OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth. 2006; **27**:137–160
Published online 24 November 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/oca.770

On the synthesis of time-varying LQG weights and noises along optimal control and state trajectories

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SUMMARY

A general approach to control non-linear uncertain systems is to apply a pre-computed nominal optimal control, and use a pre-computed LQG compensator to generate control corrections from the on-line measured data. If the non-linear model, on which the optimal control and LQG compensator design is based, is of sufficient quality, and when the LQG compensator is designed appropriately, the closed-loop control system is approximately optimal. This paper contributes to the selection and computation of the time-varying LQG weighting and noise matrices, which determine the LQG compensator design. It is argued that the noise matrices may be taken time-invariant and diagonal. Three very important considerations concerning the selection of the time-varying LQG weighting matrices are turned into a *concrete computational scheme*. Thereby, the selection of the time-varying LQG weighting matrices is reduced to selecting *three scalar design parameters*, each one weighting one consideration. Although the three considerations seem straightforward they may *oppose* one another. Furthermore, they usually result in time-varying weighting matrices that are *indefinite*, rather than positive (semi) definite as required for the LQG design. The computational scheme presented in this paper addresses and resolves both problems. By two numerical examples the benefits of our optimal closed-loop control system design are demonstrated and evaluated using Monte Carlo simulation. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: LQG weighting matrices; design parameters; neighbouring optimal control; quadratic indefinite expressions; square indefinite matrices

1. INTRODUCTION

In a way, the successful development of systems and control theory, in the 1950s and 1960s, culminated into an IEEE Transactions issue on the LQG problem in 1971 (AC-16, 6). This issue was opened by guest editor Michael Athans explaining excellently the role and use of the LQG

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Received 28 December 2004
Revised 13 September 2005
Accepted 14 September 2005

problem in control system design as well as the philosophy behind this [1]. The paper described a very general and practical approach to control non-linear uncertain systems, possibly in an (approximately) optimal manner. The paper revealed the importance of state-space modelling, optimal control, and last but not least the LQG problem, to arrive at a control system which is approximately optimal and only requires very few on-line computations because most computations can be performed off-line. The control system design takes into account both model and measurement uncertainty but these uncertainties should remain sufficiently small. If not then the linearized model used for the LQG design and the white noise used to describe the model and measurement uncertainty become inappropriate. Robust LQG design may be employed to relax this situation somewhat [2,3]. The associated robustness analysis however is restricted to the linearized system while in this application we have to consider the effect on the non-linear system. This is beyond the scope of this paper.

An alternative approach to control non-linear systems using LQG controllers relies on a set of time-invariant linear models and multiple quadratic criteria. The set of time-invariant linear models is used to approximately describe the non-linear system around several operating points and/or to try to capture robustness properties [4,5]. The approach proposed by Athans is more accurate in the sense that it linearizes about *every* point of the optimal control and state trajectory and uses this *time-varying* linearized model only at the level of *perturbation control*.

Most of the Athans paper deals with the use and selection of the LQG compensator. The paper presents, and discusses extensively, different considerations concerning how to select the LQG compensator design parameters. These design parameters are time-varying weighting matrices, which determine a quadratic cost function, and intensity and covariance matrices which represent the initial state, the system and the measurement errors. The results of the Athans paper, which considers a continuous-time control system design, can all be translated to a digital control system design, even if the sampling scheme is aperiodic and non-synchronous [6,7]. An important contribution of this paper is to show that this also applies to the computational procedure presented in this paper.

As Athans very clearly points out, the selection of the LQG design parameters requires and mirrors control engineering skills, and no general procedures for selecting them exist. The engineering skills relate to translating and weighting the importance of different objectives as to how to select the LQG design parameters. The main contribution of this paper is to select three very important LQG design objectives presented by Athans and to translate these into a concrete computational scheme. The three selected LQG design objectives, one of them being neighbouring optimal control, may *oppose* one another. Our computational procedure handles appropriately these possibly opposing objectives and reduces the selection of the time-varying LQG weighting matrices to the selection of *only three scalar design parameters*, each one weighting one design objective. Finally, we will argue in this paper that the intensity and covariance matrices may all be taken constant and diagonal, thereby further reducing the number of LQG design parameters.

The majority of papers concerning LQG design are concerned with the infinite horizon time-invariant case which results in a time-invariant dynamic optimal feedback controller obtained from time-invariant LQG design parameters. In case of the control of non-linear systems around nominal optimal control and state trajectories the optimal LQG feedback compensator is generally *time varying* and so are several of the LQG design parameters. Apart from the considerations presented by Athans, and well-known results concerning neighbouring optimal control [8,9] which is also considered in Athans paper, it seems that the selection and computation of the time-varying LQG design parameters for the control of non-linear systems around nominal optimal control and state trajectories have received little attention.

Neighbouring optimal control considers the influence of perturbations on the costs and aims at controlling these perturbations to minimize the costs. Depending on the cost criterion and the system properties it may be beneficial to decrease or increase certain perturbations. Roughly speaking, if it is beneficial to increase certain perturbations the LQG weighting matrices which result from neighbouring optimal control will not be positive semi-definite. For two reasons the latter presents a problem. Firstly, LQG theory requires this, although there may be exceptions [10,11]. Secondly, and more importantly, negative or indefinite weighting matrices do not ensure that the state perturbations and control corrections remain small. This, however, is necessary because the LQG design is based on a linearized model around the nominal optimal control and state trajectory. When deviations from these nominal trajectories do not remain small the linearized model, in general, becomes invalid. Therefore, we select a second and third design objective to compute the LQG weighting matrices which penalize perturbations from the nominal optimal control and state trajectories depending on the non-linearity of the state and the output equation, respectively. In other words, these two objectives aim at keeping the linearized model as honest as possible [1]. An interesting feature of our computational scheme is that it provides insight as to how the three design objectives may oppose one another or may be partly redundant.

The paper is organized as follows. In Section 2 we present the problem description. To present it we need to summarize shortly the control system design procedure. In Section 3 we present several mathematical preliminaries, which are not new, but of interest by themselves. They form the basis for the computation of the LQG weighting matrices in Section 4. This section, together with Section 5 which considers the adaptations required for digital optimal control system design, constitutes the main result of this paper. At the end of Section 5 the selection of diagonal intensity and covariance matrices is motivated. Section 6 deals with automatic scaling, because this paper assumes that scaling has been applied prior to designing the control system. In Section 7, by two numerical examples, the benefits of our optimal closed-loop control system design are demonstrated and evaluated using Monte Carlo simulation. In the conclusions, presented in Section 8, among other things we will argue that the results of this paper define what might be called ‘a minimal set of design parameters for optimal control systems’.

2. PROBLEM DESCRIPTION

The control system design in the Athans paper takes place at two levels. For clarity we will adopt, as much as possible, the same notation. At the highest level one a deterministic non-linear optimal control problem is solved. For the purpose of our paper it turns out to be very convenient to *scale* all the state variables so that their maximum absolute value becomes approximately equal to one. It is furthermore convenient to scale the costs. In Section 6 we present a procedure to *automatically* calculate the scaling. The optimal control problem, both before and after the scaling, is of the following form:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \quad x(t), f(.\,,.) \in R^n, \quad u(t) \in R^m \tag{1}$$

$$I = \phi(x(T)) + \int_{t_0}^{T} L(x(t), u(t)), \quad I \in R^1 \tag{2}$$

Equation (1) describes the non-linear dynamics and initial condition and Equation (2) the cost function to be minimized by the control $u(t)$, $t_0 \leqslant t \leqslant T$. The solution of this optimal control problem is the nominal optimal control and state trajectory denoted by $\{u_o(t), x_o(t), t_0 \leqslant t \leqslant T\}$. At the second level of the control system design an LQG compensator is computed which is obtained from the following problem description where $o(t)$ denotes $x_o(t)$, $u_o(t)$, $t_0 \leqslant t \leqslant T$ and E the expectation operator:

$$\delta \dot{x}(t) = \left.\frac{\partial f}{\partial x}\right|_{o(t)} \delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{o(t)} \delta u(t) + \xi(t) \tag{3}$$

$$\delta y(t) = \left.\frac{\partial g}{\partial x}\right|_{o(t)} \delta x(t) + \theta(t) \tag{4}$$

$$\bar{J} = E\left\{ \delta x'(T) F \delta x(T) + \int_{t_0}^{T} [\delta x'(t) \delta u'(t)] \begin{bmatrix} Q(t) & M(t) \\ M^{\mathrm{T}}(t) & R(t) \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} \mathrm{d}t \right\} \tag{5}$$

Equation (3) represents a linearization of system (1) around the optimal control and state trajectory and describes approximately the dynamic behaviour of small perturbations,

$$\delta x(t) = x(t) - x_o(t), \quad \delta u(t) = u(t) - u_o(t), \quad t_0 \leqslant t \leqslant T \tag{6}$$

from the optimal control and state trajectory. Equation (4) represents a linearization of the output equation,

$$y(t) = g(x(t)), \quad y \in R^l \tag{7}$$

of the non-linear system (1) and describes approximately the behaviour of output perturbations,

$$\delta y(t) = y(t) - y_o(t) \tag{8}$$

from the optimal output $y_o(t) = g(x_o(t))$, $t_0 \leqslant t \leqslant T$. The stochastic processes $\{\xi(t)\}$, $\{\theta(t)\}$ denote, respectively, system and measurement additive white noise used to model system and measurement uncertainty. They are completely characterized by their associated intensity matrices,

$$\Sigma(t) \in R^{n \times n}, \quad \Theta(t) \in R^{l \times l} \tag{9}$$

respectively. The initial state perturbation $\delta x(t_0)$ of the linearized system (3) is assumed to be stochastic with zero mean,

$$E\{\delta x(t_0)\} = 0, \ \operatorname{cov}(\delta x(t_0) \delta x'(t_0)) = \Sigma_0 \tag{10}$$

The stochastic processes $\{\xi(t)\}$, $\{\theta(t)\}$ and the initial state perturbation $\delta x(t_0)$ are assumed to be independent. Equation (5) describes the LQG compensator design criterion where E denotes the expectation operator and where the available information to compute the control correction $\delta u(t)$ at each time $t_0 \leqslant t \leqslant T$ consists of $\{\delta u(s), \delta y(s), t_0 \leqslant s \leqslant t\}$.

In this paper we will be concerned with the selection and computation of the matrices F, $Q(t), R(t), M(t)$ which determine the LQG compensator design criterion. The matrix $M(t)$ is associated with a quadratic cross product which Athans decided to skip, to simplify the presentation. The contribution of this paper is to show that the selection and computation of $F, Q(t), R(t), M(t)$ can be highly automated using three possibly *opposing* considerations presented by Athans. A computational scheme for both continuous-time and digital LQG compensators is presented. In the case of digital LQG compensator design the matrices F, $Q(t), R(t), M(t)$ are used to determine an equivalent discrete-time LQG problem [12].

3. INDEFINITE AND SEMI-POSITIVE DEFINITE MATRICES AND QUADRATIC EXPRESSIONS

For the LQG compensator design we will be concerned with the quadratic penalties in Equation (5). To simplify the notation in (5) introduce

$$x^{\mathrm{a}}(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad x_{\mathrm{o}}^{\mathrm{a}} = \begin{bmatrix} x_{\mathrm{o}}(t) \\ u_{\mathrm{o}}(t) \end{bmatrix}, \quad \delta x^{\mathrm{a}}(t) = \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} \tag{11}$$

$$Q^{\mathrm{a}}(t) = \begin{bmatrix} Q(t) & M(t) \\ M'(t) & R(t) \end{bmatrix} \tag{12}$$

Then (5) may be written as

$$\bar{J} = E\left\{ \delta x'(T) F \delta x(T) + \int_{t_0}^{T} \partial x^{\mathrm{a}'}(t) Q^{\mathrm{a}}(t) \partial x^{\mathrm{a}}(t)\, \mathrm{d}t \right\} \tag{13}$$

The LQG compensator design requires

$$Q^{\mathrm{a}}(t) \geqslant 0, \quad F \geqslant 0 \tag{14}$$

which due to (12) implies [13],

$$Q(t) \geqslant 0, \quad R(t) \geqslant 0, \quad Q(t) - M(t) R^{+}(t) M'(t) \geqslant 0 \tag{15}$$

Observe that the condition $R(t) \geqslant 0$ is generally sufficient for digital LQG compensator design [12], while continuous-time LQG compensator design requires $R(t) > 0$.

From now on our focus will be on (13) which should satisfy (14). Therefore, the following results concerning square matrices and the associated quadratic expressions will turn out useful in the selection and computation of $Q^{\mathrm{a}}(t)$, F.

Definition 1

Let $A \in R^{n \times n}$ be a real square matrix. Define the square symmetric matrix $S(A) = (A + A')/2$ which is called the *symmetric part* of A. Define the square anti-symmetric matrix $T(A) = (A - A')/2$ which is called the *anti-symmetric* part of A.

Lemma 1

$$A = S(A) + T(A) \tag{16}$$

$$\forall x \in R^n : x'Ax = x'S(A)x \tag{17}$$

Proof
Equation (16) holds trivially. From (16) $x'Ax = x'S(A)x + x'T(A)x = x'S(A)x + 0$. □

From (16) and (17), without loss of generality, we may replace $x'Ax$ by $x'S(A)x$. This replacement will be used throughout the rest of this paper. Given the desire to satisfy (14) for every square matrix A the question is whether $S(A)\geqslant 0$ holds. One of the problems addressed in this paper is how to deal with quadratic expressions of type (17) for which $S(A)\geqslant 0$ is not satisfied. The following two definitions and two lemmas provide two different answers.

Definition 2
Based on the following singular value decomposition of $S(A)$:

$$S(A) = UDV', U, D, V \in R^{n\times n}, U, V \text{ unitary, } D\geqslant 0 \text{ and diagonal} \tag{18}$$

define,

$$S^0(A) = UDU' \geqslant 0 \tag{19}$$

Lemma 2

$$\forall x \in R^n : |x'S(A)x| \leqslant x'S^0(A)x, \quad x'S^0(A)x \geqslant 0 \tag{20}$$

Proof
Let $x_u = D^{1/2}U'x$, $x_v = D^{1/2}V'x \in R^n$ where U, D, V are determined by (18). Since $S(A)$ is square symmetric the eigenvalues of $S(A)$ are all real and the singular values on the diagonal of D are the absolute values of these eigenvalues of $S(A)$. Furthermore, the corresponding columns of U and V are identical, apart from a possible sign change. A sign change occurs for each column that is associated with a singular value which is associated to a negative eigenvalue of $S(A)$. As a result also the corresponding components of x_u, x_v are identical, apart from a possible sign change. This implies $\|x_u\|_2 = \|x_v\|_2$. As a result,

$$\forall x \in R^n : |x'S(A)x| = |x_u'x_v| \leqslant \|x_u\|_2\|x_v\|_2 = \|x_u\|_2^2 = x'S^0(A)x, \quad x'S^0(A)x \geqslant 0 \tag{21}$$

□

Lemma 2 will be used in this paper to replace indefinite or negative (semi) definite square matrices A by $S^0(A)$. Then the quadratic expression $x'S(A)x$, that can be negative for some $x \in R^n$, is replaced by the quadratic expression $x'S^0(A)x$ that is non-negative and which upper bounds $|x'S(A)x|$ for every $x \in R^n$. Observe from the proof of Lemma 2 that when $S(A)\geqslant 0$ then

$S(A) = S^0(A)$. In that case no upper bounding takes place. The next definition and lemma involve a similar replacement but now by a diagonal non-negative matrix.

Definition 3
Define,

$$D^0(A) = \mathrm{diag}(s_1 \;\; s_2 \;\; \cdots \;\; s_n), \quad s_i = \sum_{j=1}^{n} |S(A)_{i,j}| \tag{22}$$

where $S(A)_{i,j}$ denotes element i, j of $S(A) \in R^{n\times n}$.

Lemma 3

$$\forall x \in R^n : |x'S(A)x| \leqslant x'D^0(A)x, \quad x'D^0(A)x \geqslant 0 \tag{23}$$

Proof
Let $x_i \in R$, $i = 1, 2, \ldots, n$ denote the ith component of the vector x. $\forall x \in R^n$:

$$\begin{aligned} |x'S(A)x| &\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |S(A)_{i,j}||x_i||x_j| \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |S(A)_{i,j}| \frac{1}{2}(x_i^2 + x_j^2) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} |S(A)_{i,j}| x_i^2 = x'D^0(A)x \geqslant 0 \end{aligned}$$ □

Observe that if $S(A) \geqslant 0$ and diagonal then $S(A) = D^0(A)$ and no upper bounding takes place. Since several different considerations will be taken into account to select and compute $Q^{\mathrm{a}}(t) \geqslant 0$ also the following result is important.

Lemma 4

$$\forall x \in R^n : \begin{cases} |x'S(A)x| + |x'S(B)x| \leqslant x'(S^0(A) + S^0(B))x \\ |x'S(A)x| + |x'S(B)x| \leqslant x'(D^0(A) + D^0(B))x \end{cases} \tag{24}$$

Proof
Follows immediately from Lemmas 2 and 3. □

From Lemma 4 observe that, once they have been transformed into semi-positive definite quadratic expressions, as in Lemmas 2 and 3, we may simply add up quadratic penalties associated with different considerations. Throughout the rest of this paper we will use $S^0(A)$ as the non-negative upper bound for $S(A)$. If the designer prefers diagonal matrices he may replace $S^0(A)$ by $D^0(A)$ whenever he likes.

4. SELECTION AND COMPUTATION OF THE LQG STATE AND CONTROL PENALTIES

Four types of considerations are presented in the Athans paper to select and compute $Q^{a}(t) \geqslant 0$, $F \geqslant 0$ in (13).

(a) The accuracy of the linearized model around $\{u_o(t), x_o(t), y_o(t)\}$.
(b) Second-order variations of the costs (neighbouring optimal control).
(c) Effects on the dynamic compensator behaviour.
(d) Deviations of the individual states and controls.

In this section the focus will be on (a) and (b) which may oppose one another. Consideration (c) and (d) are discussed shortly at the end of this section, together with an associated simple solution to ensure $R(t) > 0$ which is required in the continuous-time case.

As to (a) the error in $\delta\dot{x}_i(t)$, the ith component of $\delta\dot{x}(t)$, caused by using the linearized equation (3), without the white noise, rather than the non-linear equation (1), is approximately equal to the second variation,

$$\delta^2\dot{x}_i(t) = \delta x^{a'}(t)\left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)} \delta x^{a}(t) \tag{25}$$

Similarly, the error in $\delta y_i(t)$ caused by using the linearized output equation (4), without the white noise, rather than the non-linear equation (7), is approximately equal to

$$\delta^2 y_i(t) = \delta x^{a'}(t)\left.\frac{\partial^2 g_i}{\partial x^{a^2}}\right|_{o(t)} \delta x^{a}(t) \tag{26}$$

As to consideration (b), from Reference [1] we have the following result. The changes of the costs due to state and control perturbations are approximately equal to $\delta^2 I$, the second variation of the costs I, because the first variation is zeroed by the optimal control. The second variation of the costs is given by

$$\delta^2 I = \delta x'(T)\left.\frac{\partial^2\phi}{\partial x(T)^2}\right|_{o(T)} \delta x(T) + \int_{t_0}^{T} \delta x^{a}(t)'\left.\frac{\partial^2 H}{\partial x^{a^2}}\right|_{o(t)} \delta x^{a}(t)\,\mathrm{d}t \tag{27}$$

where

$$H = H(x(t), p(t), u(t)) = L(x(t), u(t)) + p'(t)f(x(t), u(t)) \tag{28}$$

is the Hamiltonian and $p(t)$ the co-state vector associated with the optimal control problem (1), (2). From (28) observe that,

$$\left.\frac{\partial^2 H}{\partial x^{a^2}}\right|_{o(t)} = \left.\frac{\partial^2 L}{\partial x^{a^2}}\right|_{o(t)} + \sum_{i=1}^{n} p_i(t)\left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)} \tag{29}$$

Note that the sum in (29) can be interpreted as a weighting of the Hessian matrices

$$\left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)}, \quad i = 1, \ldots, n$$

by the associated components $p_i(t)$ of the co-state $p(t)$. The major objective of the LQG compensator is to minimize the magnitudes (absolute values) of (25), (26) for $t_0 \leqslant t \leqslant T$ and to minimize (27). Given these objectives and the LQG criterion (5) the following associations, denoted by $\leftrightarrow$, are immediate:

$$Q^{a}(t) \leftrightarrow \left.\frac{\partial^2 H}{\partial x^{a^2}}\right|_{o(t)} = \left.\frac{\partial^2 L}{\partial x^{a^2}}\right|_{o(t)} + \sum_{i=1}^{n} P_i(t) \left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)} \tag{30}$$

$$Q^{a}(t) \leftrightarrow \left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)}, \quad i = 1, \ldots, n \tag{31}$$

$$Q^{a}(t) \leftrightarrow \left.\frac{\partial^2 g_i}{\partial x^{a^2}}\right|_{o(t)}, \quad i = 1, \ldots, l \tag{32}$$

$$F \leftrightarrow \left.\frac{\partial^2 \phi}{\partial x^2(T)}\right|_{o(T)} \tag{33}$$

Given these associations there are two problems in selecting F, $Q^{a}(t)$. (1) We have to weigh (combine) the associations in (30)–(33). (2) The Hessian matrices appearing in Equations (30)–(33), in general, are not semi-positive definite.

Problem (2) is resolved by replacing the possibly indefinite or negative (semi) definite Hessian matrices by associated semi-positive definite matrices, according to Lemmas 1–3 in Section 3. Note that these replacements upper bound the magnitude (absolute value) which is precisely what we want in the case of (25), (26). In the case of (27) the situation is different. A negative outcome of (27) reduces the costs and as such should be promoted. However, promoting a negative outcome may promote state and control perturbations to grow. Therefore, minimizing (27) may *oppose* minimizing the magnitudes of (25), (26). Too large state and control perturbations, in general, destroy the accuracy of the linearized model which may result in a completely deteriorated control system behaviour. Therefore, instead of minimizing (27) we choose to minimize the magnitude (absolute value) of (27).

As to problem (1) the variation of the costs, measured by the second variation (27), and represented by (30), is weighted by the weighting factor W_I. The non-linearity of $f(x(t), u(t))$, measured by the second variation (25), and represented by (31), is weighted by the weighting factor W_f. Finally, the non-linearity of $g(x(t))$, measured by the second variation (26), and represented by (32), is weighted by the weighting factor W_g. W_I, W_f, W_g are the only three scalar weighting factors used to determine $Q^{a}(t), F$. Now, using Lemma 4 and the fact that we have scaled all the state variables and the running costs, $Q^{a}(t)$, F are selected as follows:

$$Q^{a}(t) = W_I S^0 \left(\left.\frac{\partial^2 L}{\partial x^{a^2}}\right|_{o(t)} + \sum_{i=1}^{n} p_i(t) \left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)} \right) + \frac{W_f}{n} \sum_{i=1}^{n} S^0 \left(\left.\frac{\partial^2 f_i}{\partial x^{a^2}}\right|_{o(t)} \right) + \frac{W_g}{l} \sum_{i=1}^{l} S^0 \left(\left.\frac{\partial^2 g_i}{\partial x^{a^2}}\right|_{o(t)} \right) \tag{34}$$

$$F = W_I S^0 \left(\frac{\partial^2 \phi}{\partial x^2(T)} \bigg|_{o(t)} \right) \tag{35}$$

In Equation (34) the second and third sum on the right-hand side are pre-multiplied by W_f/n, W_g/l instead of W_f, W_g to make the scaling by W_f and W_g independent of n, l. From (30), (31) observe that minimizing the magnitudes of (25), (27) is *partly redundant*. This redundancy is reflected by the first two terms involving a sum in (34). To take into account partly this redundancy we propose the following alternative for (34):

$$Q_o^a(t) = W_I S^0 \left(\frac{\partial^2 L}{\partial x^{a^2}} \bigg|_{o(t)} \right) + \sum_{i=1}^{n} \max\left(W_I |p_i(t)|, \frac{W_f}{n} \right) S^0 \left(\frac{\partial^2 f_i}{\partial x^{a^2}} \bigg|_{o(t)} \right) + \frac{W_g}{l} \sum_{i=1}^{l} S^0 \left(\frac{\partial^2 g_i}{\partial x^{a^2}} \bigg|_{o(t)} \right) \tag{36}$$

Having addressed the considerations (a) and (b) mentioned at the start of this section we now briefly address considerations (c) and (d). These can usually be translated into semi-positive definite diagonal matrices. Given the scaling of the states and running costs, the selection of the diagonal elements directly reflect the desired weighting. According to Lemma 4, these semi-positive definite diagonal matrices can simply be added to $Q^a(t)$. If $R(t)$, which is obtained directly from the partitioning (12) of $Q^a(t)$, is only semi-positive definite, and not positive definite which it should be in the continuous time case, then from (12) and Lemma 4, this problem is resolved easily by adding the following diagonal matrix to $Q^a(t)$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 10^{-9} I_m \end{bmatrix} \in R^{n+m} \tag{37}$$

where I_m is the $m \times m$ identity matrix. Because we assumed that all problem data are scaled 10^{-9} in (37) is a very small number that turns a semi-positive $R(t)$ into a positive one. Although, from a theoretical point of view, this may be considered a rather simple brute-force regularization, from an engineering point of view the performance is hardly affected by it.

5. ADAPTATIONS REQUIRED FOR DIGITAL CONTROL

Assuming the sampling occurs synchronously, but possibly aperiodically, in the case of digital control, the control is *constrained* and described by

$$u(t) = u(t_k), \quad t_k \leqslant t < t_{k+1}, \ \ k = 0, 1, \ldots, N-1, \ \ t_N = T \tag{38}$$

where

$$t_k < t_{k+1}, \quad k = 0, 1, \ldots, N-1 \tag{39}$$

are the sampling instants. The optimal control problem at levels one and two of the design are now *digital* optimal control problems. At both levels these digital optimal control problems, with the control constraint (38) and (39), are transformed into unconstrained equivalent discrete-time problems [6,7]. At level one of the design the unconstrained equivalent discrete-time optimal control problem is described by [6]

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, 1, \ldots, N-1 \tag{40}$$

$$I = \phi(x_N) + \sum_{k=0}^{N-1} L_k(x_k, u_k) \tag{41}$$

Equation (40) is the equivalent discrete-time system that describes the state transitions $x_k = x(t_k)$ to $x_{k+1} = x(t_{k+1}),\ k = 0, 1, \ldots, N-1$ of the continuous-time system (1). Equation (41) is the equivalent discrete-time cost function to be minimized by the control $u_k = u(t_k),\ k = 0, 1, \ldots, N-1$. The outcome of (41) is identical to the outcome of (2) for any digital control sequence (38), (39). Let $\{x_o(t_k), u_o(t_k),\ k = 0, 1, \ldots, N-1\}$ denote the solution of this equivalent discrete-time optimal control problem and let $o(t_k)$ be the shorthand. Furthermore, $\{x_o(t), u_o(t), t_0 \leqslant t \leqslant t_N\}$ and the associated shorthand $o(t)$ from now on denote the solution of the associated digital optimal control problem.

At level two of the design, which concerns the digital LQG compensator, the equivalent discrete-time system represents the state transitions of the linearized system (3) from t_k to $t_{k+1},\ k = 0, 1, \ldots, N-1$ and the equivalent discrete-time cost function reads,

$$\bar{J} = E\left\{\delta x_N' F \delta x_N + \sum_{k=0}^{N-1} \delta x_k' Q_k \delta x_k + 2\delta x_k' M_k \delta u_k + \delta u_k' R_k \delta u_k\right\} \tag{42}$$

Similar to (11), (12) define,

$$x_k^{\mathrm{a}} = \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad k = 0, 1, \ldots, N-1 \tag{43}$$

$$Q_k^{\mathrm{a}} = \begin{bmatrix} Q_k & M_k \\ M_k' & R_k \end{bmatrix}, \quad k = 0, 1, \ldots, N-1 \tag{44}$$

Then,

$$\bar{J} = E\left\{\delta x_N' F \delta x_N + \sum_{k=0}^{N-1} \delta x_k'^{\mathrm{a}} Q_k^{\mathrm{a}} \delta x_k^{\mathrm{a}}\right\} \tag{45}$$

In the case of digital control the output equations (7) and (4) change into,

$$y(t_k) = g(x(t_k)), \quad k = 0, 1, \ldots, N-1 \tag{46}$$

$$\delta y(t_k) = \left.\frac{\partial g}{\partial x}\right|_{o(t_k)} \delta x(t_k) + \theta_k, \quad k = 0, 1, \ldots, N-1 \tag{47}$$

where $\{\theta_k\}$ is a discrete-time zero mean white noise process with a covariance matrix $\Theta_k \in R^{l \times l},\ k = 0, 1, \ldots, N-1$. The output equations (46) and (47) apply *at* the sampling instants only, not in between, as opposed to (7) and (4). The error due to linearizing the output equation can still be taken into account, but only *after* we have transformed the digital LQG problem into an unconstrained equivalent discrete-time problem, as we will demonstrate.

As to consideration (b), mentioned at the start of Section 4, we now have to consider, instead of (29),

$$\left.\frac{\partial^2 H_k}{\partial x_k^{\mathrm{a}^2}}\right|_{o(t_k)} = \left.\frac{\partial^2 L_k}{\partial x_k^{\mathrm{a}^2}}\right|_{o(t_k)} + \sum_{i=1}^{n} p_{k_i} \left.\frac{\partial^2 f_{k_i}}{\partial x_k^{\mathrm{a}^2}}\right|_{o(t_k)} \tag{48}$$

where H_k is the Hamitonian of the unconstrained equivalent discrete-time optimal control problem at level one,

$$H_k(x_k, u_k, p_k) = L_k(x_k, u_k) + p_k' f_k(x_k, u_k) \tag{49}$$

In Equation (48) f_{k_i} and p_{k_i} denote the ith component of $f_k(x_k, u_k)$ and p_k, respectively, where $p_k, k = 0, 1, \ldots, N$ in Equation (49) is the co-state of the solution of the equivalent discrete-time optimal control problem at level one. In the appendix it is proved that if we select,

$$Q^{\mathrm{a}}(t) = \left.\frac{\partial^2 L}{\partial x^{\mathrm{a}^2}}\right|_{o(t)} + \sum_{i=1}^{n} p_{k_i} \left.\frac{\partial^2 f_i}{\partial x^{\mathrm{a}^2}}\right|_{o(t)}, \quad t_k \leqslant t < t_{k+1}, \;\; k = 0, 1, \ldots, N-1 \tag{50}$$

where f_i denotes the ith component of $f(x, u)$ in Equation (1) then,

$$Q_k^{\mathrm{a}} = \left.\frac{\partial^2 H_k}{\partial x_k^{\mathrm{a}^2}}\right|_{o(t_k)}, \quad k = 0, 1, \ldots, N-1 \tag{51}$$

and so Equation (45) minimizes the second variation of the cost due to the variations $\delta x_k = \delta x(t_k)$, $\delta u_k = \delta u(t_k)$. Selecting $Q^{\mathrm{a}}(t)$ according to Equation (50) is almost identical to selecting,

$$Q^{\mathrm{a}}(t) = \left.\frac{\partial^2 L}{\partial x^{\mathrm{a}^2}}\right|_{o(t)} + \sum_{i=1}^{n} p_i(t) \left.\frac{\partial^2 f_i}{\partial x^{\mathrm{a}^2}}\right|_{o(t)} \tag{52}$$

the selection made in the continuous-time case, when we only take into account consideration (b) in Section 4, as can be seen from Equations (27), (29), (30). Since the error due to linearizing the output equation will be considered *after* the transformation into an unconstrained equivalent discrete-time LQG problem, the similarity between (50) and (52) immediately suggests the following modifications of selections (34), (36), of $Q^{\mathrm{a}}(t)$, respectively:

$$Q^{\mathrm{a}}(t) = W_1 S^0 \left(\left.\frac{\partial^2 L}{\partial x^{\mathrm{a}^2}}\right|_{o(t)} + \sum_{i=1}^{n} p_{ki} \left.\frac{\partial^2 f_i}{\partial x^{\mathrm{a}^2}}\right|_{o(t)} \right) + \frac{W_f}{n} \sum_{i=1}^{n} S^0 \left(\left.\frac{\partial^2 f_i}{\partial x^{\mathrm{a}^2}}\right|_{o(t)} \right)$$
$$t_k \leqslant t < t_{k+1}, \;\; k = 0, 1, \ldots, N-1 \tag{53}$$

$$Q^{\mathrm{a}}(t) = W_1 S^0\left(\frac{\partial^2 L}{\partial x^{\mathrm{a}^2}}\bigg|_{o(t)}\right) + \sum_{i=1}^{n} \max\left(W_I|p_{ki}|, \frac{W_f}{n}\right) S^0\left(\frac{\partial^2 f_i}{\partial x^{\mathrm{a}^2}}\bigg|_{o(t)}\right)$$
$$t_k \leqslant t < t_{k+1}, \quad k = 0, 1, \ldots, N-1 \tag{54}$$

Observe that Equations (53) and (54) can be computed numerically from the problem data and solution at level one. Finally, the selection of F, according to (35), is unaffected by the digital nature of our control system design.

Now we are in a position to transform the digital optimal LQG problem into an unconstrained equivalent discrete-time LQG problem [12]. Next, to keep the linearized output equation (46) accurate, instead of the cost function (44), (45), we select as the cost function,

$$\bar{J}_g = E\left\{\delta x_N' F \delta x_N + \sum_{k=0}^{N-1} \delta x_k'^{\mathrm{a}} Q_k^g \delta x_k^{\mathrm{a}}\right\} \tag{55}$$

where

$$Q_k^g = Q_k^{\mathrm{a}} + (t_{k+1} - t_k)\frac{W_g}{l}\sum_{i=0}^{l} S^0\left(\frac{\partial^2 g_i}{\partial x^{\mathrm{a}^2}}\bigg|_{o(t_k)}\right), \quad k = 0, 1, \ldots, N-1 \tag{56}$$

The terms after the sum in between the brackets in (56) are the second-order terms of the output equation *at* the sampling instants, which measure the non-linearity of the output equation *at* the sampling instants, which is precisely what we want. The sum in Equation (56) is pre-multiplied with $t_{k+1} - t_k$ to ensure equal weighting, compatible with Equations (7) and (4). This can be seen by considering $t_{k+1} - t_k \to 0$ [14]. Then the digital and continuous-time results should become identical, as they do when we apply the pre-multiplication.

Summarizing, in the case of digital control system design the LQG compensator design parameters are selected according to (53) and (54). After the transformation into an unconstrained equivalent discrete-time LQG problem the equivalent discrete-time cost function matrices Q_k^{a} in (45) are replaced by Q_k^g, as in (55), where Q_k^g are given by Equation (56). The solution of this modified unconstrained equivalent discrete-time LQG problem constitutes the digital (discrete-time) LQG compensator.

Finally, we turn our attention to the intensity and covariance matrices. In the case of a continuous-time control system these are the matrices $\Sigma(t), \Theta(t), \Sigma_0$ in Equations (9), (10) and in the case of a digital control system $\Theta(t)$ is replaced by $\Theta_k,\ k = 0, 1, \ldots, N-1$ in Equation (47). Within the control system design these matrices are used to describe the errors of model (1), measurements (7) and the initial state in (1), respectively. These errors should be sufficiently small for the control system design to be successful and if they are they may be taken care of in the design by the additive white noise in (3), (4) and (10) [1]. In general, no accurate information is available with respect to these errors but only (rough) estimates of the standard deviation of each associated variable. From them we obtain *time-invariant diagonal* intensity and covariance matrices. Furthermore, from Section 2 observe that quadratic expressions with non-diagonal matrices may be upper bounded using diagonal matrices. Because of this it makes sense to further reduce the number of LQG design parameters by using time-invariant diagonal intensity and covariance matrices.

6. AUTOMATIC SCALING

In practice, our design starts from problem data that are not yet scaled, as opposed to the problem data in this paper. In this section unscaled problem data are distinguished by adding an uppercase u to the corresponding symbol. Then the scaling is described by

$$x = D_x x^u, \quad D_x = \mathrm{diag}(d_1^x, \ldots, d_n^x), \;\; d_1^x, \ldots, d_n^x > 0 \tag{57}$$

$$u = D_u u^u, \quad D_u = \mathrm{diag}(d_1^u, \ldots, d_m^u), \;\; d_1^u, \ldots, d_m^u > 0 \tag{58}$$

$$y = D_y y^u, \quad D_u = \mathrm{diag}(d_1^y, \ldots, d_1^y), \;\; d_1^y, \ldots, d_1^y > 0 \tag{59}$$

$$f = D_x f^u \tag{60}$$

$$g = D_y g^u \tag{61}$$

$$L = d_1^I L^u, \quad d_1^I > 0 \tag{62}$$

$$\phi = d_1^I \phi^u \tag{63}$$

$$I = d_1^I I^u \tag{64}$$

where $d_1^x, \ldots, d_n^x, d_1^u, \ldots, d_m^u, d_1^y, \ldots, d_1^y, d_1^I$ are the individual scaling factors of the state variables, the control inputs, the outputs and the costs, respectively. The other relations between the unscaled and scaled problem data needed in this paper are,

$$\frac{\partial^2 f_i}{\partial x^{a^2}} = d_i^x D_{xu}^{-1} \frac{\partial^2 f_i^u}{\partial x^{a^{u2}}} D_{xu}^{-1}, \quad D_{xu} = \mathrm{diag}(d_1^x, \ldots, d_n^x, d_1^u, \ldots, d_m^u) \tag{65}$$

$$\frac{\partial^2 g_i}{\partial x^{a^2}} = d_i^y D_{xu}^{-1} \frac{\partial^2 g_i^u}{\partial x^{a^{u2}}} D_{xu}^{-1} \tag{66}$$

$$\frac{\partial^2 L}{\partial x^{a^2}} = d_1^I D_{xu}^{-1} \frac{\partial^2 L^u}{\partial x^{a^{u2}}} D_{xu}^{-1} \tag{67}$$

$$\frac{\partial^2 \phi}{\partial x^2} = d_1^I D_x^{-1} \frac{\partial^2 \phi^u}{\partial x^{u^2}} D_x^{-1} \tag{68}$$

$$p(t) = D_x^{-1} p^u(t) \Leftrightarrow p_i(t) = d_i^{x^{-1}} p_i^u(t), \quad i = 1, \ldots, n \tag{69}$$

The scaling factors $d_1^I, d_1^x, \ldots, d_n^x, d_1^u, \ldots, d_m^u, d_1^y, \ldots, d_l^y$ in (57)–(59) are computed *automatically* as follows. The unscaled optimal control problem is solved first. This gives,

$$u_o^u(t), x_o^u(t), L_o^u(t), t_0 \leqslant t \leqslant t_N = T, \quad y_o^u(t_k), \;\; k = 0, 1, \ldots, N \tag{70}$$

Whenever possible, the objective to scale every state variable, every control input, every output and the costs, such that their maximum absolute values become equal to one, is achieved by selecting

$$d_1^l = \frac{1}{\max_t (|L_o^u(t)|)} \quad \text{if } \max_t (|L_o^u(t)|) \neq 0 \quad \text{else } d_1^l = 1 \tag{71}$$

$$d_i^x = \frac{1}{\max_t (|x_{o_i}^u(t)|)} \quad \text{if } \max_t (|x_{o_i}^u(t)|) \neq 0 \quad \text{else } d_i^x = 1, \quad i = 1, \ldots, n \tag{72}$$

$$d_i^u = \frac{1}{\max_t (|u_{o_i}^u(t)|)} \quad \text{if } \max_t (|u_{o_i}^u(t)|) \neq 0 \quad \text{else } d_i^u = 1, \quad i = 1, \ldots, m \tag{73}$$

$$d_i^y = \frac{1}{\max_t (|y_{o_i}^u(t)|)} \quad \text{if } \max_t (y_{o_i}^u(t)|) \neq 0 \quad \text{else } d_i^y = 1, \quad i = 1, \ldots, l \tag{74}$$

To obtain the LQG compensator for the original unscaled problem, which is the one we have to implement in the actual control system, the following relations between the scaled and unscaled LQG problem data are also needed:

$$Q^{a^u}(t) = D_{xu} Q^a(t) D_{xu} \tag{75}$$

$$F^u(t) = D_x F D_x \tag{76}$$

In the case of digital control system design, except for (74), the same scaling applies. Because measurements are only obtained at the sampling instants $t_k, k = 0, 1, \ldots, N-1$, Equation (74) should be replaced by

$$d_i^y = \frac{1}{\max_k (|y_{o_i}^u(t_k)|)} \quad \text{if } \max_k (|y_{o_i}^u(t_k)|) \neq 0 \quad \text{else } d_i^y = 1, \; i = 1, \ldots, l, \; k = 0, 1, \ldots, N-1 \tag{77}$$

The LQG compensator obtained after scaling must of course be scaled back before it can be implemented on the real system. Then, theoretically, scaling does not affect the design. But it does affect the accuracy of numerical computations. In general, the numerical accuracy improves if scaling is applied which might be another reason to apply scaling.

7. NUMERICAL EXAMPLES

Because our main interest concerns digital control, to limit space, in this section we only consider a digital control system design example. This example is *deliberately* chosen to be artificial to be able to stress and demonstrate clearly the important features and benefits of our computational procedure to select the LQG compensator. The example is a difficult one in the sense that the linearized system around the nominal solution is unstable, and therefore requires the LQG compensator for stabilization. Furthermore, the weighting matrices resulting from neighbouring optimal control are non-semi-positive definite. Although for this example the Jacobian and Hessian matrices can all be computed analytically, our software implementation, which applies to general problems, computes numerical approximations of these matrices using finite differences.

The problem data are as follows. At level one we have

$$f = \begin{pmatrix} 0.1x_1(t) - 0.02(x_2(t) - 5)^2 + 0.03u_1^3(t) \\ 0.1x_2^2(t) - 0.5u_2(t) \end{pmatrix} \tag{78}$$

$$L = 0.05(u_1^2(t) + u_2^2(t)) \tag{79}$$

$$x(t_0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t_0 = 0, \quad t_N = T = 20 \tag{80}$$

$$\phi = -x_1(t_N) \tag{81}$$

To simplify the presentation the sampling scheme is chosen to be periodic,

$$N = 20, \quad t_{k+1} - t_k = 1, \quad k = 0, 1, \ldots, N-1 \tag{82}$$

From (81), (79) observe that the objective is to maximize $x_1(t_N)$ without using too much control effort. If the objective is to maximize certain state variables neighbouring optimal control design usually results in non-semi-positive definite LQG weighting matrices, as in this example. Note from the first component of f, given by (78), that for $x_1(t)$ to grow it is beneficial for $x_2(t)$ to stay close to 5 and for $u_1(t)$ to be maximally positive. For the optimal control problem at level one, the following control bounds apply:

$$0 \leqslant u_1(t) \leqslant 3 \tag{83}$$

$$0 \leqslant u_2(t) \leqslant 10 \tag{84}$$

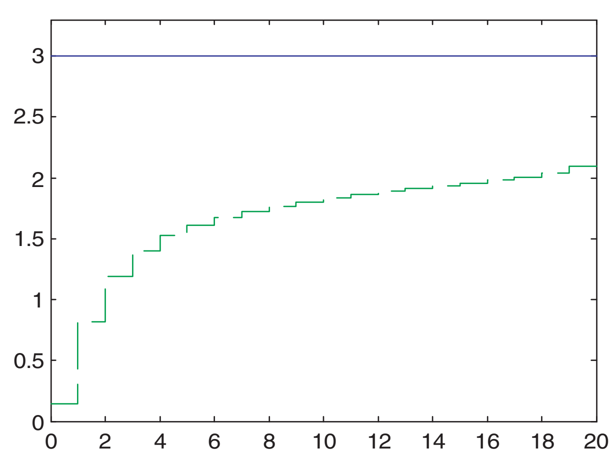

Figure 1. Optimal control, $u_1(t)$ solid, $u_2(t)$ dashed.

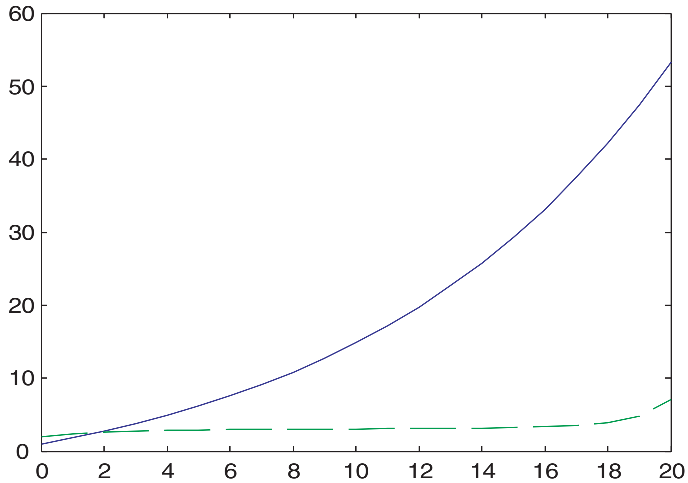

Figure 2. Optimal state trajectory, $x_1(t)$ solid, $x_2(t)$ dashed.

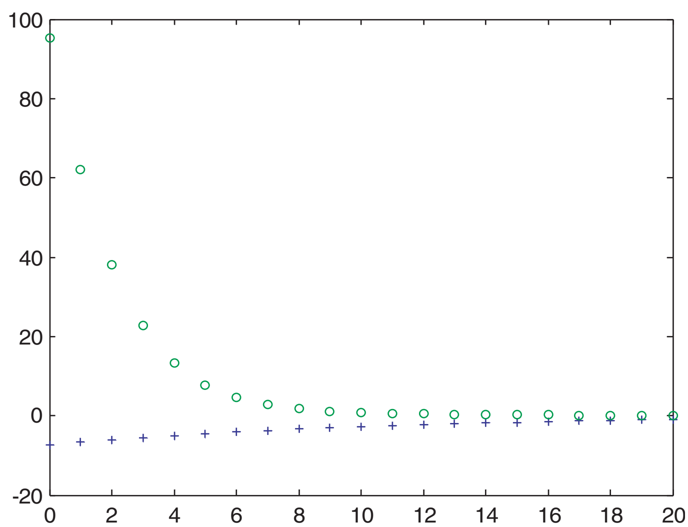

Figure 3. Optimal co-state trajectory, $p_{k_1}+, p_{k_2}\circ$.

The solution of the digital optimal control problem is displayed in Figures 1–3. The minimum costs are -41.2882. At level two of the design the following data apply:

$$g(x(t)) = x_1(t) + x_2(t) \tag{85}$$

$$\Sigma(t) = \text{diag}(0.007 \;\; 0.007), \quad t_0 \leqslant t \leqslant t_N \tag{86}$$

$$\Sigma_0 = \text{diag}(0.01 \;\; 0.01) \tag{87}$$

$$\Theta_k = 0.001, \quad k = 0, 1, \ldots, N-1 \tag{88}$$

Note that the output equation (85) is linear, and therefore all terms associated with the non-linearity of the output equation are zero and do not have to be considered. Therefore, as a second example, we consider the same problem (78)–(88) but with the output equation (85) replaced with

$$g(x(t)) = x_1(t)x_2(t) \tag{89}$$

To analyse the results of our examples it is helpful to note that,

$$\frac{\partial^2 f_1}{\partial x^{\mathrm{a}^2}} = \mathrm{diag}(0 \;\; 0.04 \;\; 0.18u_1(t) \;\; 0) \tag{90}$$

$$\frac{\partial^2 f_2}{\partial x^{\mathrm{a}^2}} = \mathrm{diag}(0 \;\; 0.2 \;\; 0 \;\; 0) \tag{91}$$

$$\frac{\partial^2 L}{\partial x^{\mathrm{a}^2}} = \mathrm{diag}(0 \;\; 0 \;\; 0.1 \;\; 0.1) \tag{92}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \mathrm{diag}(0 \;\; 0) \tag{93}$$

For example 1, Equation (85) applies and,

$$\frac{\partial^2 g_1}{\partial x^{\mathrm{a}^2}} = \mathrm{diag}(0 \;\; 0 \;\; 0 \;\; 0) \tag{94}$$

while for example 2, Equation (89) applies from which we obtain,

$$\frac{\partial^2 g_1}{\partial x^{\mathrm{a}^2}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{95}$$

Furthermore, the system matrix of the linearized system around the nominal optimal solution equals,

$$A(t) = \begin{bmatrix} 0.1 & 0.04x_{\mathrm{o}_2}(t) \\ 0 & 0.2x_{\mathrm{o}_2}(t) \end{bmatrix} \tag{96}$$

Because $x_{\mathrm{o}_2}(t) > 0, 0 \leqslant t \leqslant T$ it follows directly from (96) that this linearized system is unstable and so small perturbations will grow and *require* the LQG compensator to decay.

The selection of $\Sigma(t)$, Σ_0 and θ_k in Equations (86)–(88), which describe the system, initial state and measurement uncertainty, should reflect a worst case situation. We performed Monte Carlo simulations of the closed-loop digital control system using realizations of $\Sigma(t)$, Σ_0 and θ_k. These

Monte Carlo simulations therefore represent a worst case situation. Each Monte Carlo simulation consisted of 50 runs with 50 realizations of $\Sigma(t)$, Σ_0 and θ_k. For each Monte Carlo simulation the same 50 realizations were used. When simulating the open-loop system, i.e. without the LQG feedback compensator, for 27 out of the 50 realizations the system state tended to infinity numerically within the horizon of 20. To be more precise, in each of these 27 cases $x_2 \to +\infty$ first, which then causes $x_1 \to -\infty$, as can be seen from the first component of f given by (78). According to (81) $x_1 \to -\infty$ corresponds to an 'infinitely poor' performance.

To illustrate the effectiveness of our design, we compare our *highly automated* LQG design with a design were we *choose* $Q^{\mathrm{a}}(t)$ and F directly ourselves as follows:

$$Q^{\mathrm{a}}(t) = \mathrm{diag}(1\ \ 1\ \ 1\ \ 1), \quad t_0 \leqslant t \leqslant T, \quad F = \mathrm{diag}(0\ \ 0) \tag{97}$$

Although $Q^{\mathrm{a}}(t)$ may be selected to be time varying such a selection would be very difficult. Note that the choice (97) is only 'somewhat naïve and simple' because it is applied after the states and running costs have been scaled.

For example 1, i.e. for the example with output equation (85), and with the choice (97), 15 out of the 50 simulations resulted in a system state that tended to infinity numerically. Of the remaining $n_f = 35$ simulations the performance was evaluated by calculating

$$\overline{\Delta J} = \frac{1}{n_f} \sum_{i=1}^{n_f} |I_i - I_{\mathrm{o}}| = 17.816 \tag{98}$$

In Equation (98) I_i is the costs computed from Equation (2) for the ith realization that did not tend to infinity. So Equation (98) measures the average absolute deviation of the costs I_i from the optimal costs $I_{\mathrm{o}} = -41.2884$ obtained when there are no disturbances. Measure (98) was used because the digital control system performance is of primary importance, and is measured by the cost function (2), while a major objective of our control system design was to minimize deviations from the optimal costs I_{o} obtained when there are no disturbances.

For example 1, Table I records the outcome of (98), for the same $n_f = 35$ realizations, using different values of the design parameters W_I, W_f which, through Equations (54), (56), (35), determine $Q^{\mathrm{a}}(t), F$. Equation (54) was used instead of (53) because we prefer to consider the redundancy related to minimizing the magnitudes of (25), (27). The control bounds (83), (84), presumed for the control system design at level one, should be somewhat conservative so that the control corrections, generated by the LQG compensator, may be added on top of the optimal control. When the LQG compensator is of full-order, as in Table I, the separation property applies, and the compensator depends only on the *ratio* of the elements of $Q^{\mathrm{a}}(t), F$. In that case only the *ratio* W_I/W_f influences the control system design. If the LQG compensator is of reduced order, the separation property is lost [15]. Of the originally 50 realizations Table I also records the number of realizations n_∞ for which the system state tended to infinity numerically, within the horizon of 20. From Table I the influence of the ratio W_I/W_f is as expected. Increasing W_I reduces the deviation of the costs, possibly at the expense of large(r) variations in the state and control, or even instability. Observe from Table I that a suitable choice of W_I and W_f leads to a significant improvement over the 'somewhat naïve and simple' selection (97) illustrating another important benefit of our approach to select the LQG weighting matrices.

Table I. Closed-loop control system performance for example 1 ($W_f = 1$, W_g is irrelevant). $I_o = -41.2884$.

W_I	$\overline{\Delta J}$	n_∞
0	2.0898	0
1.0e − 9	2.0898	0
1.0e − 6	2.0895	0
1.0e − 3	1.9193	0
1.0e − 2	1.5155	3
4.0e − 2	1.3717	6
8.0e − 2	1.3512	7
1.0e − 1	1.3488	8
1.2e − 1	1.3480	8
1.4e − 1	1.3483	9
2.0e − 1	1.3522	9
3.0e − 1	1.3626	9
5.0e − 1	1.3813	9
7.0e − 1	1.3960	9
9.0e − 1	1.4074	9
5.0	1.4582	10
10.0	1.4668	10

Table II. Closed-loop control system performance for example 2 ($W_g = 1$).

W_f	W_I	$\overline{\Delta J}$	n_∞
0	1.0e − 9	0.84168	0
0	1.0e − 5	0.84366	0
0	1.0e − 4	0.85538	0
0	1.0e − 3	0.89979	0
0	1.0e − 2	1.0154	0
0	2.0e − 2	1.0581	0
0	4.0e − 2	1.1016	0
0	1.0e − 1	1.1953	0
0	2.0e − 1	1.2687	0
0	9.0E − 9	1.3999	0
0	5.0	1.4495	1
0	10.0	1.4554	1
1.0e − 9	1.0e − 9	0.84294	0
1.0e − 4	1.0e − 9	0.90908	0
1.0e − 3	1.0e − 9	0.91052	0
1.0e − 2	1.0e − 9	0.92069	0
1.0e − 1	1.0e − 9	0.96118	0
2.0e − 1	1.0e − 9	0.98108	0
9.0e − 1	1.0e − 9	1.0421	0
5.0	1.0e − 9	1.1261	0
10.0	1.0e − 9	1.1530	0

Table II displays similar results for example 2, i.e. for the example with output equation (89). Note that the optimal costs without disturbances does not depend on the output equation so $I_o = -41.2884$. In this case $n_f = 46$, the number of simulations with the full-order LQG compensator design based on (97), for which the state did not tend to infinity within the horizon of 20. The associated value of $\overline{\Delta J}$, given by (98), equals 6.6904. From Table II we conclude that the non-linearity of the output equation (89) primarily determines the accuracy since by increasing W_I and W_f, starting from zero, the performance decreases. Looking further into this matter it turned out that very small values of $R(t)$ result in the best performance while for larger values the performance soon drops. In addition to this $R(t)$ is most sensitive to both W_I and W_f due to the fact that in Equation (90)

$$\left.\frac{\partial^2 f_1}{\partial u_1^2}\right|_{o(t)} = 0.18 u_{o_1}(t) = 0.54$$

which is the largest term in Equations (90)–(93). Selecting W_I and W_f both equal to zero results in zero control penalties, even within the unconstrained equivalent discrete-time LQG problem. Therefore, very small positive values of W_I and W_f are recorded in Table II. Again observe the improved performance of the closed-loop control system compared with the 'somewhat naïve and simple' selection (97) if we select W_I, W_f and W_g appropriately.

8. CONCLUSIONS

For optimally controlled non-linear systems the selection of the LQG weighting matrix F and the time-varying LQG weighting matrices $R(t), Q(t), M(t), t_0 \leqslant t \leqslant T$ has been reduced to selecting only three scalar design parameters. At the level of perturbation control, i.e. level two of the control system design, our three scalar design parameters W_I, W_f, W_g penalize, respectively, the second-order terms which measure deviations in the costs, the second-order terms which measure errors due to linearizing the state equations, and the second-order terms which measure errors due to linearizing the output equation. The latter two are vital to keep the linearized model around the nominal solution as honest as possible [1] thereby preventing as much as possible the control system to become unstable. The first one is important from the point of view of performance. The second example in Section 7 revealed that it may sometimes be best to take into account just one of these three design objectives. The possible redundancy and also the possibility that the design objectives may oppose one another were clearly revealed as well as taken care of by our computational procedure. In the case of full-order LQG design, due to the associated separation property, only the *ratios* of the three design parameters W_I, W_f, W_g influence the LQG compensator design, thus leaving only two design parameters.

The problem that the second-order terms do not lead, in general, to semi-positive LQG weighting matrices was resolved by considering two types of semi-positive upper bounds of the absolute value of quadratic expressions. One type of upper bound results in diagonal matrices which are very easy to interpret in terms of the LQG compensator design. The other upper bound is based on a singular value decomposition. In a way the latter upper bound is somewhat less conservative, and therefore it was used to design and compute the closed-loop control systems which were simulated in Section 7. In Section 7 the closed-loop control system performance improvement over a 'somewhat naïve and simple' LQG compensator design was investigated using Monte Carlo simulation. Both the stability and performance improved significantly. Furthermore, the benefits of selecting only two or three design parameters, over selecting the four LQG weighting matrices $R(t), Q(t), M(t), F, t_0 \leqslant t \leqslant T$, themselves, the first three being time varying in principal, is obvious, especially if the number of state and/or control variables is significant.

To simplify the computation, selection and interpretation of the three design parameters, the state and running costs of the optimal control problem, as well as the LQG compensation problem, were scaled. This scaling was performed fully automatically. Furthermore, Monte Carlo simulation turned out to be a useful tool to further identify, off-line, the influence of our three remaining LQG design parameters on the closed-loop control system performance, measured by $\overline{\Delta J}$ and n_∞, introduced in this paper.

Both continuous-time and digital control system design have been considered. In the case of digital control system design our approach explicitly considers the inter-sample behaviour which removes the need to select sufficiently small sampling intervals. The case of synchronous but possibly aperiodic sampling has been considered. However, based on [6, 7, 16] the results are very likely to carry over to asynchronous sampling as well.

The three considerations which are weighted by our three remaining scalar design parameters W_I, W_f, W_g, we believe, are the most important ones to select the LQG compensator weighting matrices $R(t), Q(t), M(t), F, t_0 \leqslant t \leqslant T$. But from Athans paper [1], which motivated ours, observe that also other considerations exist. Two of these additional considerations have also been addressed shortly in this paper. We certainly stick to Athans opinion that the LQG compensator design must be based on control engineering skills, which cannot be fully automated. But this paper revealed that they can be partly automated. This is due to the fact that the considerations for selecting the LQG weighting matrices, at level two of the control system design, are linked to the design at level one [1].

Apart from the LQG weighting matrices, the intensity and covariance matrices Σ_0, $\Sigma(t)$, $\theta(t)$ (θ_k in the case of digital control) determine the LQG compensator design. In this paper we have argued to take time-invariant diagonal covariance and intensity matrices. We believe, no significant further reduction of the number of LQG compensator design parameters makes sense. Finally, at level one of the optimal control system design it is entirely up to the engineer to develop model (1) and specify the cost function (2), for which no general procedures exist. Therefore, the results of this paper define what might be called 'a minimal set of design parameters for optimal control systems'.

APPENDIX A: PROOF OF (50) $\Rightarrow$ (51)

Consider the linearized system (3) without the white noise. The state of this linearized system satisfies,

$$\delta x(t) = \Phi(t, t_k)\delta x(t_k) + \Gamma(t, t_k)\delta u(t_k), \quad t_k \leqslant t < t_{k+1} \tag{A1}$$

and the control satisfies,

$$\delta u(t) = \delta u(t_k), \quad t_k \leqslant t < t_{k+1} \tag{A2}$$

where $\Phi(t, t_k)$ is the state transition matrix of the linearized system (3) from time t_k to time t and,

$$\Gamma(t, t_k) = \int_{t_k}^{t} \Phi(t, s) \frac{\partial f}{\partial u}\bigg|_{o(s)} \mathrm{d}s \tag{A3}$$

Define,

$$\Phi^{\mathrm{a}}(t, t_k) = \begin{bmatrix} \Phi(t, t_k) & \Gamma(t, t_k) \\ 0 & I_{m\times m} \end{bmatrix} \tag{A4}$$

where $I_{m\times m}$ is the identity matrix of dimension $m \times m$. Then (A1), (A2) may be written as

$$\delta x^{\mathrm{a}}(t) = \Phi^{\mathrm{a}}(t, t_k)\delta x^{\mathrm{a}}(t_k), \quad t_k \leqslant t < t_{k+1} \tag{A5}$$

Now the second variation $\delta^2 f_{k_i}(x_k, u_k) = \delta^2 f_{k_i}(x_k^{\mathrm{a}})$ of the equivalent discrete-time system at level one of the control system design can be expressed in terms of the second variations $\delta^2 f_i(x \times (t), u(t)) = \delta^2 f_i(x^{\mathrm{a}}(t))$, $t_k \leqslant t < t_{k+l}$, $k = 0, 1, \ldots, N-1$ of the continuous-time system at level one of the control system design.

$$\delta^2 f_{k_i} = \int_{t_k}^{t_{k+1}} \delta^2 \dot{x}_i = \int_{t_k}^{t_{k+1}} \delta x^{\mathrm{a}^{\mathrm{T}}}(t) \frac{\partial^2 f_i}{\partial x^{\mathrm{a}}} \delta x^{\mathrm{a}}(t)\,\mathrm{d}t, \quad i = 1, \ldots, n \tag{A6}$$

Using (A5) this becomes,

$$\delta^2 f_{k_i} = \int_{t_k}^{t_{k+1}} \delta x^{\mathrm{a}'}(t_k)\, \Phi^{\mathrm{a}^{\mathrm{T}}}(t, t_k) \frac{\partial^2 f_i}{\partial x^{\mathrm{a}}} \Phi^{\mathrm{a}}(t, t_k)\delta x^{\mathrm{a}}(t_k)\,\mathrm{d}t, \quad i = 1, \ldots, n \tag{A7}$$

From (A7), and noting that $x_k^{\mathrm{a}} = x^{\mathrm{a}}(t_k)$, it follows that

$$\frac{\partial^2 f_{k_i}}{\partial x_k^{\mathrm{a}^2}} = \int_{t_k}^{t_{k+1}} \Phi^{\mathrm{a}'}(t, t_k) \frac{\partial^2 f_i}{\partial x^{\mathrm{a}}} \Phi^{\mathrm{a}}(t, t_k)\,\mathrm{d}t, \quad i = 1, \ldots, n \tag{A8}$$

Now consider,

$$\frac{\partial^2 H_k}{\partial x_k^{\mathrm{a}^2}} = \frac{\partial^2 L_k}{\partial x_k^{\mathrm{a}^2}} + \sum_{i=1}^{n} p_{k_i} \frac{\partial^2 f_{k_i}}{\partial x_k^{\mathrm{a}^2}} \tag{A9}$$

Note that (A6)–(A8), which relate $\delta^2 f_{k_i}$ to $\delta^2 f_i$, $i = 1, \ldots, n$, in exactly the same way relate $\delta^2 L_k$ to $\delta^2 L$. Then using (A8) we obtain

$$\frac{\partial^2 H_k}{\partial x_k^{\mathrm{a}^2}} = \int_{t_k}^{t_{k+1}} \Phi^{\mathrm{a}^{\mathrm{T}}}(t, t_k) \left(\frac{\partial^2 L}{\partial x^{\mathrm{a}^2}} + \sum_{i=1}^{n} p_{k_i} \frac{\partial^2 f_i}{\partial x^{\mathrm{a}^2}} \right) \Phi^{\mathrm{a}}(t, t_k)\,\mathrm{d}t \tag{A10}$$

Selecting $Q^{\mathrm{a}}(t)$ according to (50), i.e. equal to the term within the brackets in (A10) evaluated along the optimal trajectory $[x_{\mathrm{o}}(t)\ u_{\mathrm{o}}(t)]' = x_{\mathrm{o}}^{\mathrm{a}}(t)$, denoted by $o(t)$, Equation (A10) reads,

$$\left.\frac{\partial^2 H_k}{\partial x_k^{\mathrm{a}^2}}\right|_{o(t_k)} = \int_{t_k}^{t_{k+1}} \Phi^{\mathrm{a}^{\mathrm{T}}}(t, t_k) Q^{\mathrm{a}}(t) \Phi^{\mathrm{a}}(t, t_k)\,\mathrm{d}t \tag{A11}$$

Then from (A4) and Reference [12] observe that the integral in (A11) represents precisely the computation of the equivalent discrete-time cost function matrices Q_k^{a}, $k = 0, 1, \ldots, N-1$, given by (44). This implies (51).

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