Chapter 12

Randomized digital optimal control

Willem L. de Koning

Department of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands

P.O. Box 5031, 2600 GA Delft, the Netherlands.
E-mail w.l.deKoning@its.tudelft.nl

L. Gerard van Willigenburg
Wageningen University
Wageningen, the Netherlands

Agrotechnion, Bomenweg 4
6703 HD Wageningen
E mail Gerard.vanWilligenburg@user.aenf.wau.nl

12.1 INTRODUCTION

In sampled data systems the sampling periods are almost always assumed to be deterministic, i.e., known in advance. However, in practice, the sampling periods should often be conceived as stochastic variables. This so called stochastic sampling phenomena occurs in many areas of sampled data systems such as in biological control systems or when a human being selects the sampling instants. Data may be absent at sampling instants when the sampling is governed by a stochastic process like a radar or sonar echo, or when the sampling mechanism fails. Also technical imperfections in the instrumentation may cause stochastic sampling.

Stochastic sampling may also be applied intentionally for instance when, for economy, a digital computer is time-shared in a stochastic manner, or to prevent jamming in communications. Other applications of intentional stochastic sampling are elimination of hidden oscillations between sampling instants, decreasing the influence of intelligent disturbances, and increasing stabilizability [1].

In this chapter we consider the important class of sampled-data systems where a continuous-time system is controlled by a digital computer, called digital control systems. We are interested especially in the influence of the presence of stochastic sampling on the existence of a stationary optimal controller, on the stability of the optimal controlled system and on the control cost. Two important cases are distinguished. Firstly the case where the presence of stochastic sampling is taken into account and secondly where it is not taken into account in the determination of the optimal controller. Among other things it will be shown that in the first case stochastic sampling may increase or restore stabilizability and that in the second case stabilizability may decrease or be lost.

Bergen [2], and Leneman [3], [4] studied deterministic scalar digital control systems with unity feedback, constant gain and independent identically distributed (IID) sampling periods. Results were found concerning spectral densities, stability, and response using convolution integrals or a direct approach. Kusher and Tobias [5] and Agniel and Jury [6], [7] investigated deterministic single-input/single-output (SISO) digital control systems with unity feedback, constant gain, different types of nonlinearities and IID sampling periods. They obtained stability and boundedness results using stochastic Lyapunov functions. Darkhovsky and Leybovich [8] investigated the stability and response of deterministic digital control systems with complete state information, unity feedback and IID sampling periods using Kronecker products. An analysis of the stability in the mean sense of a deterministic SISO digital control system was conducted by Dannenberg and Melsa [9] using expected transforms in analogy with z-transforms. Optimal control in digital control systems with IID sampling periods was studied by Kalman [10], Gunckel and Franklin [11] and Davidson [12]. They all assumed complete state information and a quadratic cost criterion. Kalman [10] succeeded in finding an implicit condition for the existence of a stationary optimal controller in the case of a cost criterion without a control term and a continuous-time system which contains at least one integration. Davidson [12] considered the influence of stochastic sampling on the stationary control cost without bothering about the existence and stability of a stationary control system. Gunckel and Franklin [11] extended the result of Kalman [10] to a general continuous-time system and criterion in the finite horizon case. Chang [13] studied also the finite-horizon case but assumed incomplete state information. He considered in essence only the suboptimal filtering of continuous-time systems in the case of stochastic sampling.

In this chapter we consider digital stationary optimal control in the general case of linear stochastic continuous-time systems, quadratic integral criteria, incomplete state information and where the sampling periods are IID stochastic variables. The digital stationary optimal control problem is transformed to a discrete-time stationary optimal control problem for linear discrete-time systems and quadratic sum criteria, both with stochastic parameters, and with incomplete state information. This latter problem is then solved using the notions of mean-square stabilizability and mean-square detectability [14], [15]. Also the criterion value is determined when the feedback is not optimal but arbitrary. Conditions are stated for the existence and stability of the stationary optimal linear estimator and convergence of the sample mean of the estimator error covariance in the case of a particular stochastic sampling scheme using the notions of uniform stabilizability and uniform detectability [16]. Given our interest in randomized digital optimal control the particular sampling scheme is an intentional stochastic sampling scheme. Finally the influence of stochastic sampling on the criterion value is investigated by means of some illustrative examples. It will appear that stochastic sampling may increase or even restore stabilizability. This makes intentional stochastic sampling a useful tool in the design of digital control systems. However, if the presence of stochastic sampling is not taken into account in the determination of the optimal controller, then stability may decrease or even be lost. Thus unintentional stochastic sampling because of e.g. imperfections of instrumentation may have dramatic consequences in digital control systems if not considered in the design procedure.

12.2 OPTIMAL CONTROL PROBLEM

In this section the digital optimal control problem for linear continuous-time systems and quadratic integral criteria, in the case of stochastic sampling and incomplete state information, is stated. This problem is transformed to a discrete-time stationary optimal control problem for linear discrete-time systems and quadratic sum criteria, both with stochastic parameters and with incomplete state information.

Consider a digital control system, consisting of a continuous-time system connected with a digital computer by means of a sample-and-hold at the input and a sampler at the output, see Fig.1.

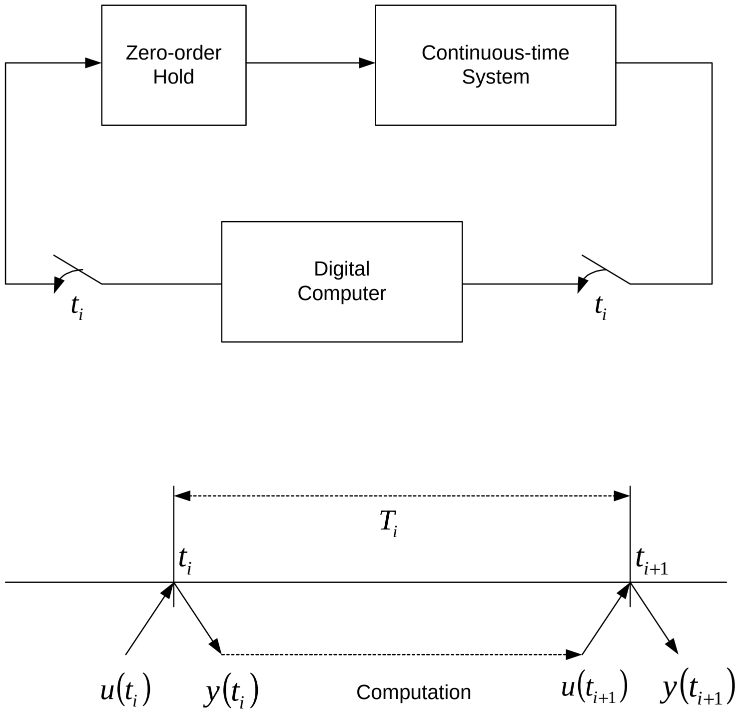

Figure 1: Digital control system and the task sequence of the computer.

The sampling instants of both samplers are $t_0, t_1, \ldots$.The continuous-time system, the sample-and-hold operation and the observations at the sampling instants are described by respectively,

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t),\ t \geq 0, \tag{1a}$$

$$u(t) = u(t_i),\ t_i \leq t < t_{i+1},\ i = 0,1,\ldots, \tag{1b}$$

$$y(t_i) = Cx(t_i) + w(t_i),\ i = 0,1,\ldots, \tag{1c}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $y(t_i) \in R^l$ is the observation, $v(t) \in R^n$ is the system noise, $w(t_i) \in R^l$ is the observation noise and A, B and C are the known real system matrices of appropriate dimensions. The initial condition $x(t_0) = x_0$ and the processes $\{v(t)\}$ and $\{w(t_i)\}$ are independent with known means and covariances

$$E\{x_0\} = \bar{x}_0,\ E\left\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right\} = G, \tag{2a}$$

$$E\{v(t)\} = 0,\ E\left\{v(t)v^T(t)\right\} = Vd(t-s), \tag{2b}$$

$$E\{w(t_i)\} = 0,\ E\left\{w(t_i)w^T(t_i)\right\} = Wd_{ij}, \tag{2c}$$

where $d(.)$ and $d_{.}$ denote, respectively, the Dirac and Kronecker delta function. G, V and W are real symmetric matrices with $G \geq 0$, $V \geq 0$ and $W \geq 0$. Equation (1a) can be defined in terms of a stochastic integral equation and $v(t)$ as the formal derivative of an independent increments process $b(t)$ [17].

Consider also the long-term average integral criterion

$$s_{\infty}(U_{\infty}) = \lim_{t_f \to \infty} \frac{1}{t_f - t_0} \int_{t_0}^{t_f} E\left\{\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt\right\}, \tag{3}$$

where Q and R are known real symmetric matrices of appropriate dimensions with $Q \geq 0$ and $R \geq 0$ and U_i denotes the control sequence $\{u(t_0),..,u(t_i)\}$.

Define the sampling periods T_i by

$$T_i = t_{i+1} - t_i,\ i = 0,1,\dots. \tag{4}$$

Then we assume that the process $\{T_i\}$ is a sequence of independent stochastic variables with known constant statistics independent of x_0, $v(t)$ and $w(t_i)$. Moreover

$$0 < a \leq T_i \leq b < \infty,\ i = 0,1,\dots, \tag{5}$$

where a and b are real scalars.

Let Y_i denote the observation sequence $\{y(t_0),..,y(t_i)\}$. Then the digital stationary optimal control problem is defined as follows.

Definition 1. Assume that $u(t_i)$ is a deterministic function of Y_{i-1}, U_{i-1}, $i = 0,1,\dots.$ and that the system (1), the criterion (3) and the sampling period process $\{T_i\}$ is given. Then the problem of finding the control sequence $U_{\infty}^* = \{u^*(t_0), u^*(t_1),\dots\}$ which minimizes $s_{\infty}(U_{\infty})$ and of finding the minimal value s_{∞}^* is called the digital stationary optimal control problem.

Form definition 1, we see that at time t_i the computer is supposed to send the control $u(t_i)$ and to receive the observation $y(t_i)$. Within the interval $[t_i, t_{i+1})$ the next control $u(t_{i+1})$ has to be calculated on the basis of the observations Y_i and the controls U_i.

We wish to transform the digital stationary optimal control problem into a discrete-time one, i.e., where only the behavior at the sampling instants is involved. First we need the following lemma concerning the sampling process.

Lemma 1. $t_i \to \infty \Leftrightarrow i \to \infty$

Proof. From (4) we have $t_i = t_0 + \sum_{k=0}^{i-1} T_i$, thus with (5) $t_0 + ia \leq t_i \leq t_0 + ib$. From the first inequality it follows that $i \to \infty \Rightarrow t_i \to \infty$ and from the second one $t_i \to \infty \Rightarrow i \to \infty$.

Let x_i, u_i, y_i and w_i denote, respectively, $x(t_i), u(t_i), y(t_i)$ and $w(t_i)$. Let an overbar denote expectation. If the statistics of a stochastic variable s_i is independent of i, then $\bar{s}_i$ is denoted by $\bar{s}$. The following theorem concerns the transformation of the system (1).

Theorem 1. The behavior of system (1) at the sampling instants is identical to the behavior of the discrete-time system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i,\ i = 0,1,\ldots, \tag{6a}$$

$$y_i = Cx_i + w_i,\ i = 0,1,\ldots, \tag{6b}$$

where

$$\Phi_i = \Phi(T_i) = e^{AT_i}, \tag{7a}$$

$$\Gamma_i = \Gamma(T_i) = \int_0^{T_i} \Phi(s)Bds, \tag{7b}$$

$$v_i = v(t_i + T_i, t_i) = \int_{t_i}^{t_i+T_i} \Phi(t_i + T_i - s)db(s). \tag{7c}$$

The initial condition x_0, $\{v_i\}$ and $\{w_i\}$ are independent with

$$E\{x_0\} = \bar{x}_0,\ E\left\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right\} = G, \tag{8a}$$

$$E\{v_i\} = 0,\ E\left\{v_i v_j^T\right\} = \bar{V}d_{ij}, \tag{8b}$$

$$E\{w_i\} = 0,\ E\left\{w_i w_j^T\right\} = Wd_{ij}, \tag{8c}$$

where $\bar{V}$ denotes the known constant matrix $E\{V_i\}$ with

$$V_i = V(T_i) = \int_0^{T_i} \Phi(s)V\Phi^T(s)ds \tag{9}$$

and $V_i \geq 0$. Also $\{v_i\}$ is a sequence of independent stochastic vectors. The processes $\{\Phi_i\}$ and $\{\Gamma_i\}$ are sequences of independent random matrices with known constant statistics, independent of $\{w_i\}$ and x_0. Moreover, Φ_i and Γ_i are independent of v_j, $i \neq j$, and uncorrelated with v_i.

Proof. See De Koning [17].

The asymptotic behavior of system (1) and system (6) expressed in theorem 1 goes still a bit further as the following theorem shows.

Theorem 2. The asymptotic behavior of system (1) at the sampling instants is identical to the asymptotic behavior of the system (6).

Proof. Follows immediately from Theorem 1 and Lemma 1.

Now we turn to the transformation of the criterion (3).

Theorem 3. Assume that u_i of system (1) is a deterministic function of Y_{i-1} and U_{i-1}. Then the value of criterion (3), if it exists, is identical to the value of the long-term average sum criterion

$$s_{\infty}(U_{\infty}) = \frac{1}{\overline{T}} \lim_{N\to\infty} \frac{1}{N} E\left\{ \sum_{i=0}^{N-1} x_i^T \overline{Q} x_i + 2x_i^T \overline{M} u_i + u_i^T \overline{R} u_i \right\} + \frac{1}{\overline{T}} \overline{g}, \tag{10}$$

where $\overline{Q}$, $\overline{M}$, $\overline{R}$, $\overline{g}$, $\overline{T}$ denote, respectively, the known constant matrices $E\{Q_i\}$, $E\{M_i\}$, $E\{R_i\}$ and the scalars $E\{g_i\}$, $E\{T_i\}$ with

$$Q_i = Q(T_i) = \int_0^{T_i} \Phi^T(s) Q \Phi(s) ds, \tag{11a}$$

$$M_i = M(T_i) = \int_0^{T_i} \Phi^T(s) Q \Gamma(s) ds, \tag{11b}$$

$$R_i = R(T_i) = \int_0^{T_i} \left[R + \Gamma^T(s) Q \Gamma(s) \right] ds, \tag{11c}$$

$$g_i = g(T_i) = \int_0^{T_i} tr\left[V(s) Q \right] ds, \tag{11d}$$

and $\overline{Q} \geq 0$, $\overline{R} \geq 0$ and $\overline{g} \geq 0$.

Proof. Consider the criterion (3). Put the factor $\frac{1}{t_f - t_0}$ inside the expectation. Then, using Lemma 1, we may replace t_f by t_N and $t_f \to \infty$ by $N \to \infty$. Hence (3) may be written as

$$s_{\infty}(U_{\infty}) = \lim_{N\to\infty} E\left\{ \frac{1}{t_N - t_0} \int_{t_0}^{t_N} \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt \right\}.$$

By the Lebesque dominated convergence theorem we may write this, if it exists, as

$$s_{\infty}(U_{\infty}) = E\left\{ \lim_{N\to\infty} \frac{1}{t_N - t_0} \int_{t_0}^{t_N} \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt \right\}. \tag{12}$$

Let a.s. denote almost surely (with probability one). By (4) and the strong law of large numbers [18] $\frac{1}{N}(t_N - t_0) = \frac{1}{N} \sum_{i=0}^{N-1} T_i \to \overline{T}$ a.s. as $N \to \infty$. Hence (12) may be written as

$$s_{\infty}\left(U_{\infty}\right)=\frac{1}{\overline{T}}E\left\{\lim_{N\to\infty}\frac{1}{N}\int_{t_0}^{t_N}\left[x^T(t)Qx(t)+u^T(t)Ru(t)\right]dt\right\},$$

and applying the Lebesque dominated convergence theorem again we have

$$s_{\infty}\left(U_{\infty}\right)=\frac{1}{\overline{T}}\lim_{N\to\infty}\frac{1}{N}E\left\{\int_{t_0}^{t_N}\left[x^T(t)Qx(t)+u^T(t)Ru(t)\right]dt\right\}.$$

From [17] we may write the expectation part as

$$E\left\{\sum_{i=0}^{N-1}\left[x_i^TQ_ix_i+2x_i^TM_iu_i+u_i^TR_iu_i\right]\right\}+N\overline{g},$$

where Q_i, M_i and R_i are independent of x_i and u_i, and might therefore be replaced by

$$E\left\{\sum_{i=0}^{N-1}\left[x_i^T\overline{Q}x_i+2x_i^T\overline{M}u_i+u_i^T\overline{R}u_i\right]\right\}+N\overline{g}.$$

The fact that $\overline{Q}\geq 0$, $\overline{R}\geq 0$ and $\overline{g}\geq 0$ follows directly from (11).

Now we define a discrete-time control problem.

Definition 2. Assume that u_i is a deterministic function of Y_{i-1}, U_{i-1}, $i=0,1,\ldots$ and given the system (6) and the criterion (10), then the problem of finding the control sequence $U_{\infty}^*=\left\{u_0^*,u_1^*,\ldots\right\}$ which minimizes $s_{\infty}\left(U_{\infty}\right)$ and of finding the minimal value s_{∞}^* is called the equivalent discrete-time stationary optimal control problem.

The relation of this problem with the one in Definition 1 is as follows.

Theorem 4. The solution of the digital stationary optimal control problem is identical to the solution of the equivalent discrete-time stationary optimal control problem
Proof. Follows from theorem 1, 2 and 3.

12.3 OPTIMAL CONTROL

In this section the stationary optimal control problem is solved via the solution of the equivalent discrete-time stationary optimal control problem. Furthermore, the criterion value is determined when the feedback is not optimal but arbitrary.

First we repeat some results concerning stabilizability and detectability from De Koning [14], [15] for easy reference.

Consider the open loop system

$$x_{i+1}=\Phi_ix_i+\Gamma_iu_i,\ i=0,1,\ldots, \tag{13}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ the control and Φ_i, Γ_i are real random matrices of appropriate dimensions. The processes $\{\Phi_i\}$ and $\{\Gamma_i\}$ are sequences of independent random matrices with constant statistics. The initial value x_0 is deterministic. System (13) is characterized by the pair (Φ_i, Γ_i). Suppose

$$u_i = -Lx_i, \tag{14}$$

where L is a real matrix with appropriate dimensions, the we have from (13) the closed loop system

$$x_{i+1} = (\Phi_i - \Gamma_i L)x_i. \tag{15}$$

Let ms denote mean square.

Definition 3. System (15) is called ms-stable if $\overline{\|x_i\|^2} \to 0$ as $i \to \infty$, $\forall x_0$.

Definition 4. (Φ_i, Γ_i) is called ms-stabilizable if $\exists L$ such that system (15) is ms-stable.

Let S^n denote the linear space of real symmetric $n \times n$ matrices and define the transformation $A_L : S^n \to S^n$ by

$$A_L X = \overline{(\Phi - \Gamma L)^T X (\Phi - \Gamma L)}, \; X \in S^n. \tag{16}$$

Note that the statistics of $(\Phi_i - \Gamma_i L)^T X (\Phi_i - \Gamma_i L)$ are independent of i , so index i may be deleted in (16) as agreed in section 2.

Lemma 2. A_L is linear. $X \ge 0 \Rightarrow A_L^i X \ge 0, \; i = 0,1,\ldots.$
Proof. See [14].

Let r denote spectral radius.

Theorem 5. System (15) is ms-stable $\Leftrightarrow r(A_L) < 1$
Proof. See [14].

Let $\otimes$ denote the Kronecker product [19] then it is easy to show that $r(A_L) = r\left(\overline{(\Phi - \Gamma L) \otimes (\Phi - \Gamma L)}\right)$ which is not difficult to calculate. Note that if $\Phi_i = \Phi$ and $\Gamma_i = \Gamma$ where Φ and Γ are deterministic and constant then $r(A_L) = r^2(\Phi - \Gamma L)$.

In view of detectability consider the system

$$x_{i+1} = \Phi_i x_i, \; i = 0,1,\ldots, \tag{17a}$$

$$y_i = C_i x_i, \; i = 0,1,\ldots, \tag{17b}$$

where $x_i \in R^n$ is the state, $y_i \in R^l$ the observation, and Φ_i, C_i are real random matrices of appropriate dimensions. The processes $\{\Phi_i\}$ and $\{C_i\}$ are sequences of independent random matrices with constant statistics. The initial value x_0 is deterministic. System (17) is characterized by the pair (Φ_i, C_i).

Definition 5. (Φ_i, C_i) is called ms-detectable if $\overline{\|y_i\|^2} = 0,\ i = 0,1,\ldots \Rightarrow \overline{\|x_i\|^2} \to 0$ as $i \to \infty$.

There exist explicit conditions for ms-stabilizability [14] and for ms-detectability [15] which are easy to calculate. Furthermore we have the following two results. Let q denote the zero matrix.

Theorem 6. $r(A_q) < 1 \Rightarrow (\Phi_i, \Gamma_i)$ ms-stabilizable, (Φ_i, C_i) ms-detectable.
Proof. Follows directly from Theorem 5 and Definitions 3, 4 and 5.

Theorem 7. If Φ_i, Γ_i and C_i are deterministic and constant then ms-stability, ms-stabilizability and ms-detectability is identical to respectively stability, stabilizability and detectability in the usual sense.
Proof. See De Koning [14], [15].

Now we turn our attention to the equivalent discrete-time stationary optimal control problem. Consider system (6) and criterion (10) which are repeated here for convenience.

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i,\ i = 0,1,\ldots, \tag{18a}$$

$$y_i = Cx_i + w_i,\ i = 0,1,\ldots, \tag{18b}$$

$$s_\infty(U_\infty) = \frac{1}{T} \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{i=0}^{N-1} x_i^T \overline{Q} x_i + 2x_i^T \overline{M} u_i + u_i^T \overline{R} u_i\right\} + \frac{1}{T}\overline{g}, \tag{19}$$

Assume that $\overline{R} > 0$. Define the transformation $B_L : S^n \to S^n$ by

$$\begin{aligned} B_L X = {} & \overline{\Phi^T X \Phi} + \overline{Q} - \left(\overline{\Phi^T X \Gamma} + \overline{M}\right)L - L^T\left(\overline{\Gamma^T X \Phi} + \overline{M}^T\right) \\ & + L^T\left(\overline{\Gamma^T X \Gamma} + \overline{R}\right)L,\ X \in S^n, \end{aligned} \tag{20}$$

and the transformation $B_* : S^n \to S^n$ by

$$B_* X = B_{L_X} X,\ X \in S^n, \tag{21}$$

where B_{L_X} is defined by (20) and L_X by

$$L_X = \left(\overline{\Gamma^T X \Gamma} + \overline{R}\right)^{-1}\left(\overline{\Gamma^T X \Phi} + \overline{M}^T\right),\ X \in S^n. \tag{22}$$

The transformation B_* is a generalized discrete-time Riccati transformation. The usual Riccati transformation is obtained after deletion of all the overbars in equation (20) and (21) [1]. Equation (21) may also be written as

$$B_* X = \overline{\Phi^T X \Phi} + \overline{Q} - L_X^T \left(\overline{\Gamma^T X \Gamma} + \overline{R} \right) L_X ,\ X \in S^n . \tag{23}$$

We will also need the transformations B_L and B_* in a different form. Define the matrices Φ_i', Q_i' and L' by

$$\Phi_i' = \Phi_i - \Gamma_i \overline{R} \overline{M}^T , \tag{24a}$$

$$\overline{Q}' = \overline{Q} - \overline{M}\, \overline{R}^{-1} \overline{M}^T , \tag{24b}$$

$$L' = L - \overline{R}^{-1} \overline{M}^T , \tag{24c}$$

and the transformation $A_{L'}' : S^n \to S^n$

$$A_{L'}' X = \overline{(\Phi' - \Gamma L')^T X (\Phi' - \Gamma L')},\ X \in S^n . \tag{25}$$

It can be proven that $\overline{Q}' \geq 0$ [20]. One may easily verify that (20) may be written as

$$B_L X = A_{L'}' X + \overline{Q}' + L'^T \overline{R} L',\ X \in S^n , \tag{26}$$

and $L_X' = L_X - \overline{R}^{-1} \overline{M}^T$ as

$$L_X' = \left(\overline{\Gamma^T X \Gamma} + \overline{R} \right)^{-1} \overline{\Gamma^T X \Phi'},\ X \in S^n . \tag{27}$$

The relation between $A_{L'}'$ and A_L is as follows.

> *Lemma 3.* Assume that $\overline{R} > 0$. Then $A_{L'}' X = A_L X,\ X \in S^n$.
> *Proof.* From (24) we have $\Phi_i' - \Gamma_i L' = \Phi_i - \Gamma_i L.$ Then the lemma follows from the definitions of A_L and $A_{L'}'$.

Finally let $\hat{x}_i$ denote the minimum variance optimal estimator of x_i given Y_{i-1} and U_{i-1}, and let the estimator error covariance P_i be defined by

$$P_i = E\left\{ \left(x_i - \hat{x}_i \right) \left(x_i - \hat{x}_i \right)^T \right\}. \tag{28}$$

Note that $\hat{x}_i$ is a deterministic function of Y_{i-1} and U_{i-1}. An arbitrary, not necessarily optimal, estimator $\hat{x}_i'$ is called variance neutral [21] if the associated estimator error covariance P_i' is not a function of U_{i-1}. The subject of estimation will be considered in section 12.4.

Now we are in a position to state the solution of the equivalent discrete-time stationary optimal control problem.

Theorem 8. Assume that $\overline{R} > 0$, that $\hat{x}_i$ is variance neutral for $i = 0,1,\ldots$ and that $P = \lim_{i\to\infty} \frac{1}{i}\sum_{k=0}^{i-1} P_k$ exists. Then (Φ_i', Γ_i) ms-stabilizable $\Rightarrow S = \lim_{i\to\infty} B_*^i q$ exists, S is the minimum nonnegative definite solution of the equation $S = B_* S$, $U_\infty^* = \{-L_S\hat{x}_0, -L_S\hat{x}_1, \ldots\}$ and

$$s_\infty^* = \frac{1}{\overline{T}} tr\left[\overline{V}S + L_S^T\left(\overline{\Gamma^T S\Gamma} + \overline{R}\right)L_S P\right] + \frac{1}{\overline{T}}\overline{g} \tag{29a}$$

$$= \frac{1}{\overline{T}} tr\left[\overline{V}S + \left(\overline{\Phi^T S\Phi} + \overline{Q} - S\right)P\right] + \frac{1}{\overline{T}}\overline{g}\,, \tag{29b}$$

where the minimization is with respect to all control sequences for which (19) exists.

Proof. The result for S follows directly from (21), (26), (27) and [14]. In fact variance neutrality of $\hat{x}_i$ for $i = 0,1,\ldots$ is not needed here. Consider the criterion

$$s_N(U_{N-1}, H) = \frac{1}{\overline{T}N} E\left\{\sum_{i=0}^{N-1}\left(x_i^T\overline{Q}x_i + 2x_i^T\overline{M}u_i + u_i^T\overline{R}u_i\right) + x_N^T H x_N\right\} + \frac{1}{\overline{T}}\overline{g}\,, \tag{30}$$

where H is a real symmetric matrix of appropriate dimensions with $H \geq 0$. This criterion is the same as criterion (19) except that N has a finite value and that the term $x_N^T H x_N$ has been inserted. The solution of the equivalent discrete-time stationary optimal control problem with criterion (10) replaced by criterion (30), assuming variance neutrality of $\hat{x}_i$ for $i = 0,1,\ldots$ gives for the minimum $s_N^*(H)$ [22]

$$s_N^*(H) = \frac{1}{\overline{T}N}\left\{\overline{x}_0^T S_0\overline{x}_0 + tr(S_0 G) + \sum_{i=0}^{N-1} tr\left[\overline{V}S_{i+1} + L_i^T\left(\overline{\Gamma_i^T S_{i+1}\Gamma_i} + \overline{R}\right)L_i P_i\right]\right\} + \frac{1}{\overline{T}}\overline{g}\,,$$

where $S_{i+1} = B_*^{N-i-1}H,\ S_0 = B_*^N H$ and $L_i = L_{S_{i+1}}$. The minimizing control sequence is $U_{N-1}^* = \{-L_0\hat{x}_0, \ldots, -L_{N-1}\hat{x}_{N-1}\}$. Let $H = q$ then, due to the existence of S and P, we have $S_{i+1} \to S$, $S_0 \to S$, $L_i \to L_S$ as $N \to \infty$ and $\lim_{N\to\infty} s_N^*(q)$ equals the right part of equation (29a), say f. For the proof of $s_\infty^* = s(U_\infty') = f$ we refer to Kushner [23]. Finally using (23) and $S = B_* S$, equation (29a) may be written as (29b).

The equation $S = B_* S$ is a generalized discrete-time Riccati equation. If complete state information is available then variance neutrality of $\hat{x}_i$ for $i = 0,1,\ldots$ and the existence of P is not needed, in fact $\hat{x}_i = x_i$ and $P_i = P = q$ for $i = 0,1,\ldots$. We see that u_i^* is a deterministic function of $\hat{x}_i$. A control problem with this property is called separable [21]. It has been proven by De Koning [22] that variance neutrality of $\hat{x}_i$, for $i = 0,1,\ldots$ is a sufficient condition for separability of the equivalent discrete-time stationary optimal control problem. The existence of P will be considered in Section 12.4.

The closed loop system may not be ms-stable. The next theorem states the conditions for which it is in the case of complete state information.

Theorem 9. Assume $\overline{R} > 0$ and that $\left(\Phi_i', \Gamma_i\right)$ is ms-stabilizable. Then $\left(\Phi_i', \overline{Q}'^{1/2}\right)$ ms-detectable $\Rightarrow r\left(A_{L_S}\right) < 1$, S is the unique nonnegative definite solution of the equation $S = B_* S$.

Proof. From (21), (26), (27) and [15] the result for S and $r\left(A_{L_S}'\right) < 1$ is obtained. Then by Lemma 3 $r\left(A_{L_S}\right) < 1$.

The statement $r\left(A_{L_S}\right) < 1$ is by Theorem 5 equivalent to the statement that the closed loop system is ms-stable in the case of complete state information with $u_i = -L_S x_i$, $i = 0,1,\ldots$. Of course the stability of the control system with incomplete state information depends also on the stability of the estimator. The estimator will be considered in Section 12.4.

Assume $\overline{R} > 0$, that $\hat{x}_i$ is variance neutral and that P exists. Then summarizing loosely we may state that if $\left(\Phi_i', \Gamma_i\right)$ is ms-stabilizable, then the equivalent discrete-time stationary optimal control problem has a solution. If in addition $\left(\Phi_i', \overline{Q}'^{1/2}\right)$ is ms-detectable, then the solution is unique and the closed loop system is ms-stable in the case of complete state information.

A useful relation between stability and convergence of $B_*^i q$ is as follows.

Corrolary 1. Assume $\overline{R} > 0$ and that $\left(\Phi_i', \overline{Q}'^{1/2}\right)$ ms-detectable. Then $\left(\Phi_i', \Gamma_i\right)$ ms-stabilizable $\Leftrightarrow B_*^i q$ converges as $i \to \infty$.

Proof. Follows from (21), (26), (27) and [15].

Thus under very weak conditions, ms-stabilizability of $\left(\Phi_i', \Gamma_i\right)$ is not only sufficient but also necessary for the convergence of $B_*^i q$.

Furthermore in this section we are interested in the value s_∞^L of $s_\infty\left(U_\infty\right)$ when the feedback matrix is not optimal but has the arbitrary value L.

Theorem 10. Assume $\overline{R} > 0$, that $P = \lim_{i\to\infty} \frac{1}{i}\sum_{k=0}^{i-1} P_k$ exists and that $U_\infty = \left\{-L\hat{x}_0, -L\hat{x}_1, \ldots\right\}$. Then $r\left(A_L\right) < 1 \Rightarrow S_L = \lim_{i\to\infty} B_L^i q$ exists, $S_L \geq 0$, S_L is the unique solution of the equation $S_L = B_L S_L$ and

$$s_\infty^L = \frac{1}{\overline{T}} tr\Big[\overline{V} S_L + \Big(\Big(\overline{\Phi^T S_L \Gamma} + \overline{M}\Big) L + L^T \Big(\overline{\Gamma^T S_L \Phi} + \overline{M}^T\Big) - L^T \Big(\overline{\Gamma^T S_L \Gamma} + \overline{R}\Big) L\Big) P\Big] + \frac{1}{\overline{T}}\overline{g}, \tag{31a}$$

$$= \frac{1}{\overline{T}} tr\Big[\overline{V} S_L + \Big(\overline{\Phi^T S_L \Phi} + \overline{Q} - S_L\Big) P\Big] + \frac{1}{\overline{T}}\overline{g}. \tag{31b}$$

Proof. Applying Lemma 2 and Lemma 3 we have from (26) by induction $B_L^i q = \sum_{k=0}^{i-1} A_L^k \left(\overline{Q}' + L'^T \overline{R} L'\right)$.

Thus $r(A_L) < 1 \Rightarrow S_L = \lim_{i\to\infty} B_L^i X = \sum_{k=0}^{\infty} A_L^k\left(\overline{Q}' + L'^T \overline{R} L'\right)$ exists.

Because $A_L^k\left(\overline{Q}' + L'^T \overline{R} L'\right) \geq 0,\ k = 0,1,\ldots,$ also $S_L \geq 0$. Moreover $B_L^{i+1} = B_L B_L^i q$, thus taking the limits we have $S_L = B_L S_L$. Suppose $\tilde{S}_L \in S^n$ is any other solution of the equation $S_L = B_L S_L$ then $S_L - \tilde{S}_L = A_L\left(S_L - \tilde{S}_L\right)$ and by induction $S_L - \tilde{S}_L = A_L^i\left(S_L - \tilde{S}_L\right)$. Let $i \to \infty$ then $S_L = \tilde{S}_L$, thus S_L is the unique solution. Note that the existence of P is not needed so far. Consider criterion (30). Choosing $u_i = -L\hat{x}_i,\ i = 0,1,\ldots$ and deleting the minimizations in the derivation of $s_N^*(H)$, [22] gives for the value $s_N^L(H)$

$$s_N^L(H) = \frac{1}{\overline{T}N}\left\{\overline{x}_0^T S_0 \overline{x}_0 + tr(S_0 G) + \sum_{i=0}^{N-1} tr\left[\overline{V} S_{i+1} \right.\right.$$
$$\left.\left. + \left(\left(\overline{\Phi^T S_{i+1}\Gamma} + \overline{M}\right)L + L^T\left(\overline{\Gamma^T S_{i+1}\Phi} + \overline{M}^T\right) - L^T\left(\overline{\Gamma^T S_{i+1}\Gamma} + \overline{R}\right)L\right)P_i\right]\right\}$$
$$+ \frac{1}{\overline{T}}\overline{g},$$

where $S_{i+1} = B_L^{N-i-1}H$. Let $H = q$ then due to the existence of S_L and P, we have $S_{i+1} \to S_L$, $S_0 \to S_L$ as $N \to \infty$ and $\lim_{N\to\infty} s_N^L(q) = s_\infty^L$ where s_∞^L is defined by (31a). Finally using (20) and $S_L = B_L S_L$, (31a) may be written as (31b).

Note that variance neutrality of $\hat{x}_i$ for $i = 0,1,\ldots$ is not needed in this theorem because there is no minimization involved as in Theorem 8. Observe also the resemblance between (29b) and (31b). If $S_L \to S$ in (31b) then $s_\infty^L = s_\infty^*$.

Finally we remark that the derivation of $s_N^*(H)$ in Theorem 8, and therefore all the results in this section are still valid if $\hat{x}_i$ for $i = 0,1,\ldots$ is restricted to be linear. Also observe that knowledge of T_i after t_i for $i = 0,1,\ldots$ does not change the result of this section. It does affect the optimal estimator as will be seen in section 12.4.

12.4 OPTIMAL ESTIMATION

In this section conditions are stated for the existence and stability of the stationary optimal linear estimator in the case of a particular sampling scheme. Moreover, convergence of the sample mean of the estimator error covariance is considered.

Consider the discrete-time system (6). Let $\hat{x}_i'$ denote an arbitrary, not necessarily optimal, estimator of x_i. Then we have the following fact concerning $\{T_i\}$.

Lemma 3. Assume $B \neq q$. Then $\hat{x}_0', \ldots, \hat{x}_{i+1}'$ variance neutral $\Rightarrow T_0, \ldots, T_i$ deterministic.
Proof. From De Groot and De Koning [24] we have $\hat{x}_0', \ldots, \hat{x}_{i+1}'$ variance neutral $\Rightarrow \Gamma_0, \ldots, \Gamma_i$ deterministic. Assuming $B \neq q$ implies $T_0, \ldots, T_i$ deterministic.

Thus whatever we do with the data, there is no hope in finding any variance neutral estimators $\hat{x}'_0,\ldots,\hat{x}'_{i+1}$ if $T_0,\ldots,T_i$ are not deterministic. To apply the solution of the digital stationary optimal control problem, we have to determine successively the variance neutral estimators $\hat{x}'_0,\hat{x}'_1,\ldots$. Then from Lemma 3 it is necessary that for $i=0,1,\ldots$ the realization of T_i is known before the determination of $\hat{x}'_{i+1}$. Due to this, and also due to our interest in randomized digital control, we assume in the case of incomplete state information that the stochastic sampling occurs intentionally where the computer generates T_i first in the interval $\left[t_i,t_{i+1}\right)$ and then determines $\hat{x}'_{i+1}$ on the basis of Y_i, U_i and $T_0,\ldots,T_i$.

Starting from the assumed intentional stochastic sampling scheme, we conclude that, as far as estimation is concerned, T_i may be assumed to be known in system (18). Thus we have the system

$$x_{i+1}=\Phi_i x_i+\Gamma_i u_i+v_i,\ i=0,1,\ldots, \tag{32a}$$

$$y_i=Cx_i+w_i,\ i=0,1,\ldots, \tag{32b}$$

which is the same as system (18) except that $\Phi_i=\Phi\left(T_i\right)$ and $\Gamma_i=\Gamma\left(T_i\right)$ are known and given by (7a) and (7b), and that

$$E\{v_i\}=0,\ E\{v_i v_j^T\}=V_i d_{ij},$$

where $V_i=V\left(T_i\right)$ is known and given by (8). The system (32) is discrete-time with time-dependent parameters.

From system (32) and its properties it follows that the standard linear estimation theory [1], [25] applies immediately. Let $\hat{x}_i$ denote the linear minimum variance or optimal linear estimator of x_i given the observations Y_{i-1} and the controls U_{i-1}, and let P_i denote the estimator error covariance defined by (28). Then the optimal linear estimator is given by

$$\begin{aligned}\hat{x}_{i+1}&=\Phi_i\hat{x}_i+\Gamma_i u_i+K_i\left(y_i-C\hat{x}_i\right)\\&=\left(\Phi_i-K_iC\right)\hat{x}_i+\Gamma_i u_i+K_i y_i,\ \hat{x}_0=\bar{x}_0,\end{aligned} \tag{34a}$$

$$K_i=\Phi_i P_i C^T\left(CP_iC^T+W\right)^+, \tag{34b}$$

$$\begin{aligned}P_{i+1}&=\Phi_i P_i\Phi_i^T+V_i-K_i\left(CP_iC^T+W\right)K_i^T\\&=\left(\Phi_i-K_iC\right)P_i\left(\Phi_i-K_iC\right)^T+V_i+K_iWK_i^T,\ P_0=G,\end{aligned} \tag{34c}$$

where the superscript + denotes the Moore-Penrose pseudo inverse [26]. It is apparent that the estimator $\hat{x}_i$ is variance neutral for $i=0,1,\ldots$. Due to the assumed intentional stochastic sampling scheme, the Riccati equation (34c) must be solved on-line, one step at each sampling interval. Suppose that x_0, $\{v(t)\}$ and $\{w(t_i)\}$ are jointly normal, then x_0, $\{v_i\}$ and $\{w_i\}$ of system (32) are also jointly normal. In that case $\hat{x}_i$ given by (34) is also the optimal estimator, i.e. not restricted to be linear.

In order to find a priori conditions for the existence and stability of the stationary optimal linear estimator, we may of course not assume that T_i is deterministic for $i = 0,1,\ldots$. However, we may conceive system (32) as a realization of system (18) dependent on a particular realization of the sequence $\{T_i\}$, and $\hat{x}_i$ given by (34) as the associated optimal linear estimator. Then, using the fact that Φ_i, Γ_i and V_i are bounded due to (5), (7) and (9), we have from [16] the following result concerning stationary behavior of the optimal linear estimator (34).

Theorem 11. (Φ_i, C) a.s. uniformly detectable $\Rightarrow P_i$ a.s. bounded.

Theorem 12. Assume $W > 0$ and that (Φ_i, C) a.s. uniformly detectable. Then $(\Phi_i, V_i^{1/2})$ a.s. uniformly stabilizable $\Rightarrow$ estimator (34) a.s. exponentiallly stable.

Corrolary 2. Assume $W > 0$ and $(\Phi_i, V_i^{1/2})$ a.s. uniformly stabilizable. Then (Φ_i, C) a.s. uniformly detectable $\Leftrightarrow P_i$ a.s. bounded.

Note that the a.s. exponential stability of system (34) means that $x_{i+1} = (\Phi_i - K_i C)x_i$ is a.s. exponentially stable. If Φ_i and C_i are deterministic and constant, then a.s. exponential stability, a.s. uniformly stabilizability and a.s. uniformly detectability is identical to, respectively, stability, stabilizability and detectability in the usual sense [1], [25].

Furthermore in this section we consider the convergence of the sample mean of P_i which was needed in Theorems 8 and 10.

Theorem 13. (Φ_i, C) a.s. uniformly detectable $\Rightarrow \lim_{i\to\infty} \frac{1}{i}\sum_{k=0}^{i-1} P_k$ a.s. exists where the limit is in the mean square sense.

Proof. Let $\overline{P}_i$ denote the expectation of P_i w.r.t. $T_0, \ldots, T_{i-1}$. From Theorem 11 P_i is a.s. bounded, thus by the weak law of large numbers [18] a.s. $\frac{1}{i}\sum_{k=0}^{i-1} P_k \to \lim_{i\to\infty} \frac{1}{i}\sum_{k=0}^{i-1} \overline{P}_k = P$ in the mean square sense as $i \to \infty$. Because P_i a.s. bounded, $\overline{P}_i$ is bounded and the right limit exists.

Note that P, if it exists, has the same value for any realization of $\{T_i\}$. So we might calculate P off-line by evaluating P_i for one single realization of $\{T_i\}$.

Lastly we remark that the assumed intentional stochastic sampling scheme in this section does not affect the control result in Section 12.3. If complete state information is available then this particular intentional sample scheme is not needed, in fact $\hat{x}_i = x_i$ and $P_i = P = q$ for $i = 0,1,\ldots$

12.5 INFLUENCE OF STOCHASTIC SAMPLING

In this section the influence of stochastic sampling on the criterion value is investigated by means of some illustrative examples.

Consider the continuous-time system (1) and the corresponding equivalent discrete-time system (5). Assume that the sampling periods T_i are uniformly distributed with mean $\overline{T}$ and variance $\mathrm{var}(T) = E\left\{\left(T_i - \overline{T}\right)^2\right\}$. Then because $T_i > 0$ we have

$$\mathrm{var}(T) < \frac{1}{3}\overline{T}^2. \tag{35}$$

Suppose $\mathrm{var}(T) = 0$ then (a.s.) $T_i = T$, $\Phi_i = \overline{\Phi}(T)$, $\Gamma_i = \overline{\Gamma}(T)$, thus T_i, Φ_i and Γ_i are deterministic and constant. Then due to Theorem 11 and the remark after Corollary 2, stabilizability (in the usual sense) and ms-stabilizability of $\left(\Phi_i, \Gamma_i\right)$ are identical, and detectability (in the usual sense) and a.s. uniform detectability of $\left(\Phi_i, C\right)$ are identical.

Assume that system (1) represents a second order single-input/single-output system and that (A, B) is reachable (in the usual sense) and (A, C) is observable (in the usual sense) [1]. Then in the case $\mathrm{var}(T) = 0$, $\left(\Phi_i, \Gamma_i\right)$ is reachable and $\left(\Phi_i, C\right)$ is observable if and only if the eigenvalues of A are real, or the eigenvalues l and l^* of A are conjugate complex and [27]

$$\overline{T} \neq \frac{kp}{\mathrm{Im}(l)},\ k = \pm 1, \pm 2, \ldots \tag{36}$$

Thus if the eigenvalues of A are complex, then reachability of $\left(\Phi_i, \Gamma_i\right)$ and observability of $\left(\Phi_i, C\right)$ are lost for values of $\overline{T}$ given by (36). If in addition these eigenvalues have positive real parts (unstable), then stabilizability of $\left(\Phi_i, \Gamma_i\right)$ and detectability of $\left(\Phi_i, C\right)$ are lost also for the values of $\overline{T}$ given by (36). In the case $\mathrm{var}(T) > 0$ there is in general no loss of ms-stabilizability of $\left(\Phi_i, \Gamma_i\right)$ and a.s. uniform detectability of $\left(\Phi_i, C\right)$ as will be illustrated by an example in this section.

In order to get more insight in the influence of stochastic sampling on the criterion value, we would like to compare this value for a particular system in three different cases. Firstly from (29b) we may calculate s_∞^* for $\mathrm{var}(T) = 0$, i.e. if the sampling is deterministic, and secondly for $\mathrm{var}(T) > 0$, i.e. if the sampling is stochastic. Thirdly from (31b) we may calculate $s_\infty^{L^D}$ where L_D is the control law L in the case $\mathrm{var}(T) = 0$. The value $s_\infty^{L^D}$ is the criterion value if the presence of stochastic sampling is not taken into account as far as the design of the control part is concerned.

Assume that system (1) is given by

$$A = \begin{bmatrix} 0.01 & 1 \\ 0 & 0.01 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ W = [0.01],$$

and criterion (2) by

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = [0].$$

The pair (A, B) is reachable and (A, C) is observable. The eigenvalues 0.01 and 0.01 of A are unstable but real. Thus in the case $\mathrm{var}(T) = 0$, $\left(\Phi_i, \Gamma_i\right)$ is reachable (and thus stabilizable) and (Φ_i, C) observable (and thus detectable). In the case $\mathrm{var}(T) > 0$ we may calculate that $\overline{R} > 0$, (Φ_i, Γ_i) is ms-stabilizable and (Φ_i, C) is a.s. uniformly detectable, which are the conditions to apply Theorem 8 and 13 in order to determine s_∞^*. Furthermore we may calculate that $r(A_L) < 1$, so Theorem 10 may be applied to determine $s_\infty^{L^D}$. In Fig. 2 s_∞^* is plotted for $\mathrm{var}(T)/\overline{T} = 0$ and for $\mathrm{var}(T)/\overline{T} = 0.1$, and $s_\infty^{L^D}$ for $\mathrm{var}(T)/\overline{T} = 0.1$ as a function of $\overline{T}$ in the case of complete state information, i.e. $P = q$. Fig. 3 is the same except that there is incomplete state information. Comparing Fig. 2 and 3 we see that incomplete state information makes s_∞^* about ten times higher w.r.t. complete state information. Compared to deterministic sampling, i.e. $\mathrm{var}(T)/\overline{T} = 0$, the stochastic sampling, i.e. $\mathrm{var}(T)/\overline{T} = 0.1$, increases s_∞^* with about 15%. From the values of $s_\infty^{L^D}$ in Fig. 2 and 3 observe that not taking into account the presence of stochastic sampling increases the criterion value with another 7%. Furthermore all the functions in Fig. 2 and 3 are monotonic increasing w.r.t. $\overline{T}$.

Now we assume the same system and criterion as above except that

$$A = \begin{bmatrix} 0.01 & -1 \\ 1 & 0.01 \end{bmatrix}.$$

The pair (A, B) is reachable and (A, C) is observable. The eigenvalues 0.01+j and 0.01-j of A are unstable and conjugate complex. Thus in the case $\mathrm{var}(T) = 0$, (Φ_i, Γ_i) is not stabilizable and (Φ_i, C) is not detectable for $\overline{T} = kp$, $k = 1, 2, \ldots$. In spite of this, in the case $\mathrm{var}(T) > 0$ we may calculate that $\overline{R} > 0$, (Φ_i, Γ_i) is ms-stabilizable and (Φ_i, C) is a.s. uniformly detectable thus s_∞^* exists for any $\overline{T} > 0$. However, $r(A_{L^D}) \geq 1$ for certain values of $\overline{T}$, so Theorem 10 may not be applied for these values to determine $s_\infty^{L^D}$. In Fig. 4 s_∞^* is plotted for $\mathrm{var}(T)/\overline{T} = 0$ and $\mathrm{var}(T)/\overline{T} = 0.1$ and $s_\infty^{L^D}$ for $\mathrm{var}(T)/\overline{T} = 0.1$ as a function of $\overline{T}$ in the case of complete state information and in Fig. 5 in the case of incomplete state information. Comparing Fig. 4 and 5 we see that incomplete state information increases s_∞^* with about 50 % w.r.t. complete state information. In the case of deterministic sampling, i.e. $\mathrm{var}(T)/\overline{T} = 0$, the criterion value s_∞^* has an asymptote for $\overline{T} = p$ corresponding with the loss of stabilizability of (Φ_i, Γ_i) and detectability of (Φ_i, C). The stochastic sampling, i.e. $\mathrm{var}(T)/\overline{T} = 0.1$, increases s_∞^* with about 10% for values of $\overline{T}$ which are not in the neighborhood of p. However, now s_∞^* does not have an asymptote for $\overline{T} = p$, which corresponds to the fact that in the case $\mathrm{var}(T) > 0$ there is no loss of ms-stabilizability of (Φ_i, Γ_i) and a.s. uniform detectability of (Φ_i, C). Thus in the neighborhood of $\overline{T} = p$, the value s_∞^* for $\mathrm{var}(T)/\overline{T} = 0.1$ is much lower than for $\mathrm{var}(T)/\overline{T} = 0$. From the values of $s_\infty^{L^D}$ in Fig. 4 and 5 observe that not taking into account the presence of stochastic sampling has almost no effect on s_∞^* for values of $\overline{T}$ which are not in the neighborhood of $\overline{T} = p$, However in the neighborhood of $\overline{T} = p$ the value $s_\infty^{L^D}$ goes to infinity which corresponds to the fact that $r(A_{L^D}) \geq 1$.

In connection with the foregoing two examples we make some remarks. It appears from the computer calculations that the effects of stochastic sampling illustrated by the two examples in Fig. 2-5 arise for general continuous-time systems with (A, B) stabilizable and

(A,C) detectable and general distributions of T_i. Especially it appears that (Φ_i,Γ_i) ms-stabilizable and (Φ_i,C) a.s. uniformly detectable for any $\overline{T}>0$. Thus there are no values of $\overline{T}$ for which s^*_∞ has an asymptote. However, if the stochastic sampling is not taken into account, then in general the area of $\overline{T}$ for which s^{LD}_∞ exists is smaller than the area for which s^*_∞ exists.

Finally in this section we make a practical remark concerning the calculation of s^*_∞ and s^{LD}_∞ from respectively (29) and (31). In these equations there arise terms like $\overline{\Phi^T X \Gamma}$ for an arbitrary matrix X, which may equally be written as $st^{-1}\left[\overline{(\Gamma\otimes\Phi)}^T st(X)\right]$, where st^{-1} means the inverse of the stack operator st [19]. Hence $\overline{(\Gamma\otimes\Phi)}$ need only be calculated once, while the st and st^{-1} operations involve only the renumbering of computer memory locations.

12.6 CONCLUSIONS

The digital stationary optimal control problem in the general case of linear stochastic continuous-time systems, quadratic integral criteria, incomplete state information and stochastic IID sampling periods has been solved, using the notions of mean-square stabilizability and mean-square detectability. Also the criterion value has been determined when the feedback is not optimal but arbitrary.

Existence and stability of the stationary optimal linear estimator and convergence of the sample mean of the estimator error covariance in the case of a particular stochastic sampling scheme have been considered, using the notions of uniform stabilizability and uniform detectability. The particular sampling scheme is an intentional stochastic sampling scheme.

It has been shown by means of two illustrative examples that stochastic sampling may increase or even restore stabilizability and may decrease or even destroy stability if the stochastic sampling is not taken into account in the determination of the optimal controller. In view of these effects intentional stochastic sampling is a useful tool in the design of digital control systems. On the other hand if stochastic sampling is present but neglected in the design procedure, then the performance of the control system may become unacceptable.

Given our interest in randomized digital optimal control, in the case of incomplete state information, a particular intentional stochastic sampling scheme was considered. As this paper demonstrated, if the state information is incomplete and if the stochastic sampling is not intentional, the associated equivalent discrete-time stationary optimal control problem is not separable. This more difficult problem has been considered and solved by De Koning [28]. The same problem has also been considered and solved by De Koning and Van Willigenburg [29], when in addition, the dimension of the controller state is constrained to be a-priori fixed, and less than that of the system state. The same authors have presented a similar result which applies to non-stationary digital optimal control problems considered over a finite horizon [30].

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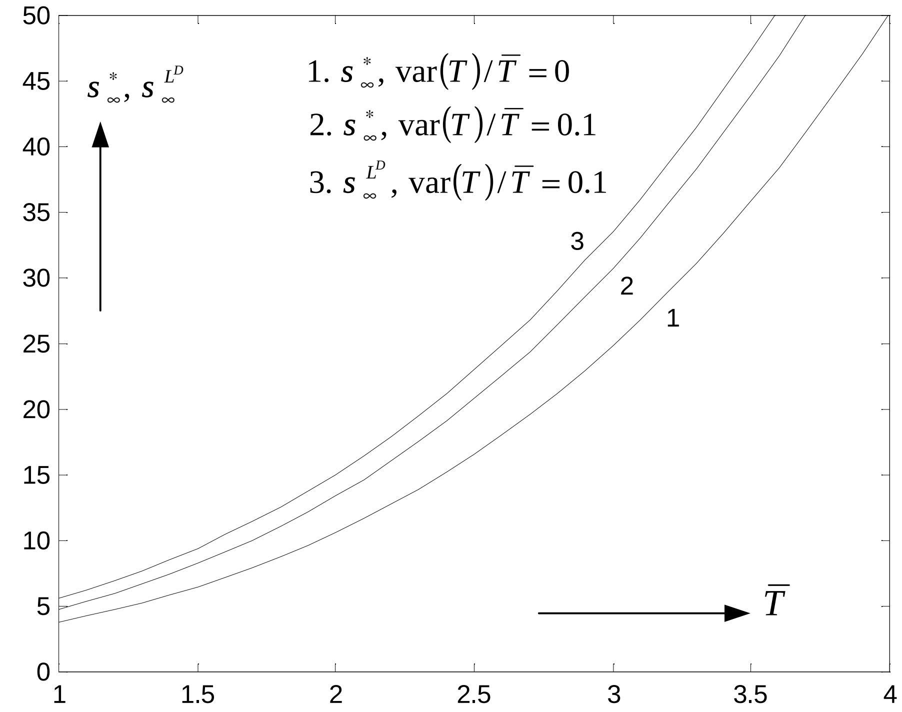

Figure 2: Complete state information

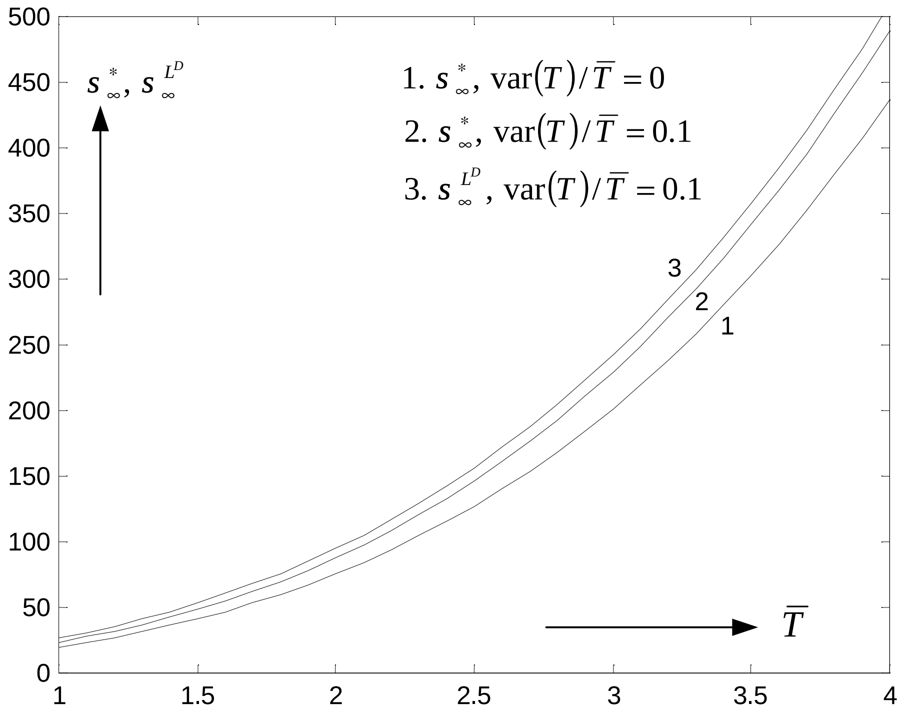

Figure 3: Incomplete state information

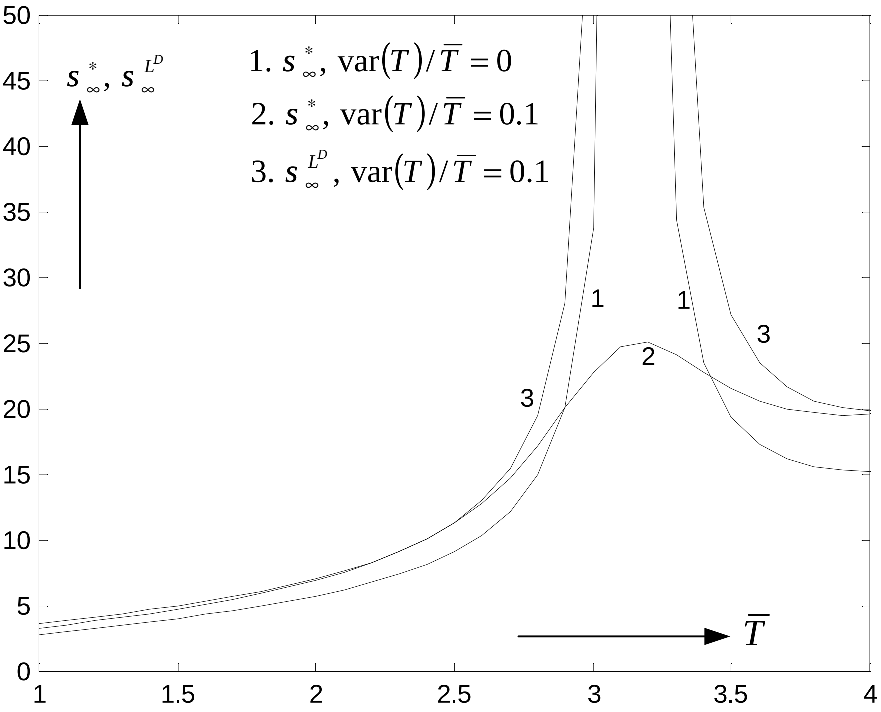

Figure 4: Complete state information

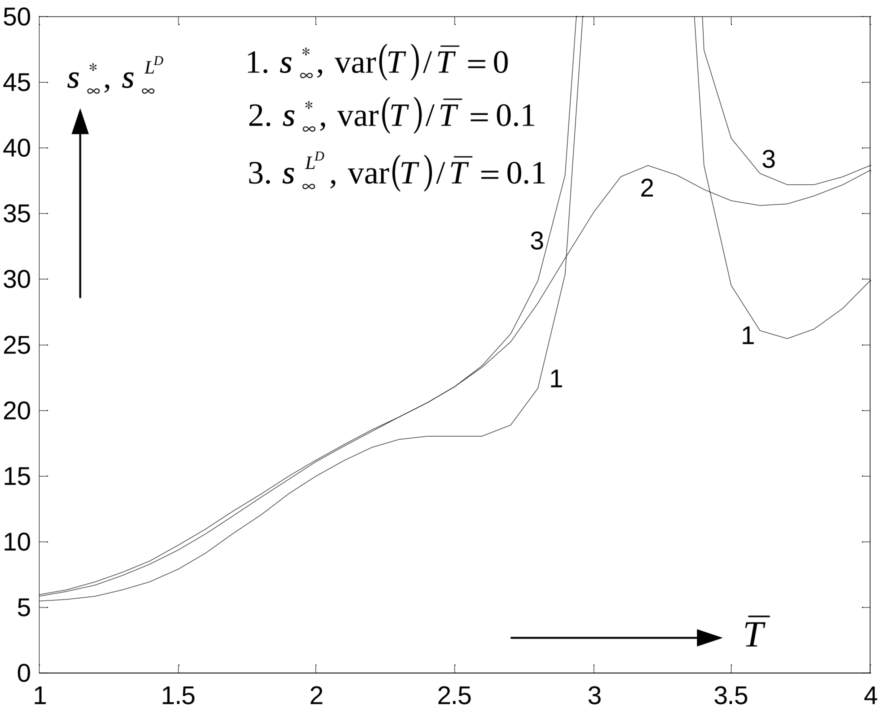

Figure 5: Incomplete state information

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