Including Errata

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Synthesis of digital optimal reduced-order compensators for asynchronously sampled systems

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The synthesis of finite-horizon digital optimal reduced-order compensators is presented, for asynchronous and aperiodically sampled continuous-time systems. The dimensions of the compensator state are a priori *fixed and may be time varying. Asynchronous and aperiodic sampling refers to a deterministic sampling scheme where an arbitrary, but* a priori *known, number of control variables is updated, and/or an arbitrary, but* a priori *known, number of outputs is sampled, at arbitrary, but* a priori *known, time instants. This sampling scheme generalizes most deterministic sampling schemes considered in the control literature. Through the use of an integral criterion the intersample behaviour is explicitly considered in the design. As a result, frequent, synchronous and periodic sampling is no longer necessary, which can be highly relevant in practice. Also the synthesis enables comparison of the optimal performance of reduced-order compensators as a function of their dimensions and the sampling scheme. The synthesis is illustrated with a numerical example.*

1. Introduction

Most digital control system design procedures put forward in the control literature assume frequent, synchronous and periodic updating of controls and observations. In practice, however, this may be undesirable, or even impossible. In the process industry, in the economy and in the area of environmental control, for example, very often not all measurements are or can be performed simultaneously, or with the same rate, but are performed at different (possibly irregular) time instants. The same holds for the updating of controls. Owing to costs associated with taking measurements and updating controls, frequent sampling may be expensive and therefore undesirable. The digital control of continuous-time systems is often actually performed in an asynchronous manner, because one analogue-to-digital (A/D) converter and one digital-to-analogue (D/A) converter are used to process different outputs and to update different control variables. Furthermore, different A/D and D/A converters are often not perfectly synchronized. The time in between consecutive measurements and control updates may not be negligible, for example in the case of the control of mechanical and electrical systems. Therefore the asynchronous and aperiodic sampling scheme considered in this paper is of significant practical importance. Furthermore it generalizes most of the deterministic sampling schemes considered in the control literature. Remarkably, most of these sampling schemes had already been put forward in 1959, by Kalman and Bertram (1959) in their theory of sampling systems.

Very often still, digital control problems are *approximated* by continuous- or discrete-time control problems. In both cases this results in a demand for a small sampling interval (Athans 1971, Van Willigenburg 1993). 'True' digital control problems involve integral criteria, and are constrained by the fact that the control is piecewise constant, assuming that zero-order holds are used (De Koning 1980, Levis *et al.* 1971, Van Willigenburg 1993). An integral criterion explicitly takes into account the inter-sample behaviour, thereby circumventing the demand for a small sampling interval. For example in the area of robot control where the computational burden on the controller is high, this is very important. Levis *et al.* (1971) (see also Dorato and Levis (1971)),

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seem to be the first who transformed a digital control problem, that is a digital linear quadratic (LQ) problem with an integral criterion, into an unconstrained *equivalent* discrete-time control problem. This turned out to be a discrete-time LQ problem in which the criterion contains a cross product even if the integral criterion does not contain a cross product (Levis *et al.* 1971). Since then the idea to transform a digital control problem into an unconstrained equivalent discrete-time problem has been used in several cases (Halyo and Caglayan 1976, De Koning 1980, Tiedemann and De Koning 1984, Van Willigenburg 1993, Van Willigenburg and De Koning 2000). All these cases concerned synchronous sampling where, at the sampling instants, all control variables are updated and all outputs are sampled.

Van Willigenburg (1995) and Van Willigenburg and De Koning (1995) considered asynchronous and aperiodic sampling. To solve these problems the digital control problem was again transformed into an unconstrained equivalent discrete-time problem. The fact that now not all the outputs are sampled at the same sampling instants can be incorporated in the equivalent discrete-time system, by adaptation of the output equation (Van Willigenburg and De Koning 1995). Also the fact that not all controls are updated at the same sampling instants implies that, in the equivalent discrete-time system, the non-updated control variables at certain sampling instants are actually not control variables any longer at these sampling instants, but state variables. As a result the dimension of the state of the equivalent discrete-time system and also the dimension of the output vary with time. Van Willigenburg (1995) and Van Willigenburg and De Koning (1995) found that, if the horizon is finite, these time-varying dimensions basically do not alter the solution.

Recent results (Van Willigenburg and De Koning 1998, 1999) have enabled the synthesis of optimal finite-horizon reduced-order discrete-time compensators, based on so-called strengthened discrete-time optimal projection equations (SDOPEs). Within the class of minimal compensators, these equations constitute first-order necessary optimality conditions. Finite-horizon minimal compensators turn out to have dimensions which vary over time (Van Willigenburg and De Koning 1998, 1999). As a result the well-known finite-horizon optimal full-order compensator is not minimal, while in general, in the infinite-horizon time-invariant case, it is (Yousuff and Skelton 1984). The above mentioned synthesis allows for discrete-time systems with time-varying dimensions, which is what we need in our case.

In the case of digital optimal reduced-order compensation the certainty equivalence property of the equivalent discrete-time problem is lost (Van Willigenburg and De Koning 1998, 1999). Therefore the results of Van Willigenburg and De Koning (1995), which possessed the certainty equivalence property, have to be extended to be applicable in our case. This is one of the contributions of this paper. The main reasons for designing reduced-order compensators are savings of computation time and computer memory and improvement in the conditioning of compensator computations. Therefore minimal compensators are preferred over non-minimal compensators. Our synthesis procedure allows the designer to prescribe arbitrary (possibly time-varying) dimensions of the compensator state. The synthesis procedure then comes up with a *minimal* compensator, with *maximal* dimensions that do not exceed the prescribed dimensions. However, to exploit maximally the available computer memory, it is beneficial to prescribe *a priori* compensator dimensions that are compatible with a minimal compensator. Another contribution of this paper is to show how the sampling scheme affects the minimal dimensions of a compensator. It provides formulae and a tool to guide the choice of the sampling scheme and the prescribed compensator dimensions. These demonstrate that this choice is not straightforward.

The synthesis of digital optimal reduced-order compensators is illustrated with a numerical example. To illustrate clearly the interesting features of the synthesis, without going into irrelevant details, the example is deliberately taken to be artificial.

2. The digital optimal reduced-order compensation problem

Consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t), \quad x(t) \in R^n, \quad u(t) \in R^m, \qquad (1a)$$

where $\{v(t)\}$ is a continuous-time zero-mean white-noise process given by

$$E\{v(t)\} = 0, \quad E\{v(t)v^{\mathrm{T}}(s)\} = V(t)\delta_D(t-s) \qquad (1b)$$

and where the initial state is stochastic:

$$E\{x(t_0)\} = \bar{x}_0, \quad E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]^{\mathrm{T}}\} = X, \qquad (1c)$$

In $(1b)$ $\delta_D(.)$ is the Dirac delta function. The system (1) may be the result of a linearization procedure of a nonlinear system about a possibly optimal trajectory in which $\{v(t)\}$ represents model uncertainty (Athans 1971). The system (1) will be sampled at sampling instances t_i, $i = 0, 1, \ldots, N-1$, $t_{i+1} > t_i$, and the output at these sampling instants is given by the following equation:

$$y(t_i) = C_i x(t_i) + w_i, \quad y(t_i) \in R^l, \quad i = 0, 1, \ldots, N-1, \tag{2a}$$

where C_i are real matrices of appropriate dimension and $\{w_i\}$ is a zero-mean discrete-time white noise process that is independent of $\{v(t)\}$:

$$E\{w_i\} = 0, \quad E\{w_i w_k^{\mathrm{T}}\} = W_i \delta_{ik}, \quad i = 0, 1, \ldots, N-1, \tag{2b}$$

where δ_{ik} is the Kronecker delta. After each sampling instant the control variables remain constant through the use of zero-order holds:

$$u(t) = u(t_i), \quad t_i \leqslant t < t_{i+1}, \quad i = 0, 1, \ldots, N-1. \tag{3}$$

Similar to Van Willigenburg and De Koning (1995) the asynchronous and aperiodic sampling can be described by the following matrix:

$$M_{\text{sampling}} = \begin{bmatrix} t_0 & t_1 & \cdot & t_{N-1} \\ m_0 & m_1 & \cdot & m_{N-1} \\ D_0 & D_1 & \cdot & D_{N-1} \\ l_0 & l_1 & \cdot & l_{N-1} \\ E_0 & E_1 & \cdot & E_{N-1} \end{bmatrix}. \tag{4a}$$

The first element t_i, $i = 0, 1, \ldots, N-1$, of each column is a sampling instant, that is a time instant at which some or all control variables are updated and/or some or all output variables are sampled (observed). The other elements in each column of M_{sampling} all refer to this sampling instant. The second element, that is

$$m_i, \quad i = 0, 1, \ldots, N-1, \quad 0 \leqslant m_i \leqslant m, \tag{4b}$$

represents the number of control variables that are updated. The column vectors

$$D_i \in R^m, \quad i = 0, 1, \ldots, N-1, \tag{4c}$$

contain the indices $\{1, 2, \ldots, m\}$ of all control variables. The order in which these indices appear is unimportant except that the indices of the updated control variables at time t_i must precede those of the non-updated control variables in D_i. Similarly,

$$l_i, \quad i = 0, 1, \ldots, N-1, \quad 0 \leqslant l_i \leqslant l, \tag{4d}$$

represents the number of sampled outputs (observed outputs) at times t_i while the column vectors

$$E_i \in R^l, \quad i = 0, 1, \ldots, N-1, \tag{4e}$$

contain the indices $\{1, 2, \ldots, l\}$ of all output variables. Again the order in which these indices appear is unimportant except that the indices of the sampled outputs must appear first. Note from our definition of a sampling instant that $m_i = l_i = 0$ cannot hold for any $i = 0, 1, \ldots, N-1$.

The following compensator is chosen to control the asynchronously sampled system (1)–(4):

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_i y_i^s, \quad i = 0, 1, \ldots, N-1, \tag{5a}$$

$$u_i^u = -L_i \hat{x}_i, \quad i = 0, 1, \ldots, N-1, \tag{5b}$$

where $\hat{x}_i \in R^{n_i^c}$ is the compensator state which has *a priori* fixed dimensions n_i^c, $i = 0, 1, \ldots, N$, specified by the designer, which may vary with time. Furthermore $u_i^u \in R^{m_i}$ is the vector of updated control variables at t_i and $y_i^s \in R^{l_i}$ is the vector of sampled outputs at t_i . Note that, if at time t_i no controls are updated, that is only outputs are sampled, u_i^u is empty and so is L_i. Similarly, if at time t_i no outputs are sampled, that is only controls are updated, y_i^s is empty and so is K_i. Furthermore the real matrices F_i, K_i and L_i have appropriate dimensions. Element k, $1 \leqslant k \leqslant m_i$ of vector $u_i^u \in R^{m_i}$ corresponds to an updated element of $u(t_i)$ the index of which equals the kth element of vector D_i. Similarly element k, $1 \leqslant k \leqslant l_i$ of vector $y_i^s \in R^{l_i}$ corresponds to a sampled element of $y(t_i)$ the index of which equals the kth element of vector E_i. Thus the matrix M_{sampling} in (4a) fully determines how the updated control variables of $u(t_i)$ are mapped on to $u_i^u \in R^{m_i}$ and how the sampled outputs of $y(t_i)$ are mapped on to $y_i^s \in R^{l_i}$. Compensator (5) is denoted by $(\hat{x}_0, F^N, K^N, L^N)$ where

$$F^N = \{F_i, \ i = 0, 1, \ldots, N-1\},$$

$$K^N = \{K_i, \ i = 0, 1, \ldots, N-1\},$$

$$L^N = \{L_i, \ i = 0, 1, \ldots, N-1\}.$$

2.1. *Problem formulation*

Given the asynchronously sampled system (1)–(4) the optimal fixed-order compensation problem is to find a compensator (5), with prescribed dimensions n_i^c, $i = 0, 1, \ldots, N$ of the compensator state, which minimizes the criterion

$$\begin{aligned} J_N(\hat{x}_0, F^N, K^N, L^N) &= E\{x^{\mathrm{T}}(t_N) Z x(t_N)\} \\ &\quad + E\left\{\int_{t_0}^{t_N} x^{\mathrm{T}}(t) Q(t) x(t) + u^{\mathrm{T}}(t) R(t) x(t)\,\mathrm{d}t\right\}, \end{aligned} \tag{6a}$$

$$Q(t) \geqslant 0, \quad R(t) > 0, \quad t_0 \leqslant t \leqslant t_N, \quad Z \geqslant 0, \tag{6b}$$

and to find the minimum value of J_N.

3. The equivalent discrete-time optimal control problem

To solve the digital optimal control problem, as in the work of Van Willigenburg and De Koning (1995), the digital optimal control problem with the piecewise constant constraint (3) on the control is first transformed into an unconstrained equivalent discrete-time optimal control problem. In the work of Van Willigenburg and De Koning (1995), which treats the case of full-order compensation, the digital optimal control problem has the certainty equivalence property. Therefore the estimation and control design were performed separately. So-called control instants, at which some or all control variables are updated, were distinguished from so-called observation instants at which some or all outputs are sampled. As a result the equivalent discrete-time optimal control problem in the paper by Van Willigenburg and De Koning (1995) consists of an equivalent discrete-time system, which describes the state transitions of the system (1)–(4) from each control instant to the next, and an equivalent discrete-time sum criterion, which describes the contribution to (6) over each time interval in between two control instants, as a function of the state and control at the start of the interval.

In the case of digital optimal reduced-order compensation, the certainty equivalence property is lost (Van Willigenburg and De Koning 1998, 1999). Therefore the control and estimation designs are now coupled and the distinction between control and observation instants, made by Van Willigenburg and De Koning (1995), must be abandoned. Our equivalent discrete-time optimal control problem consists of an equivalent discrete-time system that describes the state transitions of the system (1)–(4) from each sampling instant to the next, and an equivalent discrete-time sum criterion, which describes the contribution to (6) over each sampling interval $[t_i, t_{i+1})$, $i = 0, 1, \dots, N-1$ as a function of the state and control at t_i. The equivalent discrete-time system is described by,

$$x^a_{i+1} = \Phi^a_i x_i + \Gamma^a_i u^u_i + v^a_i, \quad i = 0, 1, \dots, N-1, \tag{7a}$$

where

$$x^a_i = \begin{bmatrix} x_i \\ u^0_i \end{bmatrix} \tag{7b}$$

with

$$x_i = x(t_i) \tag{7c}$$

and where the vectors $u^0_i \in R^{m-m_i}$ contain the non-updated control variables at times t_i. Element k, $1 \leqslant k \leqslant m - m_i$, of vector $u^0_i \in R^{m-m_i}$ is a non-updated control variable of $u(t_i)$ the index of which equals the $(m_i + k)$th element of vector D_i. Therefore the matrix M_{sampling} in (4) also fully determines how the non-updated control variables are mapped on to $u^0_i \in R^{m-m_i}$. Finally $\{v^a_i\}$ is a discrete-time zero-mean white noise process given by

$$E\{v^a_i\} = 0, \quad E\{v^a_i v^{a^{\mathrm{T}}}_k\} = V^a_i \delta_{ik} \tag{7d}$$

and the initial state x^a_0 is stochastic:

$$E\{x^a_0\} = \bar{x}^a_0, \quad E\{x^a_0 x^{a^{\mathrm{T}}}_0\} = X^a. \tag{7e}$$

The equivalent discrete-time sum criterion is given by

$$\begin{aligned} J_N(\hat{x}_0, F^N, K^N, L^N) &= E\{x^a_N Z x^a_N\} \\ &\quad + E\left\{\sum_{i=0}^{N-1} x^a_i Q^a_i x^a_i + 2x^a_i M^a_i u^u_i + u^u_i R^a_i u^u_i\right\} \\ &\quad + \sum_{i=0}^{N-1} \gamma_i \end{aligned} \tag{8a}$$

where

$$x^a_N = x_N \tag{8b}$$

since the non-updated control variables at the final time $t_N > t_{N-1}$ play no role in the problem and are therefore left out of the description.

The equivalent discrete-time system (7) describes the state transitions of the original system (1)–(4) from each sampling instant to the next, while its state is augmented with the non-updated control variables to describe there influence properly (Van Willigenburg and De Koning 1995). The matrices

$$\Phi^a_i, \ \Gamma^a_i, \ V^a_i, \quad i = 0, 1, \dots, N-1, \quad \bar{x}^a_0, \ X^a, \tag{9}$$

of the equivalent discrete-time system can all be computed from

$$A(t), \ B(t), \ V(t), \ t_0 \leqslant t \leqslant t_N, \quad x_0, \ X, \ M_{\text{sampling}}, \tag{10}$$

of the original system (1)–(4), using the procedure described by Van Willigenburg and De Koning (1995), but with control instants replaced by sampling instants. Because of this replacement, if at t_i no controls are updated, the input matrix Γ^a_i and the control penalty matrix R^a_i, are empty matrices. The matrices

$$Q^a_i, \ R^a_i, \ M^a_i \ \gamma_i, \quad i = 0, 1, \dots, N-1, \tag{11}$$

in the equivalent discrete-time sum criterion can be computed from

$$A(t), \ B(t), \ V(t), \ Q(t), \ R(t), \quad t_0 \leqslant t \leqslant t_N, \quad M_{\text{sampling}}, \tag{12}$$

of the original system (1)–(4) and the integral criterion (6) as described by Van Willigenburg and De Koning (1995), again with control instants replaced by sampling instants. Owing to the order reduction, full and perfect information about the part u^0_i, that is the non-updated control variables, of the augmented state x^a_i, is no longer available. Since we also no longer distinguish between control and observation instants, as opposed to the work of Van Willigenburg and De Koning (1995), our actual output equation must be specified in terms of the augmented state x_i^a at each time $i = 0, 1, \ldots, N-1$. This output equation is described by

$$y_i^s = C_i^a x_i^a + w_i^s, \quad i = 0, 1, \ldots, N-1, \qquad (13a)$$

where

$$C_i^a = [C_i^s \quad 0] \in R^{l_i \times (n+m-m_i)}, \quad C_i^s \in R^{l_i \times n}, \qquad (13b)$$

while $\{w_i^s\}$ is a discrete-time zero-mean white noise process:

$$E\{w_i^s\} = 0, \quad E\{w_i^s w_k^{s^{\mathrm{T}}}\} = W_i^s \delta_{ik} \qquad (13c)$$

The matrices

$$C_i^s, \ W_i^s, \quad i = 0, 1, \ldots, N-1, \qquad (14)$$

which determine the actual output equation (13) are computed from

$$C_i, W_i, \quad i = 0, 1, \ldots, N-1, \quad M_{\text{sampling}}, \qquad (15)$$

as described by Van Willigenburg and De Koning (1995), but with observation instants replaced by sampling instants. Because of this replacement, if at t_i no outputs are sampled, the output matrices C_i^a, C_i^s and the covariance matrix W_i^s in (13) are empty.

Summarizing the equivalent discrete-time optimal control problem is given by (7), (8) and (13). As in the work of Van Willigenburg and De Koning (1995), the matrices (9), (11), (14), which determine this problem, have time-varying dimensions and can all be computed from the original problem (1)–(4) and (6) using modified results of Van Willigenburg and De Koning (1995). As opposed to the work of Van Willigenburg and De Koning (1995), in the equivalent discrete-time optimal control problem formulation empty matrices may occur. In the remaining part of this paper, special attention is paid to the interpretation of equations which contain these (possibly empty) matrices.

4. First-order necessary optimality conditions and numerical solutions

The equivalent discrete-time optimal control problem (7), (8) and (13) is a discrete-time fixed-order compensation problem where the system and criterion matrices have time-varying dimensions. The final term in (8) is independent of the control and may therefore be disregarded during the minimization. In general the equivalent discrete-time optimal control problem is complicated and non-convex.

Van Willigenburg and De Koning (1998, 1999) presented first-order necessary optimality conditions for the solution of this problem in the form of so-called SDOPEs which constitute a discrete-time two point boundary value problem. The SDOPEs constitute a generalization of the well-known estimation and control Riccati equations related to full-order linear–quadratic Gaussian (LQG} design. Van Willigenburg and De Koning (1999), who treated the more general case of discrete-time systems with white parameters, presented two numerical algorithms to compute solutions that satisfy the SDOPEs. The algorithms constitute generalizations of the algorithm that iterates the estimation and control Riccati equations related to full-order LQG design and therefore efficiently solve the two point boundary value problem.

To solve the equivalent discrete-time optimal control problem (7), (8) and (13) of the two algorithms presented by Van Willigenburg and De Koning (1999) the iterative algorithm is presented here, since it is the most efficient and since it is also capable of computing multiple solutions, if these exist.

Define

$$n_i = n + m - m_i, \quad i = 0, 1, \ldots, N; \qquad (16)$$

so n_i, $i = 0, 1, \ldots, N-1$, is the dimension of the augmented state x_i^a, $i = 0, 1, \ldots, N$. Let S^{n_i, n_i} denote the space of real $n_i \times n_i$ symmetric matrices. Define

$$X_1^N = \{X_i^1, i = 0, 1, \ldots, N\}, \quad X_i^1 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (17a)$$

$$X_2^N = \{X_i^2, i = 0, 1, \ldots, N\}, \quad X_i^2 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (17b)$$

$$X_3^N = \{X_i^3, i = 0, 1, \ldots, N\}, \quad X_i^3 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (17c)$$

$$X_4^N = \{X_i^4, i = 0, 1, \ldots, N\}, \quad X_i^4 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (17d)$$

and

$$Y_1^N = \{Y_i^1, i = 0, 1, \ldots, N\}, \quad Y_i^1 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (18a)$$

$$Y_2^N = \{Y_i^2, i = 0, 1, \ldots, N\}, \quad Y_i^1 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (18b)$$

$$Y_3^N = \{Y_i^3, i = 0, 1, \ldots, N\}, \quad Y_i^1 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N, \qquad (18c)$$

$$Y_4^N = \{Y_i^4, i = 0, 1, \ldots, N\}, \quad Y_i^1 \in S^{n_i, n_i}, \quad i = 0, 1, \ldots, N. \qquad (18d)$$

Consider the following eigenvalue decompositions of $X_i^3 X_i^4$,

$$U_{X_i^3 X_i^4} \Lambda_{X_i^3 X_i^4} U_{X_i^3 X_i^4}^{-1} = X_i^3 X_i^4, \quad i = 0, 1, \ldots, N, \qquad (19)$$

arranged such that the *largest positive eigenvalues* appear first on the diagonal of the diagonal matrix $\Lambda_{X_i^3 X_i^4}$. Define

$$r_i^c = \min(n_i^c, \text{rank}(X_i^3 X_i^4)), \quad i = 0, 1, \dots, N, \quad (20a)$$

$$G_i = [I_{r_i^c} \quad 0] U^{\mathrm{T}}_{X_i^3 X_i^4}, \quad i = 0, 1, \dots, N, \quad (20b)$$

$$H_i = [I_{r_i^c} \quad 0] U^{-1}_{X_i^3 X_i^4}, \quad i = 0, 1, \dots, N, \quad (20c)$$

$$\tau_i = G_i^{\mathrm{T}} H_i = U_{X_i^3 X_i^4} \begin{bmatrix} I_{r_i^c} & 0 \\ 0 & 0 \end{bmatrix} U^{-1}_{X_i^3 X_i^4},$$

$$\tau_{\perp i} = I_{n_i} - \tau_i, \quad i = 0, 1, \dots, N, \quad (20d)$$

Consider the following nonlinear transformation:

$$(Y_1^N, Y_2^N, Y_3^N, Y_4^N) = \Re(X_1^N, X_2^N, X_3^N, X_4^N), \quad (21)$$

defined by

$$Y_{i+1}^1 = \Phi_i^a Y_i^1 \Phi_i^{a^{\mathrm{T}}} - K_{Y_i^1}(C_i^a Y_i^1 C_i^{a^{\mathrm{T}}} + W_i^s) K_{Y_i^1}^{\mathrm{T}} + V_i^a + \tau_{\perp i+1} \Psi_i^1 \tau_{\perp i+1}^{\mathrm{T}},$$
$$i = 0, 1, \dots, N-1, \quad Y_0^1 = X^a, \quad (22a)$$

$$Y_i^2 = \Phi_i^{a^{\mathrm{T}}} Y_{i+1}^2 \Phi_i^a - L_{Y_{i+1}^2}^{\mathrm{T}} (\Gamma_i^{a^{\mathrm{T}}} Y_{i+1}^2 \Gamma_i^a + R_i^a) L_{Y_{i+1}^2} + Q_i + \tau_{\perp i}^{\mathrm{T}} \Psi_{i+1}^2 \tau_{\perp i},$$
$$i = 0, 1, \dots, N-1, \quad Y_N^2 = Z, \quad (22b)$$

$$Y_{i+1}^3 = \tfrac{1}{2}(\tau_{i+1} \Psi_i^1 + \Psi_i^1 \tau_{i+1}^{\mathrm{T}}),$$
$$i = 0, 1, \dots, N-1, \quad Y_0^3 = \bar{x}_0 \bar{x}_0^{\mathrm{T}}, \quad (22c)$$

$$Y_i^4 = \tfrac{1}{2}(\tau_i^{\mathrm{T}} \Psi_{i+1}^2 + \Psi_{i+1}^2 \tau_i),$$
$$i = 0, 1, \dots, N-1, \quad Y_N^4 = 0, \quad (22d)$$

where

$$\Psi_i^1 = (\Phi_i^a - \Gamma_i^a L_{Y_{i+1}^2}) Y_i^3 (\Phi_i^a - \Gamma_i^a L_{Y_{i+1}^2})^{\mathrm{T}} + K_{Y_i^1}(C_i^a Y_i^1 C_i^{a^{\mathrm{T}}} + W_i^s) K_{Y_i^1}^{\mathrm{T}},$$
$$i = 0, 1, 2, \dots, N-1 \quad (23a)$$

$$\Psi_{i+1}^2 = (\Phi_i^a - K_{Y_i^1} C_i^a)^{\mathrm{T}} Y_{i+1}^4 (\Phi_i^a - K_{Y_i^1} C_i^a) + L_{Y_{i+1}^2}^{\mathrm{T}} (\Gamma_i^{a^{\mathrm{T}}} Y_{i+1}^2 \Gamma_i^a + R_i^a) L_{Y_{i+1}^2},$$
$$i = 0, 1, 2, \dots, N-1, \quad (23b)$$

$$K_{Y_i^1} = \Phi_i^a Y_i^1 C_i^{a^{\mathrm{T}}} (C_i^a Y_i^1 C_i^{a^{\mathrm{T}}} + W_i^s)^{-1},$$
$$i = 0, 1, \dots, N-1, \quad (23c)$$

$$L_{Y_{i+1}^2} = (\Gamma_i^{a^{\mathrm{T}}} Y_{i+1}^2 \Gamma_i^a + R_i^a)^{-1} (\Gamma_i^{a^{\mathrm{T}}} Y_{i+1}^2 \Phi_i^a + M_i^{a^{\mathrm{T}}})$$
$$i = 0, 1, \dots, N-1. \quad (23d)$$

Note that, to compute $(Y_1^N, Y_2^N, Y_3^N, Y_4^N) = \Re(X_1^N, X_2^N, X_3^N, X_4^N)$ first (22*b*) and (22*d*) are iterated backward in time using (20*d*), (23*b*), (23*c*), (23*d*) and then (22*a*) and (22*c*) are iterated forward in time using (20*d*), (23*a*), (23*c*) and (23*d*). From Van Willigenburg and De Koning (1998, 1999) observe that $(X_1^N, X_2^N, X_3^N, X_4^N) = \Re(X_1^N, X_2^N, X_3^N, X_4^N)$ satisfy the SDOPEs. Denote k *repeated* applications of $\Re$ by $(X_1^{N^k}, X_2^{N^k}, X_3^{N^k}, X_4^{N^k}) = \Re^k(X_1^{N^0}, X_2^{N^0}, X_3^{N^0}, X_4^{N^0})$. If at t_i no controls are updated, Γ_i^a and R_i^a are empty. As a result $L_{Y_{i+1}^2}$ in (23*d*) is empty and also the second term in both (22*b*) and (23*b*). Similarly, if at t_i no outputs are sampled, C_i^a and W_i^s are empty. As a result $K_{Y_i^1}$ in (23*c*) is empty and also the second term in both (22*a*) and (23*a*). To compute the equations, these empty matrices should be skipped, or considered to be zero matrices having dimensions compatible with those of the remaining part of the equations.

Algorithm:

Step 1. *Initialization.*

$$X_i^{1^0} = \Theta_{n_i}, \quad X_i^{2^0} = \Theta_{n_i}, \quad X_i^{3^0} = \Lambda_{n_i}^1,$$
$$X_i^{4^0} = \Lambda_{n_i}^2, \quad i = 0, 1, \dots, N,$$

where $\Lambda_{n_i}^1, \Lambda_{n_i}^2 \geqslant 0$ are symmetric random non-negative $n_i \times n_i$ matrices and Θ_{n_i} are zero $n_i \times n_i$ matrices.

Step 2. *Computation.* Determine, through iteration, whether

$$(X_1^N, X_2^N, X_3^N, X_4^N) = \lim_{k \to \infty} [\Re^k(X_1^{N^0}, X_2^{N^0}, X_3^{N^0}, X_4^{N^0})]$$

exists.

Then from Van Willigenburg and De Koning (1995, 1998, 1999) the following theorem is obtained.

Theorem 1: *If the algorithm converges to a non-negative solution, that is* $X_i^1 \geqslant 0$, $X_i^2 \geqslant 0$, $X_i^3 \geqslant 0$, $X_i^4 \geqslant 0$, $i = 0, 1, \dots, N$, *it generates a minimal compensator* $(\hat{x}_0, F^N, K^N, L^N)$ *given by*

$$F_i = H_{i+1}[\Phi_i^a - K_{X_i^1} C_i^a - \Gamma_i^a L_{X_{i+1}^2}] G_i^{\mathrm{T}} \in R^{n_{i+1}^{\prime c} \times n_i^{\prime c}},$$
$$i = 0, 1, \dots, N-1, \quad (24a)$$

$$K_i = H_{i+1} K_{X_i^1} \in R^{n_{i+1}^{\prime c} \times l_i}, \quad i = 0, 1, \dots, N-1, \quad (24b)$$

$$L_i = L_{X^2_{i+1}} G_i^{\mathrm{T}} \in R^{m_i \times n_i'^{\mathrm{c}}}, \quad i = 0, 1, \dots, N-1, \qquad (24\,c)$$

$$\hat{x}_0 = H_0 \bar{x}_0 \in R^{n_0'^{\mathrm{c}}}, \qquad (24\,d)$$

where the compensator dimensions $n_i'^{\mathrm{c}}$ *satisfy,*

$$n_0'^{\mathrm{c}} = 1, \quad n_i'^{\mathrm{c}} = r_i^{\mathrm{c}}, \quad i = 1, \dots, N-1, \quad n_N'^{\mathrm{c}} = 0. \quad (24\,e)$$

This compensator is a minimal realization of a local or global minimum of the optimal fixed-order compensation problem. The costs of the compensator $(\hat{x}_0, F^N, K^N, L^N)$ *are given by*

$$J_N = J_{N_1} = J_{N_2}, \qquad (25\,a)$$

$$\begin{aligned} J_{N_1} &= \mathrm{Tr}[Z(X_N^1 + X_N^3)] \\ &\quad + \sum_{i=0}^{N-1} \mathrm{Tr}[Q_i^{\mathrm{a}} X_i^1 + (Q_i^{\mathrm{a}} + L_{X^2_{i+1}}^{\mathrm{T}} R_i^{\mathrm{a}} L_{X^2_{i+1}} \\ &\quad - 2M_i^{\mathrm{a}} L_{X^2_{i+1}}) X_i^3] + \gamma_i, \end{aligned} \qquad (25\,b)$$

$$\begin{aligned} J_{N_2} &= \mathrm{Tr}[X(X_0^2 + X_0^4) + \bar{x}_0 \bar{x}_0^{\mathrm{T}} X_0^2] \\ &\quad + \sum_{i=0}^{N-1} \mathrm{Tr}[V_i^{\mathrm{a}} X_{i+1}^2 + (V_i^{\mathrm{a}} + K_{X_i^1} W_i^{\mathrm{s}} K_{X_i^1}^{\mathrm{T}}) X_{i+1}^4] \\ &\quad + \gamma_i. \end{aligned} \qquad (25\,c)$$

Proof: Van Willigenburg and De Koning (1998, 1999) treated the case of discrete-time systems with time-varying dimensions, which we have in our application, and Van Willigenburg and De Koning (1999) treated the more general case of discrete-time systems with white parameters, which we do not have in our application. In the latter paper it is explained how the results simplify if the system has deterministic parameters, as in our application. In one respect our application differs slightly from these results. If at t_i no controls are updated, Γ_i^{a}, R_i^{a} and $L_{X^2_{i+1}}$ are empty and, as a result, L_i in (24 *c*) is empty. This complies with the fact that u_i^{u} is empty. Similarly if at t_i no outputs are sampled, C_i^{a}, W_i^{s} and $K_{X_i^1}$ are empty and, as a result, K_i in (24 *b*) is empty. This complies with the fact that y_i^{s} is empty. Van Willigenburg and De Koning (1998, 1999) treated all elements of the compensator matrices as parameters that have to be optimized. Therefore, if these parameters are not present, they should be deleted in the problem description and, as a result, in the description of the first-order necessary optimality conditions. So all terms involving an empty matrix should be deleted from (19)–(25) or, alternatively, considered to be the zero matrix with dimensions compatible with the remaining part of the equations. □

If, as in theorem 1, $n_i'^{\mathrm{c}}$, $i = 0, 1, \dots, N$ are the dimensions of a minimal compensator for the system (7), then $n_i'^{\mathrm{c}}$, $i = 0, 1, \dots, N$, must satisfy (Van Willigenburg and De Koning 1998),

$$n_0'^{\mathrm{c}} = 1, \quad n_N'^{\mathrm{c}} = 0, \qquad (26\,a)$$

$$n_i'^{\mathrm{c}} - m_i \leqslant n_{i+1}'^{\mathrm{c}} \leqslant n_i'^{\mathrm{c}} + l_i, \quad i = 0, 1, \dots, N-1, \qquad (26\,b)$$

$$n_i'^{\mathrm{c}} \leqslant n_i, \quad i = 1, \dots, N-1, \qquad (26\,c)$$

Equation (26) describe constraints on the dimensions of a minimal compensator for the system (7). These constraints will be discussed in § 5. Details concerning the numerical computation of the above algorithm can be found in the paper by Van Willigenburg and De Koning (1999). There it is also reported that the algorithm seems to have the following two important properties.

(1) If the algorithm converges, it converges to a non-negative solution, as desired.
(2) Owing to (20 *a*) the algorithm finds a minimal compensator with maximal dimensions $n_i'^{\mathrm{c}}$, $i = 0, 1, \dots, N$ which do not exceed the prescribed dimensions n_i^{c}, $i = 0, 1, \dots, N$, of the compensator state.

According to Van Willigenburg and De Koning (1998) the global minimum is among these compensators, if the conjecture in the paper by Van Willigenburg and De Koning (1998) holds. Therefore, if the conjecture in the paper by Van Willigenburg and De Koning (1998) holds, (2) is also a desired property of the algorithm.

5. Design issues: choice of the sampling scheme and the prescribed compensator dimensions

The main reasons for designing reduced-order compensators are the saving of computation time, the saving of computer memory and the improvement in the conditioning of compensator computations. Therefore minimal compensators are preferred over non-minimal compensators. Although the algorithm automatically generates a minimal compensator with maximal dimensions $n_i'^{\mathrm{c}}$, $i = 0, 1, \dots, N$ that do not exceed the prescribed compensator dimensions n_i^{c}, $i = 0, 1, \dots, N$, to exploit maximally the available computer memory, it is advantageous to take into account *a priori*, the restrictions (26) on the dimensions of a minimal compensator for the system (7). In other words, it is advantageous to prescribe compensator dimensions n_i^{c}, $i = 0, 1, \dots, N$ that satisfy (26) with $n_i'^{\mathrm{c}}$ replaced by n_i^{c}.

The sampling scheme is often partly dictated by constraints imposed by the system and control equipment. Usually, however, some freedom remains as to the choice of a possibly asynchronous and aperiodic sampling scheme. One of the highly interesting features of the digital optimal compensator design procedure presented in this paper is that it allows the designer to compute the performance for all kinds of different deterministic sampling schemes. This enables optimization of the sampling scheme. Furthermore, if the sampling scheme is asynchronous, the approach reveals how the sampling scheme is linked to the dimensions of a minimal compensator. This link is described by (16) and (26).

According to (16) the dimension of the state of the augmented equivalent discrete-time system (7) varies over time with the number of non-updated control variables at each sampling instant. Equation (26 *c*) states the familiar fact that it is useless to design a compensator whose dimension of the state, at any time $i = 0, 1, \ldots, N$, exceeds that of the system (7). Equation (26 *b*) states that the *changes* of the dimension of the state of a minimal compensator for the system (7) are bounded from above and below by the number of updated control variables and sampled outputs. So besides (26 *c*), (26 *b*) and (26 *a*) further limit the possible dimensions of the state of a minimal compensator for the system (7).

Summarizing (16) and (26) together must guide the selection of the sampling scheme and the prescribed compensator dimensions. Table 1, which is introduced and explained in the next section, is a useful tool to guide this selection.

6. A numerical example

The numerical example presented in this section is chosen so that it contains the key features of the problem. The system and criterion contain time-varying matrices and the sampling scheme is asynchronous and aperiodic. The problem data are as follows:

$$A(t) = \begin{bmatrix} 0.3 + 0.2\sin(0.5\pi t) & 0 \\ 5 & 0.5 + 0.4\cos(0.5\pi t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} \sin(3t) & 1 \\ -1 & \cos(3t) \end{bmatrix},$$

$$V(t) = 0.05\begin{bmatrix} 1.5 + \cos(2\pi t) & 0.2 \\ 0.2 & 1.3 + \sin(\pi t) \end{bmatrix},$$

$$\bar{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix},$$

$$C_i = \begin{bmatrix} -\sin(2\pi t_i) & 1 \\ -2 & 3\cos(\pi t_i) \end{bmatrix},$$

$$W_i = \begin{bmatrix} 0.7 + 0.5\cos(\pi t_i) & 0.15 \\ 0.15 & 1 + 0.5\cos(4\pi t_i) \end{bmatrix},$$

$$Q(t) = \begin{bmatrix} 2 + \sin(2t) & 0.5 \\ 0.5 & 2 + \sin(2t) \end{bmatrix},$$

$$R(t) = 0.01\begin{bmatrix} 2 + \cos(2t) & -0.5 \\ -0.5 & 2 + \cos(2t) \end{bmatrix},$$

$$Z = \begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix},$$

$$M_{\text{sampling}} = \begin{bmatrix} 0.0 & 0.2 & 0.5 & 0.8 & 0.9 & 1.4 & 1.5 \\ 1 & 0 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 2 & 2 \end{bmatrix},$$

$$t_N = t_7 = 2.1.$$

The example is a modified version of the example presented by Van Willigenburg and De Koning (1995). The sampling scheme is identical. Compared with the example in the work of Van Willigenburg and De Koning (1995) the system matrix $A(t)$ now has positive real eigenvalues for all t, equal to its diagonal elements. As a result the system is unstable, the problem is numerically more challenging, and the performance is more sensitive to order reduction. Non-zero off-diagonal elements in $Q(t)$, $R(t)$ and Z have been introduced. Finally $V(t)$, $R(t)$ and Z have been scaled, again to increase the sensitivity of the performance to order reduction. From M_{sampling} observe that, for example, at $t_0 = 0.0$, only the first control variable is updated while no outputs are sampled. The non-updated control variable at $t_0 = 0.0$ is equal to zero. At $t_4 = 0.9$, for example, no control variables are updated and only the second output is sampled.

The optimal full-order compensator is computed from the algorithm in the paper by Van Willigenburg and De Koning (1995), which uses results from Van Willigenburg (1993), with an integration step size of 0.01. The optimal full-order compensator can also be obtained from our algorithm if we replace (20) by (Van Willigenburg and De Koning 1998),

$$n_i'^{c} = n_i, \tag{27 a}$$

$$G_i = H_i = \tau_i = I_{n_i}. \tag{27 b}$$

The minimum costs, in both cases, are computed to be 219.98. If we prescribe,

$$n_i^c = n_i = n + m - m_i, \quad i = 0, 1, \ldots, N, \tag{28}$$

that is 3, 4, 2, 3, 4, 4, 3, 2, 2 respectively, then our algorithm, with an integration step size of 0.01 (Van Willigenburg 1993, Van Willigenburg and De Koning 1995), computes a minimal realization of the optimal full-order compensator which achieves the same performance (Van Willigenburg and De Koning 1998). The dimensions $n_i'^{c}$, $i = 0, 1, \dots, N$ of the compensator state of this minimal realization are specified in the first row of table 2, as explained below. The compensator matrices, together with the compensator matrices of the other compensators mentioned in table 2, are listed in appendix A.

In the case of reduced-order compensator design, table 1, which relates to the numerical example, is very helpful in specifying the prescribed compensator dimensions which, as argued in the previous section, should preferably satisfy (26). The first row of table 1 contains the sampling instants and final time. The second row specifies the maximum value of the dimension of the compensator state according to (16), (26 *a*) and (26 *c*). The third and fourth row visualize the restrictions imposed on the change of the dimension of the compensator state as described by (26 *b*). The third row specifies the maximum increase in the compensator dimension when going from one sampling instant to the next, as indicated by the arrows. These numbers equal l_i, $i = 0, 1, \dots, N-1$, respectively. The final row specifies the maximum increase in the compensator dimension when going from one sampling instant to the previous sampling instant, as indicated by the arrows. These numbers equal m_i, $i = 0, 1, \dots, N-1$, respectively. From table 1, one can deduce that the dimensions of a minimal realization of the optimal full-order compensator, that is the maximal dimensions allowed by table 1, are in accordance with the first row of table 2. Table 2 lists the outcome of the algorithm for all possible compensator dimensions that satisfy (26). These can be deduced from table 1. In our example there are only three distinct possibilities because the dimension of the compensator state is not allowed to be zero, except at $i = N$. Although in table 2 everywhere $n_i^c \leqslant n$ holds, this does not hold in general. To see this, consider the same example but now with one control variable updated at $t_3 = 0.8$, that is $m_3 = 1$, and one output sampled at

Table 1. Table to determine the prescribed compensator dimensions.

0.0		0.2		0.5		0.8		0.9		1.4		1.5		2.1
1		4		2		3		4		4		3		0
	→		→		→		→		→		→		→	
	+0		+1		+2		+0		+1		+2		+0	
	←		←		←		←		←		←		←	
	+1		+0		+2		+1		+0		+0		+1	

Table 2. Dimensions and performance of all minimal reduced order compensators for the example (the compensators themselves are listed in appendix A.

n_i^c, $i = 0, 1, \dots, N$	J_N
1, 1, 2, 2, 1, 1, 1, 0	219.98
1, 1, 1, 2, 1, 1, 1, 0	221.40
1, 1, 1, 1, 1, 1, 1, 0	231.81

$t_5 = 1.4$, that is $l_5 = 1$. Then, after modification of table 1, it follows that 1, 1, 2, 3, 2, 1, 1 and 0 are the maximal dimensions of the compensator state, that satisfy (26).

Finally, if at all sampling instants all controls are updated and all outputs are sampled, the minimum costs, obtained with a minimal realization of the optimal full-order compensator, computed from our algorithm, equal 34.428. The reduction in the costs is due to the increased output information and control possibilities. These costs were also computed with the algorithm presented by Van Willigenburg (1993), which applies to synchronous sampling. The costs are identical up to the mentioned decimals.

7. Conclusions

The development of digital control system design procedures for asynchronous and aperiodically sampled systems circumvents the demand for frequent, synchronous and periodic sampling. This is of significant practical importance. The synthesis of finite-horizon digital optimal reduced-order compensators was presented in the case of asynchrounous and aperiodic sampling. Among others, the synthesis applies to situations where a digital optimal compensator is designed to control a nonlinear continuous-time system, which is sampled in an asynchronous manner about a possibly optimal state-trajectory (Athans 1971, Van Willigenburg 1991). In this case the compensator design is based on the linearized dynamics about the trajectory which constitute a continuous time-varying linear system. The digital optimal full-order design (Van Willigenburg and De Koning 1995) that possessed the certainty equivalence property was extended and modified to deal with our problem, which does not posses the certainty equivalence property. Furthermore the relations between the sampling scheme and the dimensions of a minimal compensator, which vary over time, were described explicitly. Based on these relations, a table to guide the choice of the sampling scheme and the prescribed compensator dimensions was proposed. The use of this table shows that the choice of both is not straightforward. The synthesis of optimal reduced-order compensators and the choice of the sampling scheme and the prescribed compensator dimensions was illustrated with a numerical example.

The synthesis of the digital optimal full-order compensator (Van Willigenburg and De Koning 1995), could easily be adapted to situations where the available information to compute the control updates at the current time consists of all measurements up to *and including* the current time. In our case we cannot simply replace the Kalman estimator by the Kalman filter because the certainty equivalence property is lost, so the control and estimation are coupled. Because of this, it is also not easy to incorporate the situation were measurements, performed at some sampling instant, are not yet available at the next sampling instant, e.g. due to analyses that is involved.

The choice of sampling schemes is often performed in an *ad hoc* manner or is based on rules of thumb. The synthesis presented in this paper allows the engineer to compute the influence of the choice of the sampling scheme and also of the compensator dimensions on the compensator performance. This enables selection of the sampling scheme and the prescribed compensator dimensions in an appropriate manner. Finally observe from recent results of Van Willigenburg and De Koning (1998, 1999) that the approach presented in this paper also applies to linear time-varying continuous-time system with white stochastic parameters. Plant parameters themselves may be white. They may also be assumed to be white, to design a robust compensator.

Appendix A: Optimal compensators for the example

In this appendix the three optimal compensators computed by the algorithm, the dimensions and costs of which are mentioned in table 2, are listed. Note that these compensators are unique up to basis transformations of the compensator state space at each discrete time instant $i = 0, 1, \ldots, N$, as described by Van Willigenburg and De Koning (1998).

A.1. *Compensator 1, a minimal realization of the optimal full-order compensator*

$$n_i^c, \quad i = 0, 1, \ldots, N: 1, 1, 2, 2, 1, 1, 1, 0,$$

$$\hat{x}_0 = 1.4142,$$

$$F_0 = -2.9575, \quad K_0 = [\ ], \quad L_0 = -2.5758,$$

$$F_1 = \begin{bmatrix} -1.0631 \\ 0.0462 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0.3347 \\ 0.3889 \end{bmatrix}, \quad L_1 = [\ ],$$

$$F_2 = \begin{bmatrix} -2.4613 & -1.1795 \\ -1.5739 & -0.8958 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.2334 & -0.0677 \\ 1.1471 & -0.5482 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -2.2888 & -1.5256 \\ 4.2974 & 0.7565 \end{bmatrix},$$

$$F_3 = [-0.0803 \quad 1.8175], \quad K_3 = [\ ],$$

$$L_3 = [0.1543 \quad 1.1188],$$

$$F_4 = 1.0684, \quad K_4 = 0.0116, \quad L_4 = [\ ],$$

$$F_5 = 0.8210, \quad K_5 = [-0.2819 \quad 0.3505], \quad L_5 = [\ ],$$

$$F_6 = [\ ], \quad K_6 = [\ ], \quad L_6 = 1.6985.$$

A.2. *Compensator 2*

$$n_i^c, \quad i = 0, 1, \ldots, N: 1, 1, 1, 2, 1, 1, 1, 0$$

$$\hat{x}_0 = 1.4142,$$

$$F_0 = 2.3106, \quad K_0 = [\ ], \quad L_0 = -1.7019,$$

$$F_1 = 1.3515, \quad K_1 = 0.3983, \quad L_1 = [\ ],$$

$$F_2 = \begin{bmatrix} -2.2953 \\ -1.5190 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.3552 & -0.1046 \\ 1.0789 & -0.6015 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -2.2472 \\ 3.8616 \end{bmatrix},$$

$$F_3 = [-0.0795 \quad 1.8190], \quad K_3 = [\ ],$$

$$L_3 = [0.1449 \quad 1.1364],$$

$$F_4 = 1.0576, \quad K_4 = -0.0238, \quad L_4 = [\ ],$$

$$F_5 = -0.7812, \quad K_5 = [0.2733 \quad -0.3459], \quad L_5 = [\ ],$$

$$F_6 = [\ ], \quad K_6 = [\ ], \quad L_6 = -1.7946.$$

A.3. *Compensator 3*

$$n_i^c, \quad i = 0, 1, \ldots, N: \quad 1, 1, 1, 2, 1, 1, 1, 0,$$

$$\hat{x}_0 = 1.4142,$$

$$F_0 = 2.6762, \quad K_0 = [\ ], \quad L_0 = -2.2187,$$

$$F_1 = -1.1696, \quad K_1 = 0.3705, \quad L_1 = [\ ],$$

$$F_2 = -3.3880, \quad K_2 = [1.7684 \quad -1.0885],$$

$$L_2 = [-2.1700 \quad 4.0910],$$

$$F_3 = 1.1630, \quad K_3 = [\;], \quad L_3 = 0.6197,$$

$$F_4 = 0.5841, \quad K_4 = -0.2342, \quad L_4 = [\;],$$

$$F_5 = 0.9534, \quad K_5 = [-0.4467 \quad 0.4896], \quad L_5 = [\;],$$

$$F_6 = [\;], \quad K_6 = [\;], \quad L_6 = 1.6913.$$

References

ATHANS, M., 1971, The role and use of the stochastic linear quadratic gaussian problem in control system design. *IEEE Transactions on Automatic Control*, **AC-16,** 529–552.

DE KONING, W. L., 1980, Equivalent discrete-time optimal control problem for randomly sampled digital control systems. *International Journal of Systems Science*, **11,** 841–855.

DORATO, P., and LEVIS, A. H., 1971, Optimal linear regulators: The discrete-time case. *IEEE Transactions on Automatic Control*, **AC-16,** 613–620.

HALYO, N., and CAGLAYAN, A. K., 1976, A seperation theorem for the stochastic sampled-data LQG problem. *International Journal of Control*, **23,** 237–244.

KALMAN, R. E., and BERTRAM, J. E., 1959, A unified approach to the theory of sampling systems. *Journal of the Franklin Institute*, **67,** 405–436.

LEVIS, A. H., SCHLUETER, R. A., and ATHANS, M., 1971, On the behaviour of optimal linear sampled-data regulators. *International Journal of Control*, **13,** 343–361.

TIEDEMANN, A. R., and DE KONING, W. L., 1984, The equivalent discrete-time optimal control problem for continuous-time systems with stochastic parameters. *International Journal of Control*, **40,** 449–466.

VAN WILLIGENBURG, L. G., 1991, Digital optimal control of rigid manipulators. PhD Thesis, Delft University Press, Delft, The Netherlands; 1993, Computation of the digital LQG regulator and tracker for time-varying systems. *Optimal Control Applications and Methods*, **13,** 289–300; 1995, Digital optimal control and LQG compensation of asynchronous and aperiodically sampled non-linear systems. *Proceedings of the Third European Control Conference*, Rome, Italy, 5–8 September, 1995, Vol. 1, pp. 496–500.

VAN WILLIGENBURG, L. G., and DE KONING W. L., 1995, Derivation and computation of the digital LQG regulator and tracker in the case of asynchronous and aperiodic sampling. *C-TAT*, **10,** 1–13; 1998, Minimal and non-minimal fixed-order compensators for time-varying discrete-time systems. MRS Report 98-5, Wageningen University, Systems and Control Group, Wageningen, The Netherlands; 1999, Optimal reduced-order compensators for time-varying discrete-time systems with deterministic and white parameters. *Automatica*, **35,** 129-138; 2000, The equivalent discrete-time optimal control problem for time-varying continuous-time systems with white stochastic parameters. *International Journal of Systems Science*, **31,** 479–487.

YOUSUFF, A., and SKELTON, R. E., 1984, A note on balanced controller reduction. *IEEE Transactions on Automatic Control*, **29,** 254–257.