Including Errata

International Journal of Systems Science, 2000, volume 31, number 4, pages 479–487

The equivalent discrete-time optimal control problem for time-varying continuous-time systems with white stochastic parameters

L. G. VAN WILLIGENBURG† and W. L. DE KONING‡

The transformation into discrete-time equivalents of digital optimal control problems, involving continuous-time linear systems with white stochastic parameters, and quadratic integral criteria, is considered. The system parameters have time-varying statistics. The observations available at the sampling instants are in general nonlinear and corrupted by discrete-time noise. The equivalent discrete-time system has white stochastic parameters. Expressions are derived for the first and second moment of these parameters and for the parameters of the equivalent discrete-time sum criterion, which are explicit in the parameters and statistics of the original digital optimal control problem. A numerical algorithm to compute these expressions is presented. For each sampling interval, the algorithm computes the expressions recursively, forward in time, using successive equidistant evaluations of the matrices which determine the original digital optimal control problem. The algorithm is illustrated with three examples. If the observations at the sampling instants are linear and corrupted by multiplicative and/or additive discrete-time white noise, then, using recent results, full and reduced-order controllers that solve the equivalent discrete-time optimal control problem can be computed.

1. Introduction

In order to design and study digital control systems, that is continuous-time systems controlled by digital computers, it is convenient to convert a digital control problem into an equivalent discrete-time control problem (Levis *et al.* 1971, Halyo and Caglayan 1976, De Koning 1980). The continuous-time systems considered in this paper have *white stochastic parameters* with *time-varying* statistics. In addition the systems may be corrupted by additive white noise. Plant uncertainty is often represented by additive noise only. If the parameter variations are large, then the representation of uncertainties by multiplicative noise or more generally, by stochastic parameters is more realistic. Among others, continuous-time systems with stochastic parameters arise in the field of chemical reactors (Lignon and Amudson 1981a,b, Wagenaar and De Koning 1989).

The design of compensators for nonlinear systems that have to track (optimal) trajectories, is often based on the linearized dynamics around the trajectory (Athans 1971). These dynamics constitute a *time-varying* continuous-time system and hold only on the trajectory itself. Stochastic parameters may be used to represent the uncertainty in the linearized dynamics if, during control, the state deviates from the trajectory. In this case the continuous-time system has stochastic parameters with *time-varying* statistics. The criterion is a quadratic integral criterion which explicitly considers the intersample behaviour of the (linearized) continuous-time system. The latter circumvents the demand to choose a sufficiently small sampling time which, in some applications such as robotics, may result in lack of computation time.

The transformation of digital control problems into equivalent discrete-time control problems has been considered for several types of digital optimal control problems (Levis *et al.* 1971, Halyo and Caglayan 1976, De Koning 1980, Tiedemann and De Koning 1984,

Received 22 September 1998. Revised 12 April 1999. Accepted 22 April 1999.

† Systems and Control Group, Wageningen University, Bomenweg 4, 6703 HD Wageningen, The Netherlands. Tel: 31317482841; Fax: 31317484819; e-mail: gerard.vanwilligenburg@user.aenf.wau.nl

‡ Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2628CD Delft, The Netherlands. Tel. 31152784989; Fax: 31152787209; e-mail: w.l.dekoning@math.tudelft.nl

International Journal of Systems Science ISSN 0020–7721 print/ISSN 1464–5319 online © 2000 Taylor & Francis Ltd
http://www.tandf.co.uk/journals/tf/00207721.html

Van Willigenburg, 1992, 1995, Van Willigenburg and De Koning 1992, 1995). Most digital control problems are concerned with time-invariant linear systems, having deterministic parameters, quadratic integral criteria, and deterministic sampling. Notable exceptions are the studies of Halyo and Caglayan (1976), Van Willigenburg (1992) and Van Willigenburg and De Koning (1992), who consider time-varying linear systems, Van Willigenburg (1995) who considers general nonlinear systems and integral criteria, De Koning (1980), who considers stochastic sampling and Tiedemann and De Koning (1984), who consider linear systems having white stochastic parameters with *constant* statistics. Tiedemann and De Koning (1984) focused mainly on the computation and interpretation of the equivalent discrete-time system. Two alternative computations of the equivalent discrete-time system were considered. The computation of the equivalent discrete-time sum criterion was not fully addressed. As demonstrated by Van Willigenburg (1992) the time-varying nature of the linear system complicates the numerical computation of the equivalent discrete-time optimal control problem.

In this paper it is shown how, using the results of Van Willigenburg (1992), the results of Tiedemann and De Koning (1984) can be extended to systems with time-varying statistics and integral criteria with time-varying cost matrices. A method to compute the equivalent discrete-time system and sum criterion is presented in detail. The equivalent discrete-time system has white stochastic parameters with time-varying statistics and the equivalent discrete-time sum criterion is quadratic and has time-varying cost matrices. Therefore, if the observations at the sampling instants are linear and corrupted by multiplicative and/or additive discrete-time white noise, finite-horizon full and reduced-order compensators, which solve the equivalent discrete-time optimal control problem, can be computed using the results recently presented by Van Willigenburg and De Koning (1998, 1999).

2. Digital optimal control problem

In this section the digital optimal control problem for time-varying linear continuous-time systems with white stochastic parameters and quadratic integral criteria is stated. Since our interest is mainly the design and computation of digital optimal full and reduced-order controllers, the observations at the sampling instants are assumed to be *linear*. From comparable results, Tiedemann and De Koning (1984) observed that, as far as the transformation into an equivalent discrete-time optimal control problem is concerned, they may as well be *nonlinear*.

Consider a digital control system consisting of a continuous-time system connected to a digital computer by means of a sample and hold circuit at the input and a sampler at the output. The sampling instants of both samplers are $t_0, t_1, \ldots, t_N$. At time t_i the computer sends the control $u(t_i)$ and receives the observation $y(t_i)$. Within the interval $[t_i, t_{i+1})$ the next control $u(t_{i+1})$ has to be calculated on the basis of observations $y(t_0), \ldots, y(t_i)$ and the controls $u(t_0), \ldots, u(t_i)$.

The continuous-time system, the sample and hold operation and the observations at the sampling instants are described by,

$$\mathrm{d}x(t) = \mathrm{d}A(t)x(t) + \mathrm{d}B(t)u(t) + \mathrm{d}\beta(t), \quad \times t \in [t_0, t_N], \tag{1 a}$$

$$u(t) = u(t_i), \quad t \in [t_i, t_{i+1}), \quad i = 0, 1, 2, \ldots, N-1, \tag{1 b}$$

$$y(t_i) = C_i x(t_i) + w_i, \quad i = 0, 1, 2, \ldots, N-1, \tag{1 c}$$

respectively where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $y(t_i) \in R^r$ is the output and $w_i \in R^r$ is white observation noise with *time-varying statistics*, characterized by

$$E\{w_i\} = 0, \quad E\{(w_i - E\{w_i\})(w_i - E\{w_i\})^{\mathrm{T}}\} = W_i. \tag{1 d}$$

The matrices $A(t)$ and $B(t)$ are of appropriate dimension with entries being white stochastic variables at every time t. The processes $\{A(t),\ t_0 \le t \le t_N\}$, $\{B(t),\ t_0 \le t \le t_N\}$ have independent increments and known time-varying first and second moments. The first moments of these processes are defined by $\bar{A}(t)$, $\bar{B}(t)$ where the bar denotes expectation. Let $\tilde{A}(t) = A(t) - \bar{A}(t)$ and $\tilde{B}(t) = B(t) - \bar{B}(t)$ and consider the decompositions,

$$A(t) = \bar{A}(t) + \tilde{A}(t), \quad \mathrm{d}A(t) = \bar{A}(t)\,\mathrm{d}t + \mathrm{d}\tilde{A}(t), \tag{2 a}$$

$$B(t) = \bar{B}(t) + \tilde{B}(t), \quad \mathrm{d}B(t) = \bar{B}(t)\,\mathrm{d}t + \mathrm{d}\tilde{B}(t), \tag{2 b}$$

Then the second moments of the processes $\{A(t),\ t_0 \le t \le t_N\}$, $\{B(t),\ t_0 \le t \le t_N\}$ are defined by,

$$E\{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{A}(t)\} = V^{AA}(t)\,\mathrm{d}t, \tag{2 c}$$

$$E\{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{B}(t)\} = V^{AB}(t)\,\mathrm{d}t, \tag{2 d}$$

$$E\ \mathrm{d}\tilde{B}(t) \otimes \mathrm{d}\tilde{A}(t)\} = V^{BA}(t)\,\mathrm{d}t, \tag{2 e}$$

$$E\{\mathrm{d}\tilde{B}(t) \otimes \mathrm{d}\tilde{B}(t)\} = V^{BB}(t)\,\mathrm{d}t, \tag{2 f}$$

where $\otimes$ denotes the Kronecker product (Bellman 1970). Note that our notation of a continuous-time system with white stochastic parameters is quite different from the usual one of sums of matrices multiplied by scalar stochastic processes. However, this notation is more general and more concise than the usual notation. Observe that the processes $\{A(t),\, t_0 \le t \le t_N\}$, $\{B(t),\, t_0 \le t \le t_N\}$ are not necessarily independent. The process $\{\beta(t),\, t_0 \le t \le t_N\}$ is a real zero-mean independent increment's process of appropriate dimensions with known *time-varying* first and second moment:

$$E\{\beta(t)\} = 0, \quad E\{\mathrm{d}\beta(t)\,\mathrm{d}\beta^{\mathrm{T}}(t)\} = V^{\beta}(t), \quad V^{\beta}(t) \ge 0. \tag{3a}$$

The initial condition of the state $x(t_0)$ is x_0 with known mean and covariance:

$$E\{x(t_0)\} = \bar{x}_0, \quad E\{(x(t_0) - \bar{x}_0)(x(t_0) - \bar{x}_0)^{\mathrm{T}}\} = V^{x_0}. \tag{3b}$$

The matrices C_i are real, random and of appropriate dimension. Let $\tilde{C}_i = C_i - \overline{C_i}$ then the process $\{C_i,\, i = 0, 1, \ldots, N\}$ has known time-varying first and second moments given by $\overline{C_i}$ and

$$\overline{C_i \otimes C_i} = \overline{C_i} \otimes \overline{C_i} + \overline{\tilde{C}_i \otimes \tilde{C}_i} = \overline{C_i} \otimes \overline{C_i} + V_i^{CC} \tag{4}$$

respectively. The initial condition $x(t_0)$ and the processes $\{\beta(t),\, t_0 \le t \le t_N\}$, $\{A(t),\, t_0 \le t \le t_N\}$, $\{B(t),\, t_0 \le t \le t_N\}$, $\{w_i,\, i = 0, 1, \ldots, N\}$ and $\{C_i,\, i = 0, 1, \ldots, N\}$ are mutually independent, except that $\{A(t),\, t_0 \le t \le t_N\}$ and $\{B(t),\, t_0 \le t \le t_N\}$ may be dependent.

Consider the integral criterion

$$\left.\begin{aligned} J &= E\{x^{\mathrm{T}}(t_N) Z x(t_N)\} \\ &\quad + E\left\{\int_{t_0}^{t_N} [x^{\mathrm{T}}(t) Q(t) x(t) + u^{\mathrm{T}}(t) R(t) u(t)]\, \mathrm{d}t\right\}, \\ Z &\ge 0, \quad Q(t) \ge 0, \quad R(t) \ge 0 \end{aligned}\right\} \tag{5}$$

Let Y_i denote the observation sequence $\{y(t_0), \ldots, y(t_i)\}$ and U_i the control sequence $\{u(t_0), \ldots, u(t_i)\}$. Note that the control $u(t_N)$ does not influence the value of the criterion (5). Assume that $u(t_i)$ is a deterministic function of the measurements and controls preceding t_i, that is $u(t_i, Y_{i-1}, U_{i-1})$, $i = 0, 1, 2, \ldots, N-1$. For $i = 0$, U_{-1} and Y_{-1} are both empty sets. Then the digital optimal control problem is to determine the functions (the control law) $u^*(t_i, Y_{i-1}, U_{i-1})$, $i = 0, 1, 2, \ldots, N-1$ that minimize J and to find the minimal value J^*. The difference between this optimal control problem and the time-varying finite horizon digital linear quadratic Gaussian (LQG) problem is that the parameters of the plant, instead of deterministic time functions, are white stochastic processes.

3. Equivalent discrete-time system

In this section the continuous-time system is transformed to an equivalent discrete-time system with white stochastic parameters. Explicit expressions are derived for the first and second moments of these parameters. The zero and identity matrix are denoted by θ and I.

The sampling process $\{t_i,\, i = 0, 1, 2, \ldots, N\}$ is assumed to be deterministic and known and the sampling periods $T_i = t_{i+1} - t_i$, $i = 0, 1, 2, \ldots, N-1$ are positive, that is $T_i > 0$. The solution of the system (1 a) with control (1 b) is given by

$$x(t) = \Phi(t, t_i) x_i + \Gamma(t, t_i) u_i + v(t, t_i), \tag{6}$$

where $t \in [t_i, t_{i+1})$, $i = 0, 1, 2, \ldots, N-1$, $x_i = x(t_i)$, $u_i = u(t_i)$ and Φ is the transition matrix corresponding to the homogeneous stochastic system

$$\mathrm{d}x(t) = \mathrm{d}A(t) x(t), \quad t \in [t_0, t_N], \tag{7}$$

and defined as the solution of the homogeneous stochastic matrix differential equation,

$$\mathrm{d}\Phi(t, s) = \mathrm{d}A(t)\, \Phi(t, s), \quad \Phi(s, s) = I, \quad t, s \in [t_0, t_N]. \tag{8}$$

Furthermore,

$$v(t, t_i) = \int_{t_i}^{t} \Phi(t, s)\, \mathrm{d}\beta(s) \tag{9}$$

If $t = t_{i+1}$ then (6) may be written as,

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i, \quad i = 1, 2, \ldots, N-1, \tag{10}$$

where $x_{i+1} = x(t_{i+1})$, $u_i = u(t_i)$, $\Phi_i = \Phi(t_{i+1}, t_i)$, $\Gamma_i = \Gamma(t_{i+1}, t_i)$ and $v_i = v(t_{i+1}, t_i)$. The system (10) is the equivalent discrete-time system. Furthermore, similar to the work of Tiedemann and De Koning (1984),

$$E\{v(t, t_i)\} = 0, \tag{11a}$$

$$E\{v(t, t_i) v^{\mathrm{T}}(t, t_i)\} = \int_{t_i}^{t} \overline{\Phi(t, s) V^{\beta}(s) \Phi^{\mathrm{T}}(t, s)}\, \mathrm{d}s = V^{\beta}(t, t_i), \tag{11b}$$

and, with $t = t_{i+1}$,

$$E\{v_i\} = 0, \tag{12a}$$

$$E\{v_i v_i^{\mathrm{T}}\} = V_i^{\beta}, \tag{12b}$$

where $V_i^{\beta} = V^{\beta}(t_{i+1}, t_i)$. From (12) note that $\{v_i,\, i = 0, 1, 2, \ldots, N-1\}$ is discrete-time white noise with covariance V_i^{β}.

In the remaining part of this section a matrix K with dimensions $n \times m$ is denoted by K_{nm}. The dimension indices are deleted if no confusion is possible. To derive explicit expressions for the first and second moments of Φ_i and Γ_i define the following matrices:

$$P_{n_1n_1}(t) = \bar{A}(t) \oplus \bar{A}(t) + V^{AA}(t), \qquad (13a)$$

$$W^1_{n_1n_2}(t) = I_n \otimes \bar{B}(t) + V^{AB}(t), \qquad (13b)$$

$$W^2_{n_1n_2}(t) = \bar{B}(t) \otimes I_n + V^{BA}(t), \qquad (13c)$$

$$X^1_{n_2n_2}(t) = \bar{A}(t) \otimes I_m, \qquad (13d)$$

$$X^2_{n_2n_2}(t) = I_m \otimes \bar{A}(t), \qquad (13e)$$

$$L^1_{n_2n_3}(t) = \bar{B}(t) \otimes I_m, \qquad (13f)$$

$$L^2_{n_2n_3}(t) = I_m \otimes \bar{B}(t), \qquad (13g)$$

$$Z_{n_1n_3}(t) = V^{BB}(t), \qquad (13h)$$

where $\oplus$ in (13 a) is the Kronecker sum (Bellman 1970), that is

$$\bar{A}(t) \oplus \bar{A}(t) = \bar{A}(t) \otimes I_n + I_n \otimes \bar{A}(t). \qquad (14a)$$

and where

$$n_1 = nn, \quad n_2 = nm, \quad n_3 = mm. \qquad (14b)$$

Define

$$F(t) = \begin{bmatrix} P_{n_1n_1}(t) & W^1_{n_1n_2}(t) & W^2_{n_1n_2}(t) & Z_{n_1n_3}(t) \\ \theta_{n_2n_1} & X^1_{n_2n_2}(t) & \theta_{n_2n_2} & L^1_{n_2n_3}(t) \\ \theta_{n_2n_1} & \theta_{n_2n_2} & X^2_{n_2n_2}(t) & L^2_{n_2n_3}(t) \\ \theta_{n_3n_1} & \theta_{n_3n_2} & \theta_{n_3n_2} & \theta_{n_3n_3} \end{bmatrix} \qquad (15)$$

and let Φ_F denote the transition matrix of the homogeneous time-varying deterministic system,

$$\dot{x}_F(t) = F(t)x_F(t), \quad t \in [t_0, t_N]. \qquad (16)$$

Then, according to Graham (1981), the first and second moments of Φ_i and Γ_i, as in the work of Tiedemann and De Koning (1984), follow from the equality, (see equation 17), where $\overline{(\Phi \otimes \Gamma)}_{n_1n_2}(t_{i+1}, t_i)$ for example denotes $\overline{\Phi(t_{i+1}, t_i) \otimes \Gamma(t_{i+1}, t_i)}_{n_1n_2}$. Since $F(t)$ in (15), and therefore $\Phi_F(t_{i+1}, t_i)$, are fully determined by the known first and second moments of the original continuous-time system (1) the computation of the first and second moments of Φ_i and Γ_i comes down to the computation of the transition matrix of a homogeneous deterministic time-varying system. From Van Willigenburg (1992) this computation can be performed recursively forward in time.

$$\begin{bmatrix} \overline{(\Phi \otimes \Phi)}_{n_1n_1}(t_{i+1}, t_i) & \overline{(\Phi \otimes \Gamma)}_{n_1n_2}(t_{i+1}, t_i) & \overline{(\Gamma \otimes \Phi)}_{n_1n_2}(t_{i+1}, t_i) & \overline{(\Gamma \otimes \Gamma)}_{n_1n_3}(t_{i+1}, t_i) \\ \theta_{n_2n_1} & (\overline{\Phi}(t_{i+1}, t_i) \otimes I_m)_{n_2n_2} & \theta_{n_2n_2} & (\overline{\Gamma}(t_{i+1}, t_i) \otimes I_m)_{n_2n_3} \\ \theta_{n_2n_1} & \theta_{n_2n_2} & (I_m \otimes \overline{\Phi}(t_{i+1}, t_i))_{n_2n_2} & (I_m \otimes \bar{\Gamma}(t_{i+1}, t_i))_{n_2n_3} \\ \theta_{n_3n_1} & \theta_{n_3n_2} & \theta_{n_3n_2} & \theta_{n_3n_3} \end{bmatrix} = \Phi_F(t_{i+1}, t_i) \qquad (17)$$

4. The equivalent discrete-time optimal control problem and its numerical computation

In this section the continuous-time integral criterion of the digital optimal control problem is transformed to an equivalent discrete-time sum criterion. Subsequently the equivalent discrete-time optimal control problem (EDOCP) is stated. Finally the numerical computation of the EDOCP is treated.

Define, similar to the work of Tiedemann and De Koning (1984),

$$Q(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \overline{\Phi^{\mathrm{T}}(t, t_i)Q(t)\Phi^{\mathrm{T}}(t, t_i)}\,\mathrm{d}t, \quad Q(t_{i+1}, t_i) \geq 0, \qquad (18a)$$

$$M(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \overline{\Phi^{\mathrm{T}}(t, t_i)Q(t)\Gamma(t, t_i)}\,\mathrm{d}t, \qquad (18b)$$

$$R(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} [R(t) + \overline{\Gamma^{\mathrm{T}}(t, t_i)Q(t)\Gamma(t, t_i)}]\,\mathrm{d}t, \qquad (18c)$$

$$R(t_{i+1}, t_i) \geqslant 0$$

$$\eta(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} V^{\beta}(t, t_i)Q(t)\,\mathrm{d}t. \qquad (18d)$$

Then the equivalent discrete-time sum criterion is given by

$$J = E\{x_N^{\mathrm{T}}Zx_N\} + E\left\{\sum_{i=0}^{N-1} x_i^T Q_i x_i + 2x_i^T M_i u_i + u_i^T R_i u_i\right\} + \sum_{i=0}^{N-1} \eta_i, \qquad (19)$$

where $Q_i = Q(t_{i+1}, t_i)$, $M_i = M(t_{i+1}, t_i)$, $R_i = R(t_{i+1}, t_i)$ and $\eta_i = \eta(t_{i+1}, t_i)$. Although the integral criterion (5) does not contain a cross-product, similar to the work of Levis *et al.* (1971), Halyo and Caglayan (1976), and Van Willigenburg and De Koning (1992), in (19) a cross-product $2x_i^{\mathrm{T}}M_iu_i$ appears. This is due to the piecewise constant constraint (1 b) on the control. The last term in (19) is not a function of $\{u_0, \ldots, u_{N-1}\}$ and therefore may be dropped in the minimization. Denote $y(t_i)$ by y_i and let Y_i denote the observation sequence $\{y_0, \dots, y_{i-1}\}$ and U_i the control sequence $\{u_0, \dots, u_{i-1}\}$. Assume that u_i is a deterministic function of Y_{i-1}, U_{i-1}, that is $u_i(Y_{i-1}, U_{i-1})$, $i = 0, 1, 2, \dots, N-1$. Then the following control problem is obtained. Given the system (10) find the deterministic functions (control law) $u_i^*(Y_{i-1}, U_{i-1})$, $i = 0, 1, 2, \dots, N-1$ for the system (10) that minimize J, given by (19), and find the minimum value J^*. This control problem has exactly the same solution as the original digital optimal control problem and therefore is called the EDOCP. The difference between this optimal control problem and the time-varying finite-horizon discrete-time LQG problem is that the parameters of the plant, instead of deterministic time functions, are white stochastic processes.

To compute numerically the EDOCP from the original digital optimal control problem the integrals (11 b), (18) and the right hand side of (17) have to be evaluated. As in the work of Van Willigenburg (1992), which treats the numerical computation of the EDOCP for digital LQG problems where the system has deterministic parameters, these evaluations can be performed recursively forward in time, using successive equidistant evaluations of the matrices of the original digital LQG problem. The method proposed by Van Willigenburg (1992) uses a piecewise constant approximation of the deterministic time-varying system, the scaling and squaring method in combination with a second-order Taylor expansion to compute the matrix exponential and the trapezoidal numerical integration scheme. The error using the trapezoidal integration scheme is of equal order to that related to scaling and squaring. Furthermore the combination of these computational schemes allows for the recursive forward-in-time computation. Therefore a more sophisticated integration scheme is not used in this case.

In the following the computation of the matrices of the EDOCP over one sampling period $[t_i, t_{i+1})$ will be considered. Within this time interval S_i time steps of length Δt are taken where S_i, which determines the accuracy, is a sufficiently large integer.

First note that in the integrals (11 b) and (18) terms like $\overline{\Phi X \Gamma^{\mathrm{T}}}$ and $\overline{\Phi^{\mathrm{T}} X \Gamma}$ for some matrix X occur. They may be written as,

$$\overline{\Phi X \Gamma^{\mathrm{T}}} = \mathrm{st}^{-1}\,[(\overline{\Phi \otimes \Gamma})\,\mathrm{st}\,(X)], \qquad (20\,a)$$

$$\overline{\Phi^{\mathrm{T}} X \Gamma} = \mathrm{st}^{-1}\,[(\overline{\Gamma \otimes \Phi})^{\mathrm{T}}\,\mathrm{st}\,(X)] \qquad (20\,b)$$

where st denotes the stack and st^{-1} the inverse stack operator. The stack operator changes every matrix into a column vector in which the columns of the matrix are stacked in their order of appearance, with the first column on top. The inverse stack operator, denoted by st^{-1}, recovers the original matrix from this column vector (Bellman 1970). To do this it must be given the column length, that is the number of rows, of the original matrix.

The trapezoidal integration rule is based on the following approximation of the integral of a general matrix function $X(t)$:

$$\int_{t_i}^{t_i + L\Delta t} X(t)\,\mathrm{d}t \approx \frac{\Delta t}{2}\left(X(t_i) + X(t_i + L\,\Delta t) + 2\sum_{k=1}^{L-1} X(t_i + k\,\Delta t)\right), \quad L = 2, 3, \dots, S_i, \qquad (21\,a)$$

and, for $L = 1$,

$$\int_{t_i}^{t_i + \Delta t} X(t)\,\mathrm{d}t \approx \frac{\Delta t}{2}[X(t_i) + X(t_i + \Delta t)]. \qquad (21\,b)$$

From (21) the integral can be computed recursively forwards in time as follows,

$$\int_{t_i}^{t_i+(L+1)\Delta t} X(t)\,\mathrm{d}t \approx \int_{t_i}^{t_i + L\Delta t} X(t)\,\mathrm{d}t + \frac{\Delta t}{2} \times (X(t_i + L\,\Delta t) + X[t_i + (L+1)\Delta t]) \quad L = 2, 3, \dots, S_{i-1}. \qquad (22)$$

Consider the following piecewise constant approximation $F'(t)$ of $F(t)$ in (15):

$$F'(t) = \tfrac{1}{2}\{F(t_i + L\,\Delta t) + F[t_i + (L+1)\,\Delta t]\}, \quad t \in [t_i + L\,\Delta t, t_i + (L+1)\,\Delta t), \quad L = 0, 1, 2, \dots, S_i - 1. \qquad (23)$$

Then the transition matrix of the homogeneous time-varying deterministic system,

$$\dot{x}_{F'}(t) = F'(t) x_{F'}(t), \quad t \in [t_i, t_{i+1}), \qquad (24)$$

which approximates the system (16) equals,

$$\Phi_{F'}(t_i, t_i + L\,\Delta t) = \prod_{k=1}^{L-1} \exp\,([F'(t_i + k\,\Delta t)\,\Delta t], \quad L = 1, 2, \dots, S_i. \qquad (25)$$

If Δt is sufficiently small, the following second-order Taylor expansion may be used to compute (25):

$$\exp\,[F'(t_i + k\,\Delta t)\,\Delta t] \approx I + F'(t_i + k\Delta t)\Delta t + 0.5 F'^2(t_i + k\,\Delta t)\,\Delta t^2. \qquad (26)$$

As can be seen from equations (15), (17), (22), (23), (25) and (26) the first and second moments of Φ_i, and Γ_i and the integrals (18) can be computed recursively forwards in time. The computation of the integral in (11 b) deserves further attention since this integral is not of the form (21). In the case of (11 b) the trapezoidal integration rule results in the following approximation,

$$\begin{aligned} V^{\beta}(t_i + L\,\Delta t, t_i) \approx \frac{\Delta t}{2}\Big(& \mathrm{st}^{-1}\{\overline{(\Phi \otimes \Phi)} \\ & \times (t_i + L\,\Delta t, t_i)\,\mathrm{st}\,[V^{\beta}(t_i)]\} \\ & + V^{\beta}(t_i + L\,\Delta t) \\ & + 2\sum_{k=1}^{L-1} \mathrm{st}^{-1}\,\{\overline{(\Phi \otimes \Phi)} \\ & \times (t_i + L\,\Delta t, t_i + k\,\Delta t) \\ & \times \mathrm{st}\,[V^{\beta}(t_i + k\Delta t)]\}\Big), \\ & \qquad L = 2, 3, \ldots, S_i, \end{aligned} \qquad (27\,a)$$

and, for $L = 1$,

$$\begin{aligned} V^{\beta}(t_i + \Delta t, t_i) \approx \frac{\Delta t}{2}(&\mathrm{st}^{-1}\{\overline{(\Phi \otimes \Phi)}(t_i + \Delta t, t_i)\,\mathrm{st}\,[V^{\beta}(t_i)]\} \\ & + V^{\beta}(t_i + \Delta t)). \end{aligned} \qquad (27\,b)$$

Consider the following well known property of transition matrices,

$$\begin{aligned} &\Phi(t_i + (L+1)\Delta t, t_i) \\ &\quad = \Phi(t_i + (L+1)\,\Delta t, t_i + L\,\Delta t)\Phi(t_i + L\,\Delta t, t_i). \end{aligned} \qquad (28)$$

The property (28) implies the following property,

$$\begin{aligned} \overline{(\Phi \otimes \Phi)}(t_i + (L+1)\Delta t, t_i) = \overline{(\Phi \otimes \Phi)} & \\ \times (t_i + (L+1)\Delta t, t_i + L\,\Delta t) & \\ \times \overline{(\Phi \otimes \Phi)}(t_i + L\,\Delta t, t_i). & \end{aligned} \qquad (29)$$

Using (27) and (29) the integral (11 b) is computed recursively forward in time as follows,

$$\begin{aligned} V^{\beta}(t_i + (L+1)\,\Delta t, t_i) = \mathrm{st}^{-1}\,\{ & \overline{(\Phi \otimes \Phi)}(t_i + (L+1) \\ & \times \Delta t, t_i + L\,\Delta t)\,\mathrm{st} \\ & \times [V^{\beta}(t_i + L\,\Delta t, t_i)]\} \\ & + \frac{\Delta t}{2}(\mathrm{st}^{-1}\,\{\overline{(\Phi \otimes \Phi)} \\ & \times (t_i + (L+1)\Delta t, t_i + L\,\Delta t)\,\mathrm{st} \\ & \times [V^{\beta}(L\,\Delta t)]\} + V^{\beta}[(L+1)\Delta t]) \end{aligned} \qquad (30)$$

From (17) and (23) observe that $\overline{\Phi \otimes \Phi}(t_i + (L+1)\Delta t, t_i + L\,\Delta t)$ in (30) is approximately the first diagonal block of the matrix $\exp\,[F'(t_i + L\,\Delta t)\,\Delta t]$ which is computed in order to compute (25). Summarizing, the integrals (11 b), and (18) and the first and second moments of Φ_i and Γ_i, and the EDOCP, can be computed recursively forwards in time using successive equidistant evaluations of the matrices which make up the original digital control problem.

5. Numerical examples

In this section, three numerical examples are presented. The first example concerns a *time-invariant* digital control problem and is used to check the results of our algorithm against the results obtained with the algorithm for time-invariant digital control problems (Tiedemann and De Koning 1984). The second example concerns a *time-varying* digital control problem, where the system has *deterministic* parameters. Systems with deterministic parameters are a special case of systems with white stochastic parameters. This example is used to check the computation of the EDOCP against the results obtained with the algorithm presented by Van Willigenburg (1992). Finally the third example deals with a time-varying digital control problem where the system has white stochastic parameters.

In example 1 the following choices of $\mathrm{d}\tilde{A}(t)$ and $\mathrm{d}\tilde{B}(t)$ in (2) have been made,

$$\mathrm{d}\tilde{A}(t) = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.4 \end{bmatrix} \mathrm{d}\gamma(t), \quad \mathrm{d}\tilde{B}(t) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \mathrm{d}\gamma(t). \qquad (31\,a)$$

In (31 a) the process $\{\gamma(t),\ t_0 \le t \le t_N\}$ is a scalar-independent increments process with zero first moment and unity second moment:

$$E\{\gamma(t)\} = 0, \quad E\{\mathrm{d}\gamma(t)\,\mathrm{d}\gamma^{\mathrm{T}}(t)\} = 1. \qquad (31\,b)$$

As a result the stochastic processes $\{A(t),\ t_0 \le t \le t_N\}$, and $\{B(t),\ t_0 \le t \le t_N\}$ are fully correlated. Note that (31) is an example of a description that uses sums of matrices multiplied by scalar stochastic processes. The two matrices in (31) translate into the matrices V^{AA}, V^{AB}, V^{BA} and V^{BB} in example 1. Although in this case (31) might be considered a simpler description, note that the sums of matrices and also the number of scalar stochastic processes that they multiply, can be extended arbitrarily, while the dimensions of V^{AA}, V^{AB}, V^{BA} and V^{BB} remain the same.

Example 1:

$$\bar{A}(t)=\begin{bmatrix}1 & 0.5\\ 0 & 2\end{bmatrix},\quad \bar{B}(t)=\begin{bmatrix}1\\ 2\end{bmatrix},$$

$$V^{AA}(t)=\begin{bmatrix}0.04 & 0.04 & 0.04 & 0.04\\ 0 & 0.08 & 0 & 0.08\\ 0 & 0 & 0.08 & 0.08\\ 0 & 0 & 0 & 0.16\end{bmatrix},$$

$$V^{BB}(t)=\begin{bmatrix}0.04\\ 0.06\\ 0.06\\ 0.09\end{bmatrix},$$

$$V^{AB}(t)=\begin{bmatrix}0.04 & 0.04\\ 0.06 & 0.06\\ 0 & 0.08\\ 0 & 0.12\end{bmatrix},\quad V^{BA}(t)=\begin{bmatrix}0.04 & 0.04\\ 0 & 0.08\\ 0.06 & 0.06\\ 0 & 0.12\end{bmatrix},$$

$$V^{\beta}(t)=\begin{bmatrix}0.08 & 0\\ 0 & 0.08\end{bmatrix},$$

$$\bar{x}_0=\begin{bmatrix}1\\ 1\end{bmatrix},\quad V^{x_0}=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},$$

$$Q(t)=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},\quad R(t)=0.5,\quad Z=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},$$

$$\bar{C}_i=[1\quad 0],\quad V_i^{CC}=[0.1\quad 0\quad 0\quad 0],$$

$$W_i=0.1,\quad i=0,1,2,\ldots,N-1$$

$$t_i=0.05i,\quad i=0,1,2,\ldots,N.$$ □

The EDOCP of this time-invariant digital control problem, computed using the algorithm put forward in the previous section with $S_i=50$, $i=0,1,2,\ldots,N$, is given by,

$$\bar{\Phi}_i=\begin{bmatrix}1.051 & 0.0269\\ 0 & 1.105\end{bmatrix},\quad \bar{\Gamma}_i=\begin{bmatrix}0.0526\\ 0.105\end{bmatrix},$$

$$\overline{\Phi_i\otimes\Phi_i}=\begin{bmatrix}1.107 & 0.0307 & 0.0307 & 0.00325\\ 0 & 1.166 & 0 & 0.348\\ 0 & 0 & 1.166 & 0.348\\ 0 & 0 & & 1.231\end{bmatrix},$$

$$\overline{\Phi_i\otimes\Gamma_i}=\begin{bmatrix}0.0577 & 0.00395\\ 0.114 & 0.00663\\ 0 & 0.0631\\ 0 & 0.124\end{bmatrix},$$

$$\overline{\Gamma_i\otimes\Phi_i}=\begin{bmatrix}0.0577 & 0.00395\\ 0 & 0.0631\\ 0.114 & 0.00663\\ 0 & 0.124\end{bmatrix},$$

$$\overline{\Gamma_i\otimes\Gamma_i}=\begin{bmatrix}0.00531\\ 0.00934\\ 0.00934\\ 0.0168\end{bmatrix}$$

$$Q_i=\begin{bmatrix}0.0526 & 0.000736\\ 0.000736 & 0.111\end{bmatrix},\quad R_i=0.0257,$$

$$M_i=\begin{bmatrix}0.00139\\ 0.00596\end{bmatrix},\quad \eta_i=0.000318$$

$$\bar{C}_i=[1\quad 0]\quad ,V_i^{CC}=[0.1\quad 0\quad 0\quad 0],$$

$$W_i=0.1,\quad i=0,1,2,\ldots,N-1$$

Observe that the outcome of the EDOCP is identical with that presented by Tiedemann and De Koning (1984). From (2 *d*), and (2 *e*) observe that the elements of V^{BA} in example 1 are obtained after rearrangement of the elements of V^{AB}. Similarly the elements of $\overline{\Gamma_i\otimes\Phi_i}$ are obtained after rearrangement of the elements of $\overline{\Phi_i\otimes\Gamma_i}$. Therefore in the following examples only one of the two will be specified.

Example 2: This example is the same as example 1, with the following exceptions. All the matrices in example 1, except for $\bar{x}_0$, V^{x_0} and $\bar{C}_i$, V_i^{CC}, W_i, $i=0,1,2,\ldots,N-1$, are pre-multiplied by $\cos(10\pi t)$. Furthermore V^{AA}, V^{AB}, V^{BA} and V^{BB} are zero matrices, that is the digital control problem is time varying and the system has deterministic parameters. □

The matrices of the EDOCP of example 2, for $i = 0$, are given by

$$\bar{\Phi}_0 = \begin{bmatrix} 1.032 & 0.0167 \\ 0 & 1.066 \end{bmatrix}, \quad \bar{\Gamma}_0 = \begin{bmatrix} 0.0329 \\ 0.0657 \end{bmatrix},$$

$$\overline{\Phi_0 \otimes \Phi_0} = \begin{bmatrix} 1.066 & 0.0172 & 0.0172 & 0.000\,279 \\ 0 & 1.100 & 0 & 0.178 \\ 0 & 0 & 1.166 & 0.178 \\ 0 & 0 & 0 & 1.136 \end{bmatrix},$$

$$\overline{\Phi_0 \otimes \Gamma_0} = \begin{bmatrix} 0.0339 & 0.000\,549 \\ 0.679 & 0.001\,10 \\ 0 & 0.0350 \\ 0 & 0.0701 \end{bmatrix},$$

$$\overline{\Gamma_0 \otimes \Gamma_0} = \begin{bmatrix} 0.001\,08 \\ 0.002\,16 \\ 0.002\,16 \\ 0.004\,32 \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} 0.0329 & 0.000\,267 \\ 0.000\,267 & 0.0679 \end{bmatrix}, \quad R_0 = 0.0160,$$

$$M_0 = \begin{bmatrix} 0.000\,529 \\ 0.002\,17 \end{bmatrix}, \quad \eta_0 = 0.000\,126$$

$$\bar{C}_0 = [1 \quad 0], \quad V_0^{CC} = [0.1 \quad 0 \quad 0 \quad 0]$$

$$W_0 = 0.1.$$

Since the system is deterministic the EDOCP may also be computed using the algorithm described by Van Willigenburg (1992). Taking identical values S_i, that is the same step size Δt for the numerical integration, and realizing that for systems with deterministic parameters,

$$\overline{\Phi_i \otimes \Phi_i} = \bar{\Phi}_i \otimes \bar{\Phi}_i,$$

$$\overline{\Phi_i \otimes \Gamma_i} = \bar{\Phi}_i \otimes \bar{\Gamma}_i,$$

$$\overline{\Gamma_i \otimes \Gamma_i} = \bar{\Gamma}_i \otimes \bar{\Gamma}_i$$

the outcomes of both algorithms are identical.

Example 3: This example is the same as example 2, except that V^{AA}, V^{AB}, V^{BA} and V^{BB} are not zero matrices, but equal to those specified in example 1, when pre-multiplied by $100 \cos(10\pi t)$ □

Note that example 3 concerns a digital time-varying control problem where the system has white stochastic parameters with time-varying first and second moments. The matrices of the EDOCP of example 3, for $i = 0$, are given by

$$\bar{\Phi}_0 = \begin{bmatrix} 1.032 & 0.0167 \\ 0 & 1.066 \end{bmatrix}, \quad \bar{\Gamma}_0 = \begin{bmatrix} 0.0329 \\ 0.0657 \end{bmatrix},$$

$$\overline{\Phi_0 \otimes \Phi_0} = \begin{bmatrix} 1.210 & 0.188 & 0.188 & 0.252 \\ 0 & 1.419 & 0 & 0.445 \\ 0 & 0 & 1.419 & 0.445 \\ 0 & 0 & 0 & 1.890 \end{bmatrix},$$

$$\overline{\Phi_0 \otimes \Gamma_0} = \begin{bmatrix} 0.0200 & 0.235 \\ 0.310 & 0.324 \\ 0 & 0.440 \\ 0 & 0.641 \end{bmatrix},$$

$$\overline{\Gamma_0 \otimes \Gamma_0} = \begin{bmatrix} 0.220 \\ 0.309 \\ 0.309 \\ 0.437 \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} 0.0351 & 0.002\,74 \\ 0.002\,74 & 0.0923 \end{bmatrix}, \quad R_0 = 0.0310,$$

$$M_0 = \begin{bmatrix} 0.002\,95 \\ 0.0212 \end{bmatrix}, \quad \eta_0 = 0.000\,147,$$

$$\bar{C}_0 = [1 \quad 0], \quad V_0^{CC} = [0.1 \quad 0 \quad 0 \quad 0],$$

$$W_0 = 0.1$$

Since the digital control problem is time varying, even if the sampling intervals $T_i = t_{i+1} - t_i$, $i = 0, 1, 2, \dots, N-1$ are constant, as in examples 1–3, the matrices of the EDOCP will be time varying and therefore need to be computed for each sampling interval. Having computed the EDOCP, straightforward application of the algorithms, recently presented by Van Willigenburg and De Koning (1999), enables the computation of full and reduced-order controllers that solve the EDOCP. Owing to the *integral* criterion (5), these digital optimal controllers explicitly take into account the *intersample behaviour* of the system (1).

6. Conclusions

The results of this paper enable the computation of finite-horizon digital optimal full and reduced-order controllers for continuous-time systems with white stochastic parameters, having *time-varying* statistics. In addition the systems may be corrupted by additive white noise. The sampling instants are *a priori* fixed but the sampling interval may change over time. The observations at the sampling instants must be linear and corrupted by multiplicative and/or additive discrete-time white noise. The digital optimal controllers are obtained after transformation of the digital optimal control problem into an equivalent discrete-time optimal control problem. This transformation and its numerical computation were the subject of this paper. Full and reduced-order controllers that solve the EDOCP are obtained after straightforward application of the results and algorithms presented by Van Willigenburg and De Koning (1999). Owing to the integral criterion of the original digital optimal control problem, the resulting digital full and reduced-order optimal controllers explicitly take into account the intersample behaviour of the controlled system. The principal application is the design and computation of robust digital compensators for nonlinear continuous-time systems that have to track (optimal) trajectories.

The transformation into an EDOCP is not affected by the equations which describe the observations at the sampling instants. Therefore, if we restrict our attention to the transformation only, the observations at the sampling instants may also be nonlinear and corrupted by discrete-time white noise.

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