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COMPUTATION OF THE DIGITAL LQG REGULATOR AND TRACKER FOR TIME-VARYING SYSTEMS

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SUMMARY

Digital optimal control problems, i.e. problems where a continuous-time system is controlled by a digital computer, are very often approximated by either discrete-time or continuous-time optimal control problems. A digital controller based on one of these approximations requires a *small sampling time* and constitutes only an *approximate solution*. The digital LQG regulator and tracker constitute solutions to real digital control problems which involve sampled-data, piecewise constant controls and integral criteria. Until now only the numerical computation of the digital LQG regulator in the case of time-invariant system and criterion matrices has been considered in the literature. The control of non-linear stochastic systems about state trajectories is very often performed by an LQG regulator based on the linearized dynamics about the trajectory, which constitute a *time-varying system*. We present a numerical procedure to compute the digital LQG regulator and tracker in the case where the system and criterion matrices are time-varying. Finally we present a numerical example.

KEY WORDS Digital LQG controllers Sampled-data controllers Regulator Tracker Time-varying systems Numerical computation

1. INTRODUCTION

Industrial processes very often constitute continuous-time systems. The automatic control of industrial processes is performed by digital computers. The resulting automatic control system is a digital control system schematically represented by Figure 1. The continuous-time system has a sampler at the output and a sampler and zero-order hold circuit at the input.

The design of a digital controller for a continuous-time system is called a *digital control problem*. The term *digital* refers to the following facts.

(a) We have sampled measurements, since a computer cannot deal with continuous-time measurements.
(b) The control is of piecewise constant nature (a staircase function), since a sampler and zero-order hold circuit connect the computer to the input of the system.
(c) We consider the continuous-time behaviour of the system.

Although these all seem very straightforward considerations, very often at least one of these considerations is not met in the design of digital controllers for continuous-time systems. Very often the digital control problem is *approximated* by a discrete-time control problem which *completely disregards the intersample behaviour.*[1,2] In this case consideration (c) is not met. In other cases continuous-time control algorithms are designed which then somehow have to

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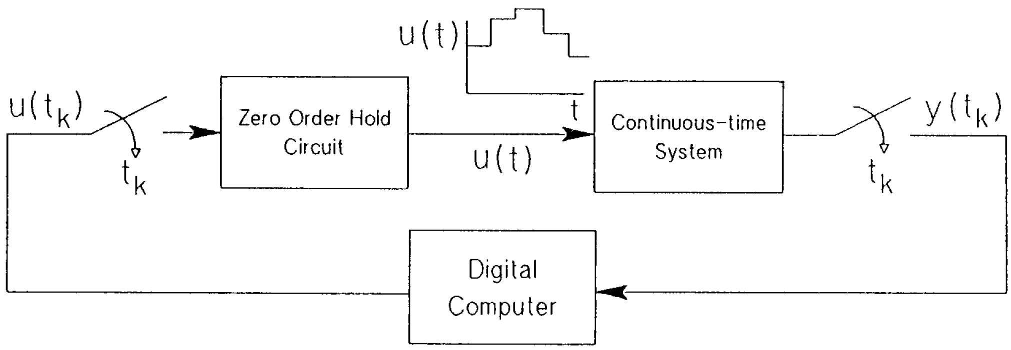

Figure 1. Digital control system

be *approximated* by a digital control algorithm.[3] In these cases both considerations (a) and (b) are not met. In both cases there is a demand for a *small sampling time*, in the former case to prevent undesirable intersample behaviour and in the latter case to properly approximate the continuous-time algorithm. This demand, e.g. in the case of robot control where the computational burden on the computer is high, results in computational difficulties. Even if the sampling time is chosen to be small, these digital controllers will only constitute *approximate solutions*.

The digital LQ regulator, sometimes called the optimal sampled-data regulator, constitutes the solution to a real digital control problem which involves sampled-data, piecewise constant controls and an integral criterion and was initially presented by Levis *et al.*[4] They considered time-invariant system and criterion matrices and equidistant sampling. Nour Eldin[5] and Dorato and Levis[6] considered time-varying system and criterion matrices and non-equidistant sampling. Halyo and Caglayan[7] were the first to consider the digital LQG regulator, i.e. with the state and the incomplete state information at the sampling instants corrupted by additive white Gaussian noise. They also considered time-varying system and criterion matrices but did not derive an expression for the minimum cost of the problem. This was done by De Koning,[8] who considered time-invariant system and criterion matrices and both deterministic and random sampling. While assuming non-equidistant deterministic sampling and time-varying system and criterion matrices, Van Willigenburg[9,19] completely derived both the digital LQG regulator and tracker, i.e. including expressions for the minimum cost explicit in the system and criterion matrices. In the case of the digital LQG tracker, state deviations from a prescribed state trajectory are penalized.

The digital LQ and LQG regulators, compared with the discrete-time and continuous-time versions, have received little attention in the literature. This is remarkable, since a well-known approach to control non-linear continuous-time stochastic systems about precomputed (possibly optimal) state trajectories is through the use of an LQG regulator. On the basis of deviations from the desired state trajectory, the LQG regulator computes *on-line* control corrections to control the deviations to zero. The LQG regulator is based on the linearized dynamics about the precomputed state trajectory, which constitute a *time-varying* linear system.[3] Therefore in the case of computer control this constitutes the digital LQG regulator for time-varying systems!

Although the continuous-time and discrete-time LQ and LQG trackers are very well known,[10] except for Nour Eldin,[5] who considered the digital LQ tracker, and Van

Willigenburg,[9,19] who considered the digital LQG tracker, the digital LQ and LQG trackers seem to be unpublished, which in the light of computer control again is remarkable!

None of the authors is concerned with the *numerical computation* of the digital LQ or LQG regulator and tracker in the case of *time-varying systems*. Only the computation of the digital LQ regulator for time-invariant system and criterion matrices has been considered.[11] In this paper we present a numerical procedure to compute the digital LQG regulator and tracker for time-varying system and criterion matrices.

2. THE DIGITAL LQG REGULATOR AND TRACKER

Consider the stochastic continuous-time linear time-varying system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{v}(t) \tag{1a}$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are the system matrices and $\{\mathbf{v}(t)\}$ a Gaussian white noise process, with

$$\mathrm{E}\{\mathbf{v}(t)\} = 0, \qquad \operatorname{cov}(\mathbf{v}(t), \mathbf{v}(s)) = \mathbf{V}(t)\delta(t-s) \tag{1b}$$

and

$$\mathrm{E}\{\mathbf{x}(t_0)\} = \bar{\mathbf{x}}(t_0), \qquad \operatorname{cov}(\mathbf{x}(t_0), \mathbf{x}(t_0)) = \mathbf{G} \geqslant 0 \tag{1c}$$

The system is controlled by a digital computer, so measurements are taken at the sampling instants, i.e.

$$\mathbf{Y}(t_k) = \mathbf{C}(t_k)\mathbf{x}(t_k) + \mathbf{w}(t_k), \quad k = 0, 1, 2, 3, \ldots \tag{1d}$$

where t_k, $k = 0, 1, 2, \ldots$, are the (not necessarily equidistant) sampling instants and $\{\mathbf{w}(t_k)\}$ is a discrete-time Gaussian white noise process independent of $\{\mathbf{v}(t)\}$, with

$$\mathrm{E}\{\mathbf{w}(t_k)\} = 0, \qquad \operatorname{cov}(\mathbf{w}(t_k), \mathbf{w}(t_l)) = \mathbf{W}(t_k)\delta(k-\ell) \tag{1e}$$

The control is piecewise constant, i.e.

$$\mathbf{u}(t) = \mathbf{u}(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, 3 \ldots \tag{1f}$$

The information available to compute the control $\mathbf{u}_k$ consists of the measurements and the controls up to t_{k-1}, i.e. $\{\mathbf{y}(t_i),\ i = 0, 1, 2, \ldots, k-1\}$ and $\{\mathbf{u}(t_i),\ i = 0, 1, 2, \ldots, k-1\}$. In this case the time available for the computer to compute $\mathbf{u}(t_k)$ equals $t_k - t_{k-1}$. Although not treated in this paper, we may also assume the information to be $\{\mathbf{y}(t_i),\ i = 0, 1, 2, \ldots, k\}$ and $\{\mathbf{u}(t_i),\ i = 0, 1, 2, \ldots, k-1\}$, in case the computation time is negligible compared with $t_k - t_{k-1}$. In the latter case all the results of this paper still hold if we substitute the discrete-time Kalman one-step-ahead predictor by the discrete-time Kalman filter.

The *digital LQG regulator problem* for system (1) is to minimize

$$J = \mathrm{E}\left(\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{H}\mathbf{x}(t_{\mathrm{f}}) + \int_{t_0}^{t_{\mathrm{f}}} \mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)\, \mathrm{d}t\right) \tag{2}$$

where E denotes the expectation operator and $\mathbf{Q}(\cdot)$, $\mathbf{R}(\cdot)$ and $\mathbf{H}$ are all symmetric semipositive definite matrices. Furthermore,

$$t_{\mathrm{f}} = t_N \tag{3}$$

where N is a positive integer.

The *digital LQG tracking problem* takes the following form. Given system (1) and a reference trajectory

$$\mathbf{x}_r(t), \quad t_0 \leqslant t \leqslant t_f \tag{4}$$

minimize

$$J = \mathrm{E}\left((\mathbf{x}(t_f) - \mathbf{x}_r(t_f))^{\mathrm{T}} \mathbf{H} (\mathbf{x}(t_f) - \mathbf{x}_r(t_f)) + \int_{t_0}^{t_f} (\mathbf{x}(t) - \mathbf{x}_r(t))^{\mathrm{T}} \mathbf{Q}(t) (\mathbf{x}(t) - \mathbf{x}_r(t)) + \mathbf{u}^{\mathrm{T}}(t) \mathbf{R}(t) \mathbf{u}(t) \, \mathrm{d}t \right) \tag{5}$$

where furthermore (3) holds and again $\mathbf{Q}(\cdot)$, $\mathbf{R}(\cdot)$ and $\mathbf{H}$ are all symmetric semipositive definite matrices. Obviously the digital LQG regulator problem is a special case of the digital LQG tracking problem, i.e. the case where

$$\mathbf{x}_r(t) = 0, \quad t_0 \leqslant t \leqslant t_f \tag{6}$$

The solution to the digital LQG tracking problem, i.e. the *digital LQG tracker*, is given by[9,19]

$$\mathbf{u}_k = -\mathbf{K}_k \hat{\mathbf{x}}_k + \mathbf{K}_k^1 \mathbf{n}_{k+1} + \mathbf{K}_k^2 \mathbf{d}_k \tag{7a}$$

$$\mathbf{K}_k = (\mathbf{R}_k + \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Gamma}_k)^{-1} (\mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Phi}_k + \mathbf{M}_k^{\mathrm{T}}) \tag{7b}$$

$$\mathbf{K}_k^1 = (\mathbf{R}_k + \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Gamma}_k)^{-1} \mathbf{\Gamma}_k^{\mathrm{T}} \tag{7c}$$

$$\mathbf{K}_k^2 = (\mathbf{R}_k + \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Gamma}_k)^{-1} \tag{7d}$$

$$\mathbf{S}_k = \mathbf{Q}_k + \mathbf{\Phi}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Phi}_k - \mathbf{K}_k^{\mathrm{T}} (\mathbf{R}_k + \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Gamma}_k) \mathbf{K}_k, \qquad \mathbf{S}_N = \mathbf{H} \tag{7e}$$

$$\mathbf{n}_k = (\mathbf{\Phi}_k - \mathbf{\Gamma}_k \mathbf{K}_k)^{\mathrm{T}} \mathbf{n}_{k+1} - \mathbf{K}_k^{\mathrm{T}} \mathbf{d}_k + \mathbf{l}_k, \qquad \mathbf{n}_N = \mathbf{H} \mathbf{x}_r(t_f) \tag{7f}$$

and the minimum cost is

$$\begin{aligned} J = {} & \bar{\mathbf{x}}_0^{\mathrm{T}} \mathbf{S}_0 \bar{\mathbf{x}}_0 - 2 \bar{\mathbf{x}}_0^{\mathrm{T}} \mathbf{n}_0 + \mathbf{x}_r^{\mathrm{T}}(t_f) \mathbf{H} \mathbf{x}_r(t_f) + \mathrm{tr}(\mathbf{S}_0 \mathbf{G}) + \sum_{k=0}^{N-1} \{ \mathrm{tr}[\mathbf{K}_k^{\mathrm{T}} (\mathbf{R}_k + \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{S}_{k+1} \mathbf{\Gamma}_k) \mathbf{K}_k \mathbf{P}_k] \\ & + \mathrm{tr}(\mathbf{V}_k \mathbf{S}_{k+1}) + \mathbf{z}_k + \gamma_k - (\mathbf{K}_k^1 \mathbf{n}_{k+1})^{\mathrm{T}} (2 \mathbf{d}_k + \mathbf{\Gamma}_k^{\mathrm{T}} \mathbf{n}_{k+1}) - \mathbf{d}_k^{\mathrm{T}} \mathbf{K}_k^2 \mathbf{d}_k \} \end{aligned} \tag{7g}$$

where

$$\mathbf{u}_k = \mathbf{u}(t_k) \tag{8a}$$

$$\mathbf{x}_k = \mathbf{x}(t_k) \tag{8b}$$

$$\mathbf{\Phi}_k = \mathbf{\Phi}(t_{k+1}, t_k) \tag{8c}$$

$$\mathbf{\Gamma}_k = \mathbf{\Gamma}(t_{k+1}, t_k) \tag{8d}$$

$$\mathbf{V}_k = \mathbf{V}(t_{k+1}, t_k) \tag{8e}$$

$$\mathbf{Q}_k = \int_{t_k}^{t_{k+1}} \mathbf{\Phi}^{\mathrm{T}}(t, t_k) \mathbf{Q}(t) \mathbf{\Phi}(t, t_k) \, \mathrm{d}t \tag{8f}$$

$$\mathbf{M}_k = \int_{t_k}^{t_{k+1}} \mathbf{\Phi}^{\mathrm{T}}(t, t_k) \mathbf{Q}(t) \mathbf{\Gamma}(t, t_k) \, \mathrm{d}t \tag{8g}$$

$$\mathbf{R}_k = \int_{t_k}^{t_{k+1}} (\mathbf{R}(t) + \mathbf{\Gamma}^{\mathrm{T}}(t, t_k) \mathbf{Q}(t) \mathbf{\Gamma}(t, t_k)) \, \mathrm{d}t \tag{8h}$$

$$\mathbf{l}_k = \int_{t_k}^{t_{k+1}} \mathbf{\Phi}^{\mathrm{T}}(t, t_k) \mathbf{Q}(t) \mathbf{x}_r(t) \, \mathrm{d}t \tag{8i}$$

$$\mathbf{d}_k = \int_{t_k}^{t_{k+1}} \mathbf{\Gamma}^{\mathrm{T}}(t, t_k)\mathbf{Q}(t)\mathbf{x}_{\mathrm{r}}(t)\,\mathrm{d}t \tag{8j}$$

$$\mathbf{z}_k = \int_{t_k}^{t_{k+1}} \mathbf{x}_{\mathrm{r}}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{x}_{\mathrm{r}}(t)\,\mathrm{d}t \tag{8k}$$

$$\gamma_k = \int_{t_k}^{t_{k+1}} \mathrm{tr}(\mathbf{V}(t, t_k)\mathbf{Q}(t))\,\mathrm{d}t \tag{8l}$$

in which $\Phi(t,s)$ is the state transition matrix considered over the interval $[t,s]$ of system (1a),

$$\mathbf{\Gamma}(t, t_k) = \int_{t_k}^{t} \mathbf{\Phi}(t,s)\mathbf{B}(s)\,\mathrm{d}s \tag{8m}$$

and

$$\mathbf{V}(t, t_k) = \int_{t_k}^{t} \mathbf{\Phi}(t,s)\mathbf{V}(s)\mathbf{\Phi}^{\mathrm{T}}(t,s)\,\mathrm{d}s \tag{8n}$$

The minimum variance state estimate $\hat{\mathbf{x}}_k$ is generated by the well-known discrete-time Kalman one-step-ahead predictor for the so-called equivalent discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k\mathbf{x}_k + \mathbf{\Gamma}_k\mathbf{u}_k + \mathbf{v}_k \tag{9a}$$

$$\mathbf{u}_k = \mathbf{C}_k\mathbf{x}_k + \mathbf{w}_k \tag{9b}$$

where $\{\mathbf{v}_k\}$ is a Gaussian discrete-time white noise process characterized by

$$\mathrm{E}\{\mathbf{v}_k\} = 0, \qquad \mathrm{cov}(\mathbf{v}_k, \mathbf{v}_l)) = \mathbf{V}_k\delta(k-l) \tag{9c}$$

and

$$\Phi_K = \Phi(t_{K+1}, t_K), \quad \Gamma_K = \Gamma(t_{K+1}, t_K)$$

$$\mathbf{C}_k = \mathbf{C}(t_k) \tag{9d}$$

$$\mathbf{w}_k = \mathbf{w}(t_k) \tag{9e}$$

Also, $\mathbf{P}_k$ in (7g) is obtained from the Kalman one-step-ahead predictor which is given by

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{\Phi}_k - \mathbf{H}_k\mathbf{C}_k)\hat{\mathbf{x}}_k + \mathbf{H}_k\mathbf{y}_k + \mathbf{\Gamma}_k\mathbf{u}_k, \qquad \hat{\mathbf{x}}_0 = \bar{\mathbf{x}}_0 \tag{10a}$$

$$\mathbf{H}_k = \mathbf{\Phi}_k\mathbf{P}_k\mathbf{C}_k^{\mathrm{T}}(\mathbf{C}_k\mathbf{P}_k\mathbf{C}_k^{\mathrm{T}} + \mathbf{W}_k)^{-1} \tag{10b}$$

$$\mathbf{P}_{k+1} = (\mathbf{\Phi}_k - \mathbf{H}_k\mathbf{C}_k)\mathbf{P}_k(\mathbf{\Phi}_k - \mathbf{H}_k\mathbf{C}_k)^{\mathrm{T}} + \mathbf{H}_k\mathbf{W}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k, \qquad \mathbf{P}_0 = \mathbf{G} \tag{10c}$$

where

$$\mathbf{W}_k = \mathbf{W}(t_k) \tag{10d}$$

The digital LQG regulator is obtained by setting $\mathbf{l}_k$, $\mathbf{d}_k$, $\mathbf{z}_k$, $k = 0, 1, 2, \ldots, N-1$, and $\mathbf{n}_k$, $k = 1, 2, \ldots, N$, to zero.

3. COMPUTATION OF THE DIGITAL LQG REGULATOR AND TRACKER

3.1. Computation of the state transition matrix

The main difficulty in the numerical computation of the digital LQG tracker (7)–(10) is the computation of (8c)–(8l), since the other formulae constitute backward recursions. Some of them are also involved in the well-known discrete-time LQG regulator problem,[12] and have received a lot of attention in the literature (see e.g. Reference 13). In the following we will therefore focus on the numerical computation of (8c)–(8l) and use the following well-known facts concerning the state transition matrix:

$$\boldsymbol{\Phi}(t_1, t_3) = \boldsymbol{\Phi}(t_1, t_2)\boldsymbol{\Phi}(t_2, t_3) \quad \forall t_1, t_2, t_3, \quad t_0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant t_f \tag{11a}$$

$$\boldsymbol{\Phi}(t_1, t_1) = \mathbf{I} \quad \forall t_1, \quad t_0 \leqslant t_1 \leqslant t_f \tag{11b}$$

where $\mathbf{I}$ is the identity matrix. Furthermore, if

$$\mathbf{A}(t) = \mathbf{A}_1, \quad t \in [t_1, t_2], \quad t_0 \leqslant t_1 \leqslant t_2 \leqslant t_f \tag{12}$$

i.e. $\mathbf{A}(t)$ is a constant matrix within $[t_1, t_2]$, then

$$\boldsymbol{\Phi}(t_1, t_2) = \exp(\mathbf{A}_1(t_2 - t_1)) \tag{13}$$

To numerically compute (8c)–(8l), $\mathbf{A}(t)$, $t \in [t_k, t_{k+1})$, $k = 0, 1, 2, \ldots, N-1$, in system (1a) is approximated by a series of constant matrices in the following way:[14]

$$\mathbf{A}'(t) = \mathbf{A}_i, \quad t \in [t_k + (i-1)\,\Delta t_k, t_k + i\,\Delta t_k), \quad i = 1, 2, \ldots, I_k \tag{14a}$$

where Δt_k and I_k are related through

$$\Delta t_k = (t_{k+1} - t_k)/I_k \tag{14b}$$

and Δt_k must be chosen such that I_k is a positive integer. Finally,

$$\mathbf{A}_i = \int_{t_k+(i-1)\,\Delta t_k}^{t_k + i\,\Delta t_k} \mathbf{A}(t)\,\mathrm{d}t, \quad i = 1, 2, \ldots, I_k \tag{15}$$

so $\mathbf{A}(t)$ over $[t_k + (i-1)\,\Delta t_k, t_k + i\,\Delta t_k)$ is approximated by its average value. By reducing Δt_k, the approximation can be made arbitrarily close.[14] Summarizing, we approximate system (1a), $t \in [t_k, t_{k+1})$, by

$$\dot{\mathbf{x}}(t) = \mathbf{A}'(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{v}(t) \tag{16}$$

Since $\mathbf{A}'(t)$ is piecewise constant according to (11)–(13), we have for the state transition matrix of (16)

$$\boldsymbol{\Phi}(t_k + L\,\Delta t_k, t_k) = \prod_{i=1}^{L} \exp(\mathbf{A}_i\,\Delta t_k), \quad L = 1, 2, \ldots, I_k \tag{17}$$

Note that when the time-varying system (1a) belongs to the class of commutative systems, i.e.

$$\mathbf{A}(t_1)\mathbf{A}(t_2) = \mathbf{A}(t_2)\mathbf{A}(t_1) \quad \forall t_1, t_2 \in [t_0, t_f] \tag{18}$$

then (17) also holds for the original system (1a), since for a commutative system[15]

$$\boldsymbol{\Phi}(t_1, t_2) = \exp\left(\int_{t_1}^{t_2} \mathbf{A}(t)\,\mathrm{d}t\right), \quad t_0 \leqslant t_1 \leqslant t_2 \leqslant t_f \tag{19}$$

Only if system (1a) does not belong to this class, which according to (18) is very restrictive, does (17) constitute an approximation with regard to system (1a). In this case Δt_k should be sufficiently small that (16) approximates (1a) close enough. Note that the computation of (17) can be performed as a forward-in-time recursion. If $\Delta t_k \ll 1$, then (16) can be very well approximated by a second-order Taylor expansion[16]

$$\exp(\mathbf{A}_i\,\Delta t_k) \approx \mathbf{I} + \mathbf{A}_i\,\Delta t_k + 0{\cdot}5\mathbf{A}_i^2\,\Delta t_k^2 \tag{20}$$

Using (20), the error during each time step Δt_k is of the order Δt_k^3. As it will turn out later, the error using the trapezoidal rule is of the same order. If the requirement $\Delta t_k \ll 1$ is too severe considering the approximation of system (1a) by system (16), then (20) can be replaced by another method to compute $\exp(\mathbf{A}_i\, \Delta t_k)$. A well-known method is the scaling and squaring method,[16]

$$\exp(\mathbf{A}_i\, \Delta t_k) = [\exp(\mathbf{A}_i\, \Delta t_k/m)]^m \tag{21}$$

where m is a positive, sufficiently large integer so that $\Delta t/m \ll 1$, and we can use (20) to obtain

$$\exp(\mathbf{A}_i\, \Delta t_k/m) \approx \mathbf{I} + \mathbf{A}_i\, \Delta t_k/m + 0{\cdot}5\mathbf{A}_i^2(\Delta t_k/m)^2 \tag{22}$$

Note, however, that (21), (22) is exactly the same as (17), (20) with Δt_k replaced by $\Delta t_k/m$ and where m consecutive values of $\mathbf{A}_i$ are kept constant. If the evaluation of $\mathbf{A}(t)$ is cheap, choosing $\Delta t_k \ll 1$ initially and using (17), (20) is preferable from the point of view of accuracy and simplicity. In the following we will choose $\Delta t_k \ll 1$, such that (16) approximates (1a) properly, and use (15), (17) and (20) to compute the state transition matrix of the time-varying system (1a). The error during each integration step is then of the order Δt_k^3. This procedure is a natural extension of the scaling and squaring method originally designed to compute the state transition matrix for time-invariant systems. In Section 4 an example is presented which demonstrates that the error can be made arbitrarily small.

3.2. *Numerical integration*

Consider a general matrix function

$$\mathbf{F}(t), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots, N-1 \tag{23}$$

The trapezoidal numerical integration rule, where the integrand is evaluated at equidistant times given by (14b), is based on the approximation

$$\int_{t_k}^{t_k + L\,\Delta t_k} \mathbf{F}(t)\,\mathrm{d}t \approx (\Delta t_k/2)\left(\mathbf{F}(t_k) + \mathbf{F}(t_k + L\,\Delta t_k) + 2\sum_{i=1}^{L-1} \mathbf{F}(t_k + i\,\Delta t_k)\right), \quad L = 2, 3, \ldots, I_k \tag{24a}$$

while for $L = 1$ we have

$$\int_{t_k}^{t_k + \Delta t_k} \mathbf{F}(t)\,\mathrm{d}t \approx (\Delta t_k/2)(\mathbf{F}(t_k) + \mathbf{F}(t_k + \Delta t_k)) \tag{24b}$$

The error during each time step Δt_k caused by approximation (24) is of the order Δt_k^3,[17] i.e. of the same order as the error caused by approximation (20). From (24) observe that each integral, using the trapezoidal rule, can be computed as a forward-in-time recursion using successive equidistant evaluations of the integrand. All matrices appearing in the integrands of (8d)–(8l) are *a priori* known except for $\mathbf{\Phi}$ and $\mathbf{\Gamma}$. Given (15), (17), (20) and (24b), $\mathbf{\Phi}$ can also be computed as a forward-in-time recursion using successive equidistant evaluations of $\mathbf{A}(t)$. Consider $\mathbf{\Gamma}$ given by (8d). If $\mathbf{\Gamma}_N$ denotes the value of $\mathbf{\Gamma}$ obtained via numerical integration with the trapezoidal rule, from (8d), (11b) and (24a) we have

$$\mathbf{\Gamma}_N(t_k + L\,\Delta t_k, t_k) = \Delta t_k/2)\left(\mathbf{\Phi}(t_k + L\,\Delta t_k, t_k)\mathbf{B}(t_k) + \mathbf{B}(t_k + L\,\Delta t_k)\right.$$

$$\left. + 2\sum_{i=1}^{L-1} \mathbf{\Phi}(t_k + L\Delta t_k, \underset{+i\Delta t_k}{t_k})\mathbf{B}(t_k + i\,\Delta t_k)\right), \quad L = 2, 3, \ldots, I_k \tag{25a}$$

while for $L = 1$ we have

$$\boldsymbol{\Gamma}_N(t_k + \Delta t_k, t_k) = (\Delta t_k/2)(\boldsymbol{\Phi}(t_k + \Delta t_k, t_k)\mathbf{B}(t_k) + \mathbf{B}(t_k + \Delta t_k)) \tag{25b}$$

Furthermore,

$$\boldsymbol{\Gamma}_N(t_k + (L+1)\,\Delta t_k, t_k) = (\Delta t_k/2)\Big(\boldsymbol{\Phi}(t_k + (L+1)\,\Delta t_k, t_k)\mathbf{B}(t_k) + \mathbf{B}(t_k + (L+1)\,\Delta t_k) + 2\sum_{i=1}^{L} \boldsymbol{\Phi}(t_k + L\Delta t_k, \underset{+i\Delta t_K}{t_k})\mathbf{B}(t_k + i\,\Delta t_k)\Big), \quad L = 1, 2, 3, \ldots, I_k - 1 \tag{26}$$

From (25), (26), (11) and (12) we obtain

$$\boldsymbol{\Gamma}_N(t_k + (L+1)\,\Delta t_k, t_k) = \boldsymbol{\Phi}(t_k + (L+1)\,\Delta t_k, t_k + L\,\Delta t_k)\boldsymbol{\Gamma}_N(t_k + L\,\Delta t_k, t_k) + (\Delta t_k/2) \times (\boldsymbol{\Phi}(t_k + (L+1)\,\Delta t_k, t_k + L\,\Delta t_k)\mathbf{B}(\underset{+L\Delta t_K}{t_k}) + \mathbf{B}(t_k + (L+1)\,\Delta t_k)), \quad L = 2, 3, \ldots, I_k - 1 \tag{27}$$

Same equation for V_K

From (28) and the previous results it follows that $\boldsymbol{\Gamma}$ can also be computed as a forward-in-time recursion using successive equidistant evaluations of $\mathbf{A}(t)$ and $\mathbf{B}(t)$. Summarizing, the trapezoidal numerical integration rule as described in this section together with the results of the previous section allow us to compute equations (8c)–(8l) as forward-in-time recursions using successive equidistant evaluations of the system and criterion matrices within $[t_k, t_{k+1}]$.

4. A NUMERICAL EXAMPLE

On the basis of the procedures outlined in Section 3, the digital LQG regulator and tracker have been programmed by the author using PC-Matlab. Two approximations are made within the proposed computation scheme: firstly we compute the state transition matrix of system (16) instead of (1a) and secondly we compute all integrals using the trapezoidal rule. When the time-varying system (1a) is commutative, the state transition matrix for system (1a) and (16) is the same at the time instants given by (14) (see Section 3.2). We first demonstrate through a numerical example that for a non-commutative system we can approximate the state transition matrix arbitrarily close by choosing Δt_k in (14) sufficiently small. Consider the non-commutative time-varying linear system characterized by

$$\mathbf{A}(t) = \begin{pmatrix} -3t^2 & 0 \\ 3t^5 & -6t^2 \end{pmatrix} \tag{28}$$

for which an analytical expression for the state transition matrix is known and is given by[18]

$$\boldsymbol{\Phi}(t, 0) = \begin{pmatrix} \exp(-t^3) & 0 \\ t^3\exp(-t^3) & \exp(-t^3) \end{pmatrix} \tag{29}$$

We computed the components of $\boldsymbol{\Phi}(t, 0)$ using the exact solution (29) and the numerical procedure based on (15), (17), (20) and (25) with $t_k = 0$, $t_{k+1} = 2$, $L = 1, 2, \ldots, I_k$ and $\Delta t_k = 0{\cdot}1$. The maximum error relative to the maximum value attained within the interval $[t_k, t_{k+1}]$ of each element of $\boldsymbol{\Phi}(t, 0)$ was computed. The maximum of these four values equals 47% and holds for the non-zero off-diagonal element. If we do the same experiment with $\Delta t_k = 0{\cdot}02$, we obtain $0{\cdot}28\%$, which also holds for the non-zero off-diagonal element. This demonstrates that by reduction of Δt_k the error can be made arbitrarily small.

Next consider the non-commutative system (1) with

$$\mathbf{x}_0 = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{30a}$$

$$\mathbf{A}(t) = \begin{pmatrix} t^2 - 1 & 0 \\ 5 & 2(t^2 - 1) \end{pmatrix} \tag{30b}$$

$$\mathbf{B}(t) = \begin{pmatrix} \sin(3t) & 1 \\ -1 & \cos(3t) \end{pmatrix} \tag{30c}$$

$$\mathbf{V}(t) = 0{\cdot}2 \begin{pmatrix} 1 + \sin(t) & 0 \\ 0 & 1 + \cos(3t) \end{pmatrix} \tag{30d}$$

$$\mathbf{G} = \begin{pmatrix} 0{\cdot}2 & 0 \\ 0 & 0{\cdot}2 \end{pmatrix} \tag{30e}$$

$$\mathbf{C}(t_k) = \begin{pmatrix} 2\sin(4t_k) & -2 \\ -1 & \cos(3t_k) \end{pmatrix} \tag{30f}$$

$$\mathbf{W}(t_k) = 0{\cdot}2 \begin{pmatrix} 1 + \cos(2t_k) & 0 \\ 0 & 1 + \sin(t_k) \end{pmatrix} \tag{30g}$$

$$t_0 = 0, \quad t_1 = 0{\cdot}2, \quad t_2 = 0{\cdot}7, \quad t_3 = 1{\cdot}1, \quad t_4 = 1{\cdot}4, \quad t_f = t_5 = 2 \tag{30h}$$

the reference state trajectory

$$\mathbf{x}_r(t) = 10[\sin(t) \ \cos(t)]^{\mathrm{T}}, \quad t_0 \leqslant t \leqslant t_f \tag{31}$$

and finally the tracking criterion (5) with

$$\mathbf{Q}(t) = \begin{pmatrix} 2 + \sin(2t) & 0 \\ 0 & 2 + \sin(2t) \end{pmatrix} \tag{32a}$$

$$\mathbf{R}(t) = \begin{pmatrix} 2 + \cos(2t) & 0 \\ 0 & 2 + \cos(2t) \end{pmatrix} \tag{32b}$$

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{32c}$$

The example is a general one in the sense that we consider a multi-input/multi-output system, non-equidistant sampling, while all system and criterion matrices are time-varying. We confine ourselves to mentioning the value of the minimum cost given by (7g), since the computation of (7g) requires all computations within our numerical algorithm to be performed. The minimum value of the cost computed by our numerical procedure for example (30)–(32) is

$$J = \cancel{472{\cdot}4500} \quad 658.6473 \tag{33}$$

The computation is performed with the step size $\Delta t_k = 0{\cdot}02$, $k = 0, 1, 2, 3, 4$. To verify most of the outcome of our numerical procedure, we now consider a special case of the LQG tracker, i.e. the LQ tracker. In this case we omit the white noise process $\{\mathbf{v}(t)\}$ in (1a) and replace $\hat{\mathbf{x}}_k$ by $\mathbf{x}_k$, $k = 0, 1, 2, \ldots, N-1$, which we assume to be deterministic and available. Note that in this case, given the deterministic initial state $\mathbf{x}_0$, the criterion value J is uniquely determined by the *finite control sequence* (1f). Thus J may be regarded as a complicated

Table I. Optimal controls for the LQ tracker example and their differences in percentages (see text)

t_0	t_1	t_2	t_3	t_4
$-1\cdot1936$	$-2\cdot9236$	$2\cdot7360$	$8\cdot3248$	$5\cdot7943\times10^{-1}$
$-3\cdot6922$	$-4\cdot8013$	$-6\cdot3647\times10^{-1}$	$6\cdot6531$	$4\cdot5704$
$-1\cdot1741$	$-2\cdot9223$	$2\cdot7327$	$8\cdot3218$	$5\cdot6305\times10^{-1}$
$-3\cdot6940$	$-4\cdot7945$	$-6\cdot5215\times10^{-1}$	$6\cdot6524$	$4\cdot5668$
$1\cdot6267$	$4\cdot3618\times10^{-2}$	$1\cdot2220\times10^{-1}$	$3\cdot6309\times10^{-2}$	$2\cdot8279$
$5\cdot0631\times10^{-2}$	$1\cdot4099\times10^{-1}$	$2\cdot4633$	$9\cdot1988\times10^{-3}$	$8\cdot0255\times10^{-2}$

function of the finite sequence (1f) From (5) observe that given this sequence, J may be computed through numerical integration. Application of the IMSL library routine BCPOL, which minimizes a function of a finite number of variables, i.e. function (5) of the finite sequence (1f) together with a numerical integration procedure to compute (5) given the sequence (1f), i.e. the routine IVPRK from the IMSL library, allows us to compute the optimal control which minimizes J.

The optimal control that results from our numerical scheme is obtained through simulation of the equivalent discrete-time system, where the white noise $\mathbf{v}_k$ is omitted, while applying the control (7a), where $\hat{\mathbf{x}}_k$ is replaced by $\mathbf{x}_k$. The result is shown in the first two rows of Table I, the first row corresponding to the first control variable. Given this control the value of J computed through numerical integration of the deterministic system (1a) using the IMSL routine IVPRK is

$$J = 422\cdot6672 \tag{34}$$

If in equation (7g) we omit γ_k and all terms involving a trace operator, we determine the cost of the digital LQ tracker. By doing so, we obtain

$J =$ ~~[illegible]~~ 422.6099 (35)

This result matches (36) within 0·04%. Having initiated the minimization with the optimal control computed from our numerical scheme which results in the cost (34), we finally arrive at a minimum cost value of

$$J = 422\cdot6556 \tag{36}$$

so the improvement is negligible. The control corresponding to (36) is represented by the third and fourth rows of Table I. Finally, the last two rows of Table I represent the differences in percentages with the optimal control obtained from our numerical scheme. This verifies the 'LQ part', i.e. the feedback, feedforward and the deterministic part of the cost, computed from our numerical scheme.

5. CONCLUSIONS

Very often digital control problems are approximated by either discrete-time or continuous-time control problems. Digital controllers obtained from these approximations require the sampling time to be small and constitute only approximate solutions. The digital LQG regulator and tracker constitute solutions to real digital control problems which involve sampled-data, piecewise constant controls and integral criteria. The numerical computation of the digital LQG regulator for *time-varying systems* for the first time allows the *computation* of digital LQG regulators to control non-linear stochastic systems about pre-specified state trajectories, e.g. a robot performing a prescribed motion or a batch fermentation process. The numerical computation of the digital LQG tracker has never been considered before in the literature, which in the light of computer control is remarkable. The LQG tracker can be applied in all situations where the aim is to let a linear stochastic system track a pre-specified state trajectory, e.g. a Cartesian robot performing a prescribed motion. The computation of the digital LQG regulator and tracker has been programmed using PC-Matlab. The software is available on request.

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