representation time-*increments* of $P, S, \hat{P}, \hat{S}$ are computed and *added* to the previous value, instead of calculating the next value directly as in the discrete-time case. The latter is numerically less favourable if the increments become relatively small as $T \rightarrow 0$. Because *(8.3), (8.4)* are numerically unstable the following equations, instead of *(8.3), (8.4)*, are used to update $\hat{P}_i$ and $\hat{S}_i$ in the algorithms (see also [4], [10], [11]),

$$\frac{\hat{P}'_{i+1} - \hat{P}_i}{T} = \Psi^{1\delta}_i, \; i = 0,1,2,..,N-1, \; \hat{P}_0 = \bar{x}_0 \bar{x}^T_0 \qquad (15.1)$$

$$\hat{P}_{i+1} = \tfrac{1}{2}\left(\tau_{i+1} \hat{P}'_{i+1} + \hat{P}'_{i+1} \tau^T_{i+1}\right), \; i = 0,1,..,N \qquad (15.2)$$

$$\frac{\hat{S}'_i - \hat{S}_{i+1}}{T} = \Psi^{2\,\delta}_{i+1}, \; i = 0,1,2,..,N-1, \; \hat{S}_N = \Theta_n \qquad (15.3)$$

$$\hat{S}_i = \tfrac{1}{2}\left(\tau^T_i \hat{S}'_i + \hat{S}'_i \tau_i\right), \; i = 0,1,..,N \qquad (15.4)$$

To see that *(15.1)*, *(15.2)* are equivalent to *(8.3)* substitute the expression for $\hat{P}'_{i+1}$, obtained from *(15.1)*, into *(15.2)*. Then substract $\hat{P}_i$ on both sides and divide by T to obtain *(8.3)*. Similarly *(15.3), (15.4)* are equivalent to *(8.4)*. Two numerical algorithms will be presented. They will be illustrated with numerical examples in section 6. The first algorithm computes a solution to the finite-horizon time-varying fixed-order LQG problem. Let Λ_n^1 and Λ_n^2 denote two random semi-positive definite symmetric $n \times n$ matrices.

Algorithm 1

1 Set $P_0 = X$, $\hat{P}_0 = \bar{x}_0\bar{x}_0^T$, $P_i = \Theta_n$, $\hat{P}_i = \Lambda_n^1$, $i = 1,2,..N$, $S_i = \Theta_n$, $\hat{S}_i = \Lambda_n^2$, $i = 0,1,..,N-1$, $S_N = Z$, $\hat{S}_N = \Theta_n$

2 Compute τ_i , $i = 0,1,..,N$ according to *(8.6)* with a rank $\le n^c$

3 Compute $S_i, \hat{S}_i^{'}, i = 0,1,..,N-1$ from equations *(8.2), (15.3)* multiplied by T

4 Compute $P_i, \hat{P}_i^{'}$, $i = 1,2,..N$ from equations *(8.1), (15.1)* multiplied by T

5 Compute $\hat{P}_i, \hat{S}_i$, $i = 0,1,..N$ from equations *(15.2), (15.4)*

6 Goto 2 unless the difference between $tr\left[P_N + S_0\right]$ and its previous value falls below a certain tolerance ●

Since multiple solutions may exist, the algorithm is initialised randomly in step 1 so that the algorithm is capable of finding multiple solutions, if these exist [4], [10], [11]. Step 2 is performed in the same way as described in [4], [10], [11], i.e. by applying an eigenvalue decomposition to $\hat{P}_i\hat{S}_i$. Then from comparable theorems in [4], [10], [11], the following theorem is obtained.

Theorem 4

If algorithm 1 converges to a solution with the property $P_i, \hat{P}_i, S_i, \hat{S}_i \ge 0$ it generates a finite-horizon time-varying (locally) optimal fixed-order LQG compensator ●

The next algorithm computes solutions to the infinite-horizon time-invarant fixed-order LQG compensation problem. It is based on equations *(8.1),(8.2),(15.1)-(15.4)* after removal of the final terms in *(8.1), (8.2)*, the boundary conditions and all time-indices except those associated to $P, \hat{P}, S, \hat{S}, \tau$. The dimension of the fixed-order compensator equals $n^c \le n^m \le n$ where n^m denotes the dimension of a minimal realisation of the full-order optimal LQG compensator.

Algorithm 2

1 Set $k = 0$ and $P_k = \Theta_n$, $\hat{P}_k = \Lambda_n^1$, $S_k = \Theta_n$, $\hat{S}_k = \Lambda_n^2$

2 Compute τ_k according to *(8.6)* with a rank $\le n^c$

3 Associate $S_k \leftrightarrow S_{i+1}$, $\hat{S}_k \leftrightarrow \hat{S}_{i+1}$, $S_{k+1} \leftrightarrow S_i$, $\hat{S}_{k+1} \leftrightarrow \hat{S}_i$ and compute $S_{k+1}, \hat{S}_{k+1}^{'}$ from equations *(8.2), (15.3)* multiplied by T

4 Associate $P_k \leftrightarrow P_i$, $\hat{P}_k \leftrightarrow \hat{P}_i$, $P_{k+1} \leftrightarrow P_{i+1}$, $\hat{P}_{k+1} \leftrightarrow \hat{P}_{i+1}$, and compute $P_{k+1}, \hat{P}_{k+1}^{'}$ from equations *(8.1), (15.1)* multiplied by T

5 Compute $\hat{P}_k, \hat{S}_k$, $i = 0,1,..N$ from equations *(15.2), (15.4)*

6 $k+1 \to k$

7 Goto 2 unless the difference between $tr\left[P_k + S_k\right]$ and $tr\left[P_{k-1} + S_{k-1}\right]$ falls below a certain tolerance ●

Again from comparable theorems in [4], [10], [11], the following theorem is obtained.

Theorem 5

Assume that the continuous or discrete-time system is both detectable and stabilisable. Then if algorithm 2 converges to a solution with the property $P, \hat{P}, S, \hat{S} \ge 0$ it generates an infinite-horizon time-invariant (locally) optimal fixed-order LQG compensator ●

6. NUMERICAL EXAMPLES

The first example concerns an infinite horizon discrete-time fixed-order LQG problem and is taken from [4].

Example 1 :

$$\Phi = \begin{bmatrix} -0.7336 & 0.6036 \\ -0.6036 & -0.7336 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.4492 \\ 0.1784 \end{bmatrix}, C^T = \begin{bmatrix} 0.6171 \\ 0.3187 \end{bmatrix}$$

$V = diag\left(0.7327 \quad 0.8612\right)$, $W = 0.9334$,

$Q = diag\left(0.0437 \quad 0.1108\right)$, $R = 0.3311$ ●

When example 1, with $n^c = 1$, is converted to the delta-domain according to *(6.6)-(6.8)* with $T > 0$, then irrespective of the choice of T , the two expressions for the costs *(11.2), (11.3)* must be identical and equal to 0.9957 or 1.1315, the two solutions of the problem [4]. This was verified for $T = 0.5$, $T = 1$ and $T = 2$ using algorithm 2.

Example 2 : (taken partly from [10])

Consider the finite-horizon discrete-time compensation problem characterized by $\Phi_i = \Phi$, $i = 0,1,..,10$ and the other matrices also time-invariant and equal to those in example 1 and $\bar{x}_0 = \left[1 \quad 1\right]^T$, $X = diag\left(0.1 \quad 0.1\right)$, $Z = diag\left(0.1 \quad 0.1\right)$ ●

Again using $T = 0.5$, $T = 1$ and $T = 2$ the problem was converted to the delta domain and solved using algorithm 1. The outcome of both *(11.2), (11.3)* was found to be 6.2730 or 6.2988, two solutions that are also obtained with the discrete-time algorithm described in [10].

Example 3 :

$$A(t) = \begin{bmatrix} 0.2500 & 3.0171 \\ -3.0171 & 0.2500 \end{bmatrix}, B(t) = \begin{bmatrix} -0.0799 \\ 0 \end{bmatrix},$$

$C^c(t) = \left[1.2867 \quad 0.4218\right], V^c(t) = diag\left[0.7136 \quad 0.4492\right]$,

$W^c(t) = 0.1784$,

$Q^c(t) = diag\left(0.6171 \quad 0.3187\right)$, $R^c(t) = 0.4603$ ●

Example 3 concerns a continuous-time time-invariant LQG problem. The system is unstable and of second order. The problem is converted to the delta domain using *(7.6)-(7.8)*. When the horizon is infinite and $n^c = 1$, the minimum costs computed from algorithm 2, with $T = 0$ on the right hand side of equations *(8.1), (8.2), (15.1), (15.3)* and $T = 0.005$ on the left hand side, equal 115.2.

Example 4 :
This example is equal to example 3 with a finite horizon $t_f = 1$ and $\bar{x}_0 = [1 \quad 1]^T$, $X = diag(0.1 \quad 0.1)$ and $Z = X$ •

Figure 1 shows the costs as a function of the "Euler frequency" $1/T$. As can be seen if T decreases, and the Euler approximation becomes more accurate, the costs converge to the costs of the continuous-time LQG problem. The final two values of the costs in the plot are 2.1554 and 2.1451. Decreasing T for fixed t_f implies increasing N and the associated computation time.

7. CONCLUSIONS

The strengthened discrete-time optimal projection equations for finite-horizon fixed-order LQG compensation, recently developed in [3], [4], [10] have been presented in a form based on the delta operator. It was shown how the result unifies continuous and discrete-time finite and infinite horizon fixed-order LQG compensation results. In the infinite-horizon time-invariant case the *equivalence* of the (strengthened) optimal projection equations with first-order necessary optimality conditions together with the condition that the compensator is minimal was established. In the finite-horizon time-varying case the same equivalence holds in both continuous and discrete time if the discrete-time minimality definition of [11] is applied. Based on the delta-operator analysis this minimality definition was generalised to the continuous-time case. To obtain these results the optimal projection approach must be based on the Moore-Penrose instead of the standard inverse, which constitutes a major generalisation [11].

Efficient numerical algorithms, recently developed in discrete-time, were modified based on the delta operator description. These algorithms perform superior, compared to discrete-time algorithms, for equivalent discrete-time LQG problems associated to digital control with a small sampling interval [9]. Furthermore they enable the computation of continuous-time optimal fixed-order LQG compensators. To the best knowledge of the authors for the first time a numerical solution of a finite-horizon continuous-time fixed-order LQG problem was presented (example 4). The algorithms iterate (integrate) the (strengthened) optimal projection equations repeatedly, forward and backward in time.

As explained in [3], [10], and opposite to [2], due to the possible existence of multiple solutions homotopy degree theory cannot be used to prove the convergence of the algorithms to a solution that satisfies $P, \hat{P}, S, \hat{S} \geq 0$. As in [3], [4], [10] the algorithms turn out to work very well in practice and seem to have the property that, if they converge, they converge to a solution with the property $P, \hat{P}, S, \hat{S} \geq 0$, as desired.

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Figure 1