Including Errata

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The digital optimal regulator and tracker for stochastic time-varying systems

L. G. VAN WILLIGENBURG† and W. L. DE KONING‡

The general approach to solve digital control problems is to approximate them by discrete-time control problems, which consider the system behaviour at the sampling instants only, completely disregarding the inter-sample behaviour. In this paper we consider digital control problems without making any approximations, i.e. we solve problems involving continuous-time criteria taking explicitly into account the inter-sample behaviour, which relaxes the demand for a 'small' sampling time. We solve what we call the digital optimal regulator and tracking problem where the continuous-time system is linear time-varying, and disturbed by additive white noise, and the state information at the sampling instants incomplete, and corrupted by additive white noise. The control is piecewise constant, and the continuous-time criteria are quadratic. Both the regulator and tracking problem turn out to be certainty equivalent. The solutions to both the regulator and tracking problem therefore consist of the well-known discrete-time Kalman one step ahead predictor, and a feedback generated by a Riccati type recursion that runs backward in time. In the case of the tracking problem the feedforward is also generated by a recursion that runs backward in time. Both recursions can be computed off-line. Expressions for the minimum cost of both problems, explicit in the system, criterion and covariance matrices, are derived. In a companion paper we treat the numerical computation which is not straightforward.

1. Introduction

In many practical situations we are faced with a continuous-time plant, controlled by a digital computer. It is common practice to approximate the associated digital control problems by discrete-time control problems which only consider the system behaviour at the sampling instants (Ackermann 1985). In these cases the inter-sample behaviour is completely disregarded. Therefore the sampling time has to be chosen 'small' to prevent undesirable inter-sample behaviour. For instance in the case of robot control, where the computational burden on the computer is high, this presents a serious limitation. Furthermore a discrete-time criterion has to be searched for, which leads to a satisfactory continuous-time behaviour. Both the choice of this criterion and the choice of the sampling time are often reported to be a problem (Franklin and Powell 1980, Åström and Wittenmark 1984). In this paper the digital control problems are solved without making any approximations since we consider continuous-time criteria taking explicitly into account the inter-sample behaviour which relaxes the demand for a 'small' sampling time. Furthermore the choice of a continuous-time criterion is a natural choice for a continuous-time system.

The digital optimal regulator for time-varying systems has already been considered by Halyo and Caglayan (1976). They however did not derive an expression for the minimum cost of the problem; neither did they specify a numerical solution. The

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† Department of Agricultural Engineering and Physics, University of Wageningen, AGROTECHNION, Bomenweg 4, 6701 HD Wageningen, The Netherlands.

‡ Faculty of Technical Mathematics and Informatics, Delft University of Technology, P.O. Box 356, 2600 AJ Delft, The Netherlands.

numerical solution is not straightforward since it involves the computation of integrals involving the state-transition matrix of time-varying systems. The computation of these matrices has only been considered for time-invariant systems (Van Loan 1978). De Koning (1980 *a, b*) considered the digital optimal regulator for time-invariant systems and derived an expression for the minimum cost of the problem, explicit in the system, criterion and covariance matrices.

The digital optimal tracker for stochastic linear systems has never been considered in the literature before. This is remarkable since it can be applied in all situations where a continuous-time system, controlled by a digital computer, has to track a reference state trajectory, e.g. a cartesian robot performing a prescribed motion.

We will present both the digital optimal regulator and tracker for linear time-varying stochastic systems including expressions for the minimum cost, explicit in the system, criterion and covariance matrices. In a companion paper (Van Willigenburg 1991) we treat the numerical computation. It is believed that the digital optimal regulator for linear time-varying stochastic systems permits for the first time the computation of a digital optimal compensator, based on the linearized dynamics about the trajectory, for nonlinear systems that have to track reference state trajectories. Important applications are e.g. a robot performing a prescribed motion or a batch fermentation process, where in both cases the linearized dynamics about the trajectory constitute a time-varying linear system.

2. Problem formulations

Consider the stochastic continuous time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t) \tag{1 a}$$

where $A(t)$ and $B(t)$ are the system matrices and $\{v(t)\}$ a white noise process with possibly time-varying intensity, with

$$E\{v(t)\} = 0 \quad \operatorname{cov}(v(t), v(s)) = V(t)\delta(t - s) \tag{1 b}$$

and

$$E\{x(0)\} = \bar{x}(0) \quad \operatorname{cov}(x(0), x(0)) = G \quad G \geqslant 0 \tag{1 c}$$

The system is controlled by a digital computer, so measurements are taken at the sampling instants, i.e.

$$y(t_k) = C(t_k)x(t_k) + w(t_k) \quad k = 0, 1, 2, 3, \ldots \tag{1 d}$$

where t_k, $k = 0, 1, 2, \ldots$ are the, not necessarily equidistant, sampling instants and $\{w(t_k)\}$ a discrete-time white noise process independent of $\{v(t)\}$, with

$$E\{w(t_k)\} = 0 \quad \operatorname{cov}(w(t_k), w(t_l)) = W(t_k)\delta(t_k - t_l) \tag{1 e}$$

The control is piecewise constant i.e.

$$u(t) = u(t_k) \quad t \in [t_k, t_{k+1}) \quad k = 0, 1, 2, 3, \ldots \tag{1 f}$$

The information available to compute the control u_k consists of the measurements and the controls up to t_{k-1}, i.e., $\{y(t_i), i = 0, 1, 2, \ldots, k - 1\}$ and $\{u(t_i), i = 0, 1, 2, \ldots, k - 1\}$. In that case the time available for the computer to compute $u(t_k)$ equals $t_k - t_{k-1}$. We may also assume the information to be $\{y(t_i), i = 0, 1, 2, \ldots, k\}$ and $\{u(t_i), i = 0, 1, 2, \ldots, k - 1\}$ in case the computation time is negligible compared to $t_k - t_{k-1}$. All the results of this paper are still valid in that case.

The digital optimal regularor problem for the system (1) is to minimize

$$J = E\left(\int_{t_0}^{t_f} x^{\mathrm{T}}(t)Q(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t)\,\mathrm{d}t + x^{\mathrm{T}}(t_f)Hx(t_f)\right) \tag{2}$$

where E denotes the expectation operator, and $Q(t) \geqslant 0$, $H \geqslant 0$ and $R(t) > 0$. We will pay special attention to situations where $R(t) \geqslant 0$. Furthermore

$$t_f = t_N \tag{3}$$

where N is a positive integer.

The digital optimal tracking problem takes the following form. Given the system (1) and a reference trajectory

$$x_r(t) \quad t_0 \leqslant t \leqslant t_f \tag{4}$$

minimize

$$J = E\left(\int_{t_0}^{t_f} (x(t) - x_r(t))^{\mathrm{T}} Q(t)(x(t) - x_r(t)) + u^{\mathrm{T}}(t)R(t)u(t)\,\mathrm{d}t + (x(t_f) - x_r(t_f))^{\mathrm{T}} H(x(t_f) - x_r(t_f))\right) \tag{5}$$

where furthermore (3) holds, and again $Q(t) \geqslant 0$, $H \geqslant 0$ and $R(t) > 0$. As in the regulator case we will pay special attention to situations where $R(t) \geqslant 0$.

3. The equivalent discrete-time regulator and tracking problem

To solve the digital optimal regulator and tracking problem presented in the previous chapter, we first transform them into so-called equivalent discrete-time problems (Levis *et al.* 1971, Halyo and Caglayan 1976, De Koning 1980*a*, Van Willigenburg 1991), with unconstrained control. The derivation of the equivalent discrete-time regulator problem for stochastic continuous-time linear time-varying systems resembles the derivation in case of time-invariant systems, given by De Koning (1980*a*). Equation (1) is defined in terms of the stochastic integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} A(s)x(s)\,\mathrm{d}s + \int_{t_0}^{t} B(s)u(s)\,\mathrm{d}s + \int_{t_0}^{t} \mathrm{d}\beta(s) \tag{6a}$$

where $\beta(t)$ is a zero-mean process with independent increments and

$$E\{\beta(t)\} = 0 \quad \mathrm{cov}(\mathrm{d}\beta(t), \mathrm{d}\beta(t)) = V(t)\,\mathrm{d}t \tag{6b}$$

The solution of system (1) is given by

$$x(t) = \Phi(t, t_k)x(t_k) + \Gamma(t, t_k)u(t_k) + v(t, t_k) \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots, N-1 \tag{7}$$

where Φ is the state transition matrix of system (1),

$$\Gamma(t, t_k) = \int_{t_k}^{t} \Phi(t, s)B(s)\,\mathrm{d}s \tag{8}$$

and

$$v(t, t_k) = \int_{t_k}^{t} \Phi(t, s)\,\mathrm{d}\beta(s) \tag{9}$$

From (9) and (6 *b*) it follows that

$$E\{v(t, t_k)\} = 0 \tag{10a}$$

and

$$E\{v(t, t_k)v^{\mathrm{T}}(t, t_k)\} = \iint_{t_k}^{t} \Phi(t, \lambda)E\{\mathrm{d}\beta(\lambda)\,\mathrm{d}\beta^{\mathrm{T}}(\mu)\}\Phi^{\mathrm{T}}(t, \mu)$$

$$= \int_{t_k}^{t} \Phi(t, \lambda)V(t)\Phi^{\mathrm{T}}(t, \lambda)\,\mathrm{d}\lambda = V(t, t_k) \tag{10b}$$

For $t = t_{k+1}$ we have

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + v_k \tag{11a}$$

and furthermore

$$y_k = C_k x_k + w_k, \tag{11b}$$

where

$$x_k = x(t_k) \tag{11c}$$

$$y_k = y(t_k) \tag{11d}$$

$$u_k = u(t_k) \tag{11e}$$

$$\Phi_k = \Phi(t_{k+1}, t_k) \tag{11f}$$

$$\Gamma_k = \Gamma(t_{k+1}, t_k) \tag{11g}$$

$$C_k = C(t_k) \tag{11h}$$

$$v_k = v(t_{k+1}, t_k) \tag{11i}$$

$$w_k = w(t_k) \tag{11j}$$

The system (11) is called the equivalent descrete-time system since the behaviour of this system is exactly the same as that of system (1) at the sampling instants, for $k = 0, 1, 2, \ldots, N-1$.

The stochastic regulator criterion (2) may be written as

$$J = E\left(\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} x^{\mathrm{T}}(t)Q(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t)\,\mathrm{d}t + x_{\mathrm{N}}^{\mathrm{T}}Hx_N\right) \tag{12}$$

which, given (7), equals

$$J = E\Bigg(\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} x_k^{\mathrm{T}}\Phi^{\mathrm{T}}(t, t_k)Q(t)\Phi(t, t_k)x_k + 2x_k^{\mathrm{T}}\Phi^{\mathrm{T}}(t, t_k)Q(t)\Gamma(t, t_k)u_k$$

$$+ u_k^{\mathrm{T}}(R_k + \Gamma^{\mathrm{T}}(t, t_k)Q(t)\Gamma(t, t_k))u_k + v^{\mathrm{T}}(t, t_k)Q(t)v(t, t_k)$$

$$+ 2x_k^{\mathrm{T}}\Phi^{\mathrm{T}}(t, t_k)Q(t)v(t, t_k) + 2u_k^{\mathrm{T}}\Gamma^{\mathrm{T}}(t, t_k)Q(t)v^{\mathrm{T}}(t, t_k)\,\mathrm{d}t\Bigg) \tag{13}$$

Introducing

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi^{\mathrm{T}}(t, t_k)Q(t)\Phi(t, t_k)\,\mathrm{d}t \tag{14a}$$

$$M_k = \int_{t_k}^{t_{k+1}} \Phi^{\mathrm{T}}(t, t_k)Q(t)\Gamma(t, t_k)\,\mathrm{d}t \tag{14b}$$

$$R_k = \int_{t_k}^{t_{k+1}} [R(t) + \Gamma^{\mathrm{T}}(t, t_k)Q(t)\Gamma(t, t_k)]\,\mathrm{d}t \tag{14c}$$

and splitting up the integral results in

$$\begin{aligned} J = E\Bigg(\sum_{k=0}^{N-1} & x_k^{\mathrm{T}} Q_k x_k + 2x_k^{\mathrm{T}} M_k u_k + u_k^{\mathrm{T}} R_k u_k \\ & + \int_{t_k}^{t_{k+1}} 2x_k^{\mathrm{T}} \Phi^{\mathrm{T}}(t, t_k) Q(t) v(t, t_k)\,\mathrm{d}t \\ & + \int_{t_k}^{t_{k+1}} 2u_k^{\mathrm{T}} \Gamma^{\mathrm{T}}(t, t_k) v(t, t_k)\,\mathrm{d}t \\ & + \int_{t_k}^{t_{k+1}} v^{\mathrm{T}}(t, t_k) Q(t) v(t, t_k)\,\mathrm{d}t \Bigg) \end{aligned} \tag{15}$$

The state x_k depends only on the increments $\mathrm{d}\beta(t)$, $t \in [t_0, t_k]$ and on $w_0, w_1, \ldots, w_{k-1}$, while $v(t, t_k)$, $t \geqslant t_k$ depends only on the increments $\mathrm{d}\beta(t)$, $t \geqslant t_k$, so x_k and $v(t, t_k)$, $t \geqslant t_k$ are independent. Because u_k depends only on $y_0, y_1, y_2, \ldots, y_{k-1}$, thus on $x_0, v_0, v_1, \ldots, v_{k-1}, w_0, w_1, \ldots, w_{k-1}$ and since $\{v_k\}$ is independent of $\{w_k\}$, u_k is also independent of $v(t, t_k)$, $t \geqslant t_k$. Therefore

$$E\left(\int_{t_k}^{t_{k+1}} 2x_k^{\mathrm{T}} \Phi_k^{\mathrm{T}}(t, t_k) Q(t) v(t, t_k)\,\mathrm{d}t \right) = \int_{t_k}^{t_{k+1}} 2E\{x_k^{\mathrm{T}}\} \Phi_k^{\mathrm{T}}(t, t_k) Q(t) E\{v(t, t_k)\}\,\mathrm{d}t = 0 \tag{16}$$

and

$$E\left(\int_{t_k}^{t_{k+1}} 2u_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}}(t, t_k) Q(t) v(t, t_k)\,\mathrm{d}t \right) = \int_{t_k}^{t_{k+1}} 2E\{u_k^{\mathrm{T}}\} \Gamma_k^{\mathrm{T}}(t, t_k) Q(t) E\{v(t, t_k)\}\,\mathrm{d}t = 0 \tag{17}$$

Furthermore,

$$\begin{aligned} E\left(\int_{t_k}^{t_{k+1}} v(t,\, t_k)\, Q(t) v^{\mathrm{T}}(t,\, t_k)\,\mathrm{d}t \right) &= \int_{t_k}^{t_{k+1}} \operatorname{tr}\, [E(v(t,\, t_k) v^{\mathrm{T}}(t,\, t_k))\, Q(t)]\,\mathrm{d}t \\ &= \int_{t_k}^{t_{k+1}} \operatorname{tr}\, [V(t,\, t_k)\, Q(t)]\,\mathrm{d}t \\ &= \gamma(t_{k+1}, t_k) = \gamma_k \end{aligned} \tag{18}$$

Now we are in a position, after having stated the equivalent discrete-time system (11), to state the equivalent discrete-time criterion for the regulator problem.

$$J = E\left(\sum_{k=0}^{N-1} (x_k^{\mathrm{T}} Q_k x_k + 2x_k^{\mathrm{T}} M_k u_k + u_k^{\mathrm{T}} R_k u_k) + x_{\mathrm{N}}^{\mathrm{T}} H x_N \right) + \sum_{k=0}^{N-1} \gamma_k \tag{19}$$

where Q_k, M_k and R_k are given by (14). If $Q(t) \geqslant 0$, and $R(t) > 0$, as assumed in § 2, by inspection of (14) it can be seen that $Q_k \geqslant 0$, and $R_k > 0$. However by inspection of (14 *c*) it can be seen that if $R(t) \geqslant 0$ and $\Gamma^{\mathrm{T}}(t,\, t_k)Q(t)\Gamma(t,\, t_k) + R(t)$ is positive definite over some open time interval within $[t_k,\, t_{k+1})$ then also $R_k > 0$. In the sequel of the paper we will assume $R_k > 0$. Finally we have

$$\gamma_k = \int_{t_k}^{t_{k+1}} \operatorname{tr}\,[V(t, t_k)\, Q(t)]\, \mathrm{d}t \tag{20}$$

So the original digital optimal regulator problem is equivalent to the discrete-time regulator problem given by (11) and (19), where the equivalent discrete-time criterion matrices are given by (14) and (20). Note that the part involving γ_k in (19) is deterministic and independent of the control, so the problem of minimizing J, with respect to the control, is equivalent to minimizing

$$J' = E\left(\sum_{k=0}^{N-1} \left(x_k^{\mathrm{T}} Q_k x_k + 2x_k^{\mathrm{T}} M_k u_k + u_k^{\mathrm{T}} R_k u_k \right) + x_N^{\mathrm{T}} H x_N \right) \tag{21}$$

In deriving the solution of the digital optimal regulator problem we will consider the minimization of (21).

The procedure to derive the equivalent discrete-time tracking problem proceeds along exactly the same lines. Substituting the solution (7) of system (1) into the tracking criterion (5) results in

$$\begin{aligned} J = E\Bigg(\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} & x_k^{\mathrm{T}} \Phi^{\mathrm{T}}(t, t_k) Q(t) \Phi(t, t_k) + 2x_k^{\mathrm{T}} \Phi^{\mathrm{T}}(t, t_k) Q(t) \Gamma(t, t_k) u_k \\ & + u_k^{\mathrm{T}} (R_k + \Gamma^{\mathrm{T}}(t, t_k) Q(t) \Gamma(t, t_k)) u_k)) u_k + v^{\mathrm{T}}(t, t_k) Q(t) v(t, t_k) \\ & + 2x_k^{\mathrm{T}} \Phi^{\mathrm{T}}(t, t_k) Q(t) v(t, t_k) + 2u_k^{\mathrm{T}} \Gamma^{\mathrm{T}}(t, t_k) Q(t) v^{\mathrm{T}}(t, t_k) \\ & - 2x_r^{\mathrm{T}}(t) Q(t) \Phi(t, t_k) x_k - 2x_r^{\mathrm{T}}(t) Q(t) \Gamma(t, t_k) u_k - 2x_r^{\mathrm{T}}(t) Q(t) v(t, t_k) \\ & + x_r^{\mathrm{T}}(t) Q(\mathrm{t}) x_r(t)\, \mathrm{d}t + x_N^{\mathrm{T}} H x_N - 2x_r^{T}(t_f) H x_N \Bigg) + x_r^{\mathrm{T}}(t_f) H x_r(t_f) \end{aligned} \tag{22}$$

Comparing the tracking criterion (22) to the regulator criterion (13) we see that, except for terms involving the reference trajectory x_r, they are exactly the same. Since the reference trajectory $x_r(t)$, $0 \leqslant t \leqslant t_f$ is deterministic,

$$E\left(\int_{t_k}^{t_{k+1}} 2x_r^{\mathrm{T}}(t) Q(t) v(t, t_k)\, \mathrm{d}t \right) = \int_{t_k}^{t_{k+1}} 2x_r^{\mathrm{T}}(t) Q(t) E\{v(t, t_k)\}\, \mathrm{d}t = 0 \tag{23}$$

Introducing again R_k, M_k and Q_k given by (14) and also

$$L_k = \int_{t_k}^{t_{k+1}} x_r^{\mathrm{T}}(t) Q(t) \Phi(t, t_k) \mathrm{d}t \tag{24 a}$$

$$T_k = \int_{t_k}^{t_{k+1}} x_r^{\mathrm{T}}(t) Q(t) \Gamma(t, t_k)\, \mathrm{d}t \tag{24 b}$$

$$X_k = \int_{t_k}^{t_{k+1}} x_r^{\mathrm{T}}(t) Q(t) x_r(t)\, \mathrm{d}t \tag{24 c}$$

and given (16), (17), (18) and (23) the equivalent discrete-time tracking criterior J becomes

$$\begin{aligned} J = E\Bigg(& \sum_{k=0}^{N-1} (x_k^{\mathrm{T}} Q_k x_k + 2x_k^{\mathrm{T}} M_k u_k + u_k^{\mathrm{T}} R_k u_k - 2L_k x_k - 2T_k u_k) + x_N^{\mathrm{T}} H x_N \\ & - 2x_r^{\mathrm{T}}(t_f) H x_N \Bigg) + x_r^{\mathrm{T}}(t_f) H x_r(t_f) + \sum_{k=0}^{N-1} X_k + \gamma_k \end{aligned} \tag{25}$$

The equivalent discrete-time tracking problem is determined by the equivalent discrete-time system (11) and the equivalent discrete-time criterion (25), where the equivalent discrete-time criterion matrices are given by (14), (20) and (24). Note that the part outside the brackets of the expectation operator in (25) is deterministic and independent of the control. So the minimization of (25), with respect to the control, is equivalent to the minimization of

$$J' = E\left(\sum_{k=0}^{N-1} (x_k^{\mathrm{T}} Q_k x_k + 2x_k^{\mathrm{T}} M_k u_k + u_k^{\mathrm{T}} R_k u_k - 2L_k x_k - 2T_k u_k \right) + x_N^{\mathrm{T}} H x_N - 2x_r^{\mathrm{T}}(t_f) H x_N \tag{26}$$

In deriving the solution of the digital optimal tracking problem we will consider the minimization of (26).

4. Solution to the equivalent discrete-time regulator problem

The derivation of the solution of the equivalent discrete-time regulator problem resembles the one presented by De Koning (1980 *b*). He considered randomly sampled linear time-invariant systems. The case of linear time-varying systems and deterministic sampling resembles this situation where the random system matrices now become deterministic.

The conditional mean $\hat{x}_k$ and the covariance P_k of the state x_k are defined as

$$\hat{x}_k = E\{x_k | Y_{k-1}, U_{k-1}\} \tag{27}$$

where

$$Y_{k-1} = \{y_0, y_1, y_2, \ldots, y_{k-1}\} \tag{28 a}$$

$$U_{k-1} = \{u_0, u_1, u_2, \ldots, u_{k-1}\} \tag{28 a}$$

and

$$P_k = E\{\tilde{x}_k \tilde{x}_k^{\mathrm{T}}\} \tag{29 a}$$

where

$$\tilde{x}_k = x_k - \hat{x}_k \tag{29 b}$$

It is well known that $\hat{x}_k$ is the best linear estimator of x_k on the basis of Y_{k-1}, U_{k-1}, in the sense that P_k is minimal. It is well known that for the equivalent discrete-time system (11) the estimator is generated by the discrete-time Kalman one steps ahead predictor. In deriving the digital optimal regulator we will need the following facts. If Z is an arbitrary matrix and x_k a stochastic vector then

$$\begin{aligned} E\{x_k^{\mathrm{T}} Z x_k | Y_{k-1}, U_{k-1}\} &= E\{\hat{x}_k^{\mathrm{T}} Z \hat{x}_k\} + E\{\tilde{x}_k^{\mathrm{T}} Z \tilde{x}_k | Y_{k-1}, U_{k-1}\} \\ &= \hat{x}_k^{\mathrm{T}} Z \hat{x}_k + \operatorname{tr}(Z P_k^{\mathrm{c}}) \end{aligned} \tag{30}$$

where the conditional covariance P_k^{c} is given by

$$P_k^{\mathrm{c}} = E\{\tilde{x}_k \tilde{x}_k^{\mathrm{T}} | Y_{k-1}, U_{k-1}\} \tag{31}$$

Furthermore if x, y and z are arbitrary stochastic variables then

$$\sigma(y) \supset \sigma(x) \Rightarrow \mathrm{E}\{z|x\} = E\{E\{z|y\}|x\} \tag{32 a}$$

where $\sigma(x)$ denotes the σ algebra generated by x. Furthermore if $f(x, y, z)$ is an arbitrary function of x, y and z then

$$E\{E\{f(x, y, z)|x, y\}\} = E\{f(x, y, z)\} \tag{32b}$$

Finally since v_k is independent of x_i, $k \geq i$ and $\{v_k\}$ is independent of $\{w_k\}$

$$E\{(\Phi_k x_k + \Gamma_k u_k)^{\mathrm{T}} v_k | Y_{k-1}, U_{k-1}\} = 0 \tag{33a}$$

$$E\{v_k v_k^{\mathrm{T}} | Y_{k-1}, U_{k-1}\} = V_k \tag{33b}$$

Considering (21) we define the scalar function

$$C_i(Y_{i-1}, U_{i-1}) = \min_{u_i, \ldots, u_{N-1}} E\left\{\sum_{k=i}^{N-1} (x_k^{\mathrm{T}} Q_k x_k + 2x_k^{\mathrm{T}} M_k u_k + u_k^{\mathrm{T}} R_k u_k) + x_N^{\mathrm{T}} H x_N | Y_{i-1}, U_{i-1}\right\} \tag{34}$$

Under suitable existence conditions for the expectations and the minima (Meier *et al.* 1971) C in (34) satisfies the Bellman equation

$$C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} E\left\{x_i^T Q_i x_i + 2x_i^T M_i u_i + u_i^{\mathrm{T}} R_i u_i + C_{i+1}(Y_i, U_i) | Y_{i-1}, U_{i-1}\right\} \tag{35}$$

with for $i = N$ the initial condition

$$\begin{aligned} C_N(Y_{N-1}, U_{N-1}) &= \min_{u_N} E\left\{x_N^{\mathrm{T}} H x_N | Y_{N-1}, U_{N-1}\right\} = E\left\{x_N^{\mathrm{T}} H x_N | Y_{N-1}, U_{N-1}\right\} \\ &= \hat{x}_N^{\mathrm{T}} H \hat{x}_N + \operatorname{tr}(H P_N^{\mathrm{c}}) \end{aligned} \tag{36}$$

Now suppose that $C_i(Y_{i-1}, U_{i-1})$ has the form

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) &= E\left\{x_i^{\mathrm{T}} S_i x_i | Y_{i-1}, U_{i-1}\right\} + \alpha_i \\ &= \hat{x}_i^{\mathrm{T}} S_i \hat{x}_i + \operatorname{tr}(S_i P_i^{\mathrm{c}}) + \alpha_i \end{aligned} \tag{37}$$

where $S_i \geq 0$ and deterministic, and S_i and α_i are not functions of U_{i-1}. Considering the boundary condition (36), this is true for $i = N$ if

$$S_N = H \tag{38a}$$

$$\alpha_N = 0 \tag{38b}$$

Suppose it is true for $i + 1$, i arbitrary, then we must prove that it is true for i. From the Bellman equation (35) we may write

$$C_i(Y_{i-1}, U_{i-1})$$

→ $$= \min_{u_i} E\{x_i^{\mathrm{T}} Q_i x_i + 2x_i^T M_i u_i + u_i^{\mathrm{T}} R_i u_i + E\{x_{i+1}^{\mathrm{T}} S_{i+1} x_{i+1} | Y_i, U_i\} + \alpha_{i+1} | Y_{i-1}, U_{i-1}\}$$

Since $\sigma(Y_i, U_i) \supset \sigma(Y_{i-1}, U_{i-1})$, and given (32 *a*), this becomes

$$C_i(Y_{i-1}, U_{i-1})$$

$$= \min_{u_i} E\{x_i^{\mathrm{T}} Q_i x_i + 2x_i^T M_i u_i + u_i^{\mathrm{T}} R_i u_i + x_{i+1}^{\mathrm{T}} S_{i+1} x_{i+1} + \alpha_{i+1} | Y_{i-1}, U_{i-1}\}$$

Using (33) and the assumption that S_{i+1} and α_{i+1} are not functions of U_i this may be written as

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) &= \min_{u_i} E\{x_i^{\mathrm{T}}(Q_i + \Phi_i^{\mathrm{T}} S_{i+1} \Phi_i) x_i + u_i^{\mathrm{T}}(R_i + \Gamma_i^{\mathrm{T}} S_{i+1} \Gamma_i) u_i \\ &+ 2x_i^{\mathrm{T}}(M_i + \Phi_i^{\mathrm{T}} S_{i+1} \Gamma_i) u_i | Y_{i-1}, U_{i-1}\} + \operatorname{tr}(V_i S_{i+1}) + E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \end{aligned}$$

From (30) this becomes

$$C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} [\hat{x}_i^T(Q_i + \Phi_i^T S_{i+1}\Phi_i)\hat{x}_i + u_i^T(R_i + \Gamma_i^T S_{i+1} i\Gamma_i)u_i + 2\hat{x}_i^T(M_i + \Phi_i^T S_{i+1}\Gamma_i)u_i] + \operatorname{tr}((Q_i + \Phi_i^T S_{i+1}\Phi_i)P_i^c) + \operatorname{tr}(V_i S_{i+1}) + E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \quad (39)$$

The term between the brackets in equation (39) is a quadratic form in $\hat{x}_i$ and u_i. We want to find the u_i that minimizes (39) so the obvious way to complete the square for the term in between the brackets of (39) is

$$C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} [(u_i + K_i\hat{x}_i)^T(R_i + \Gamma_i^T S_{i+1}\Gamma_i)(u_i + K_i\hat{x}_i) + \hat{x}_i^T(Q_i + \Phi_i^T S_{i+1}\Phi_i - K_i^T(R_i + \Gamma_i^T S_{i+1}\Gamma_i)K_i)\hat{x}_i] + \operatorname{tr}((Q_i + \Phi_i^T S_{i+1}\Phi_i)P_i^c) + \operatorname{tr}(V_i S_{i+1}) + E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \quad (40)$$

where

$$K_i = (R_i + \Gamma_i^T S_{i+1}\Gamma_i)^{-1}(\Gamma_i^T S_{i+1}\Phi_i + M_i^T) \quad (41)$$

The minimum is attained when

$$u_i = -K_i\hat{x}_i \quad (42)$$

If P_i^c in (37) is not a function of U_{i-1}, which is true for the discrete-time Kalman one step ahead predictor for the equivalent discrete time system (11), then $C_i(Y_{i-1}, U_{i-1})$ has indeed the form assumed in (37) with

$$S_i = Q_i + \Phi_i^T S_{i+1}\Phi_i - K_i^T(R_i + \Gamma_i^T S_{i+1}\Gamma_i)K_i \quad (43)$$

$$\alpha_i = \operatorname{tr}(K_i^T(R_i + \Gamma_i^T S_{i+1}\Gamma_i)K_i P_i^c) + \operatorname{tr}(V_i S_{i+1}) + E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \quad (44)$$

where $S_i \geqslant 0$ and deterministic, and S_i and α_i not functions of U_{i-1}. The fact that $S_i \geqslant 0$ can be seen by writing (43) in the following form (Van Willigenburg 1991).

$$S_i = (\Phi_i - \Gamma_i K_i)^T S_{i+1}(\Phi_i - \Gamma_i K_i) + (K_i - R_i^{-1}M_i^T)^T R_i(K_i - R_i^{-1}M_i^T) + Q_i - M_i R_i^{-1} M_i^T \quad (45)$$

From (45) it can be seen that since $R_i \geqslant 0$, $S_N \geqslant 0$, and $Q_i - M_i R_i^{-1} M_i^T \geqslant 0$ (Van Willigenburg 1991), indeed $S_i \geqslant 0$. The solution of the digital optimal regulator problem is therefore given by (38), (41), . . . , (44). Given (35), and considering (21), the minimum value of the cost (19) equals

$$J = E\{C_0\} = E\{x_0^T S_0 x_0\} + E\{\alpha_0\} + \sum_{i=0}^{N-1} \gamma_i \quad (46)$$

Given (37), (38), (43), (44) and (1 *c*) and using (32 *b*) this becomes

$$J = \bar{x}_0^T S_0 \bar{x}_0 + \operatorname{tr}(S_0 G) + \sum_{i=0}^{N-1} \operatorname{tr}(V_i S_{i+1}) + \sum_{i=0}^{N-1} \operatorname{tr}(K_i^T(R_i + \Gamma_i^T S_{i+1}\Gamma_i)K_i P_i^c) + \sum_{i=0}^{N-1} \gamma_i \quad (47)$$

The first term on the right-hand side of (47) can be compared to the cost in the deterministic case (Van Willigenburg 1991). The second term on the right is due to uncertainty in the initial state, the third term is caused by disturbances acting on the system, and the fourth by uncertainty in the state estimation. The fifth term on the right, which showed up in deriving the equivalent discrete-time regulator problem, is also caused by disturbances acting on the system.

Summarizing the solution to the equivalent discrete-time regulator problem is given by

$$K_k = (R_k + \Gamma_k^T S_{k+1}\Gamma_k)^{-1}(\Gamma_k^T S_{k+1}\Phi_k + M_k^T) \tag{48a}$$

$$S_k = Q_k + \Phi_k^T S_{k+1}\Phi_k - K_k^T(R_k + \Gamma_k^T S_{k+1}\Gamma_k)K_k \quad S_N = H \tag{48b}$$

$$u_k = -K_k\hat{x}_k \tag{48c}$$

$$J = \bar{x}_0^T S_0 \bar{x}_0 + \operatorname{tr}(S_0 G) + \sum_{k=0}^{N-1}[\operatorname{tr}(V_k S_{k+1}) + \operatorname{tr}(K_k^T(R_k + \Gamma_k^T S_{k+1}\Gamma_k)K_k P_k^c) + \gamma_k] \tag{48d}$$

where $\hat{x}_k$ is generated by the discrete-time Kalman one step ahead predictor, for the equivalent discrete-time system (11). Replacing Y_{i-1} by Y_i only affects the state estimator, which is now generated by the Kalman filter instead of the one step ahead predictor. Finally we remark that clearly, the digital optimal regulator is certainty equivalent.

5. Solution to the equivalent discrete-time tracking problem

Consider the discrete-time tracking criterion (26).

$$J' = E\left(\sum_{k=0}^{N-1}(x_k^T Q_k x_k + 2x_k^T M_k u_k + u_k^T R_k u_k - 2L_k x_k - 2T_k u_k) + x_N^T H x_N - 2x_r^T(t_f)Hx_N\right) \tag{49}$$

Like in the regulator case we define the scalar function

$$C_i(Y_{i-1}, U_{i-1}) = \min_{u_i,\dots,u_{N-1}} E\left\{\sum_{k=0}^{N-1}(x_k^T Q_k x_k + 2x_k^T M_k u_k + u_k^T R_k u_k - 2L_k x_k - 2T_k u_k) + x_N^T H x_N - 2x_r^T(t_f)Hx_N | Y_{i-1}, U_{i-1}\right\} \tag{50}$$

Again under suitable existence conditions for the expectations and the minima (50) satisfies the Bellman equation

$$C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} E\{x_i^T Q_i x_i + 2x_i^T M_i u_i + u_i^T R_i u_i - 2L_i x_i - 2T_i u_i + C_{i+1}(Y_i, U_i)| Y_{i-1}, U_{i-1}\} \tag{51}$$

with for $i = N$ the initial condition

$$\begin{aligned} C_N(Y_{N-1}U_{N-1}) &= \min_{u_N} E\{x_N^T H x_N - 2x_r^T(t_f)Hx_N | Y_{N-1}, U_{N-1}\} \\ &= E\{x_N^T H x_N - 2x_r^T(t_f)Hx_N | Y_{N-1}, U_{n-1}\} \\ &= \hat{x}_N^T H \hat{x}_N - 2x_r^T(t_f)H\hat{x}_N + \operatorname{tr}(HP_N^c) \end{aligned} \tag{52}$$

Now suppose, according to the case of the discrete-time tracker presented by Lewis

(1986), that $C_i(Y_{i-1}, U_{i-1})$ has the form

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) &= E\{x_i^T S_i x_i - 2x_i^T W_i | Y_{i-1}, U_{i-1}\} + \alpha_i \\ &= \hat{x}_i^T S_i \hat{x}_i - 2\hat{x}_i^T W_i + tr\,(S_i P_i^c) + \alpha_i \end{aligned} \tag{53}$$

where $S_i \geqslant 0$, S_i and W_i deterministic, and S_i, W_i and α_i are not functions of U_{i-1}. Given the boundary condition (52) this is true for $i = N$ if

$$S_N = H \tag{54 a}$$

$$W_N = Hx_r(t_f) \tag{54 b}$$

$$\alpha_N = 0 \tag{54 c}$$

Suppose it is true for $i + 1$, i arbitrary, then we must prove that it is true for i. From the Bellman equation (51) we have

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} E\{&x_i^T Q_i x_i + 2x_i^T M_i u_i + u_i^T R_i u_i - 2L_i x_i - 2T_i u_i \\ &+ E\{x_{i+1}^T S_{i+1} x_{i+1} - 2x_{i+1}^T W_{i+1} | Y_i, U_i\} + \alpha_{i+1} | Y_{i-1}, U_{i-1}\} \end{aligned}$$

Since $\sigma(Y_i, U_i) \supset \sigma(Y_{i-1}, U_{i-1})$, and given (32 *a*), this becomes

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} E\{&x_i^T Q_i x_i + 2x_i^T M_i u_i + u_i^T R_i u_i - 2L_i x_i \\ &- 2T_i u_i + x_{i+1}^T S_{i+1} x_{i+1} - 2x_{i+1}^T W_{i+1} + \alpha_{i+1} | Y_{i-1}, U_{i-1}\} \end{aligned}$$

Using (33) and the assumption that S_{i+1}, W_{i+1} and α_{i+1} are not functions of U_i this may be written as

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} E\{&x_i^T(Q_i + \Phi_i^T S_{i+1} \Phi_i) x_i + u_i^T(R_i + \Gamma_i^T S_{i+1} \Gamma_i) u_i \\ &+ 2x_i^T(M_i + \Phi_i^T S_{i+1} \Gamma_i) u_i - x_i^T(2\Phi_i^T W_{i+1} + 2L_i^T) \\ &- u_i^T(2\Gamma_i^T W_{i+1} + 2T_i^T) | Y_{i-1}, U_{i-1}\} + \text{tr}\,(V_i S_{i+1}) \\ &+ E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \end{aligned}$$

From (30) this becomes

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} [&\hat{x}_i^T(Q_i + \Phi_i^T S_{i+1} \Phi_i)\hat{x}_i + u_i^T(R_i + \Gamma_i^T S_{i+1} \Gamma_i) u_i \\ &+ 2\hat{x}_i^T(M_i + \Phi_i^T S_{i+1} \Gamma_i) u_i - 2\hat{x}_i^T(\Phi_i^T W_{i+1} + L_i^T) \\ &- 2u_i^T(\Gamma_i^T W_{i+1} + T_i^T)] + \text{tr}\,((Q_i + \Phi_i^T S_{i+1} \Phi_i) P_i^c) \\ &+ \text{tr}\,(V_i S_{i+1}) + E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \end{aligned} \tag{55}$$

The term between the brackets of equation (55) is a quadratic expression in $\hat{x}_i$ and u_i. Since we want to find the u_i that minimizes (55) the obvious way to complete the square for the term in between the brackets of (55) leads to

$$\begin{aligned} C_i(Y_{i-1}, U_{i-1}) = \min_{u_i} [&(u_i + K_i \hat{x}_i - K_i^1 W_{i+1} - K_i^2 T_i^T)^T (R_i + \Gamma_i^T S_{i+1} \Gamma_i) \\ &(u_i + K_i \hat{x}_i - K_i^1 W_{i+1} - K_i^2 T_i^T) + \hat{x}_i^T(Q_i + \Phi_i^T S_{i+1} \Phi_i \\ &- K_i^T(R_i + \Gamma_i^T S_{i+1} \Gamma_i) K_i)\hat{x}_i - 2\hat{x}_i^T(\Phi_i^T W_{i+1} + L_i^T - K_i^T \Gamma_i^T W_{i+1} \\ &- K_i^T T_i^T) - 2W_{i+1}^T K_i^{1T} T_i^T - W_{i+1}^T K_i^{1T} \Gamma_i^T W_{i+1} - T_i K_i^2 T_i^T] \\ &+ \text{tr}\,((Q_i + \Phi_i^T S_{i+1} \Phi_i) P_i^c) + \text{tr}\,(V_i S_{i+1}) \end{aligned}$$

$$+ E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \tag{56}$$

where

$$K_i = (R_i + \Gamma_i^T S_{i+1} \Gamma_i)^{-1} (\Gamma_i^T S_{i+1} \Phi_i + M_i^T) \tag{57 a}$$

$$K_i^1 = (R_i + \Gamma_i^T S_{i+1} \Gamma_i)^{-1} \Gamma_i^T \tag{57 b}$$

$$K_i^2 = (R_i + \Gamma_i^T S_{i+1} \Gamma_i)^{-1} \tag{57 c}$$

The minimum is attained when

$$u_i = -K_i \hat{x}_i + K_i^1 W_i + K_i^2 T_i \tag{58}$$

If P_i^c in (37) is not a function of U_{i-1} which is true for the discrete-time Kalman one step ahead predictor for the equivalent discrete-time system (11), then $C_i(Y_{i-1}, U_{i-1})$ has indeed the form assumed in (53) with

$$S_i = Q_i + \Phi_i^T S_{i+1} \Phi_i - K_i^T (R_i + \Gamma_i^T S_{i+1} \Gamma_i) K_i \tag{59 a}$$

$$W_i = (\Phi_i - \Gamma_i K_i)^T W_{i+1} - K_i^T T_i^T + L_i^T \tag{59 b}$$

$$\begin{aligned} \alpha_i = & -(K_i^1 W_{i+1})^T (2T_i^T + \Gamma_i^T W_{i+1}) - T_i K_i^2 T_i^T \\ & + \operatorname{tr}(K_i^T (R_i + \Gamma_i^T S_{i+1} \Gamma_i) K_i P_i^c) + \operatorname{tr}(V_i S_{i+1}) + E\{\alpha_{i+1} | Y_{i-1}, U_{i-1}\} \end{aligned} \tag{59 c}$$

where $S_i \geqslant 0$, S_i and W_i deterministic, and S_i, W_i and α_i are not functions of U_{i-1}. The fact that $S_i \geqslant 0$ is obvious from the regulator case since the equations that determine the feedback, (57 *a*) and (59 *a*), are exactly the same as in the regulator case. The solution to the digital optimal tracking problem is therefore determined by (57), (58) and (59). Given (53), and considering (26), the cost (25) is given by

$$\begin{aligned} J &= E\{C_0\} + x_r^T(t_f) H x_r(t_f) + \sum_{i=0}^{N-1} \chi_i + \gamma_i \\ &= E\{x_0^T S_0 x_0 - 2x_0^T W_0\} + E\{\alpha_0\} + x_r^T(t_f) H x_r(t_f) + \sum_{i=0}^{N-1} \chi_i + \gamma_i \end{aligned} \tag{60}$$

which given (54), (59) and (1 *c*), using (32 *b*), becomes

$$\begin{aligned} J = & \bar{x}_0^T S_0 \bar{x}_0 - 2\bar{x}_0^T W_0 + x_r^T(t_f) H x_r(t_f) + \sum_{i=0}^{N-1} [\chi_i - (K_i^1 W_{i+1})^T (2T_i^T + \Gamma_i^T W_{i+1}) \\ & - T_i K_i^2 T_i^T] + \operatorname{tr}(S_0 G) + \sum_{i=0}^{N-1} \operatorname{tr}(V_i S_{i+1}) \\ & + \sum_{i=0}^{N-1} \operatorname{tr}(K_i^T (R_i + \Gamma_i^T S_{i+1} \Gamma_i) K_i P_i^c) + \sum_{i=0}^{N-1} \gamma_i \end{aligned} \tag{61}$$

The first four terms of equation (61) can be compared to the cost in the deterministic case. The remaining terms also appeared in the regulator case and were classified there.

Summarizing the solution (54), (57), . . . , (61) to the equivalent discrete-time tracking problem is given by

$$K_k = (R_k + \Gamma_k^T S_{k+1} \Gamma_k)^{-1} (\Gamma_k^T S_{k+1} \Phi_k + M_k^T) \tag{62 a}$$

$$K_k^1 = (R_k + \Gamma_k^T S_{k+1} \Gamma_k)^{-1} \Gamma_k^T \tag{62 b}$$

$$K_k^2 = (R_k + \Gamma_k^{\mathrm{T}} S_{k+1} \Gamma_k)^{-1} \quad (62c)$$

$$u_k = -K_k \hat{x}_k + K_k^1 W_k + K_k^2 T_k \quad (62d)$$

$$S_k = Q_k + \Phi_k^{\mathrm{T}} S_{k+1} \Phi_k - K_k^{\mathrm{T}} (R_k + \Gamma_k^{\mathrm{T}} S_{k+1} \Gamma_k) K_k \quad (62e)$$

$$S_N = H \quad (62f)$$

$$W_k = (\Phi_k - \Gamma_k K_k)^{\mathrm{T}} W_{k+1} - K_k^{\mathrm{T}} T_k^{\mathrm{T}} + L_k^{\mathrm{T}} \quad (62g)$$

$$W_N = H x_r(t_f) \quad (62h)$$

$$J = \bar{x}_0^{\mathrm{T}} S_0 \bar{x}_0 - 2\bar{x}_0^{\mathrm{T}} W_0 + x_r^{\mathrm{T}}(t_f) H x_r(t_f) + \operatorname{tr}(S_0 G) + \sum_{k=0}^{N-1} [\operatorname{tr}(K_k^{\mathrm{T}}(R_k + \Gamma_k^{\mathrm{T}} S_{k+1} \Gamma_k) K_k P_k^c) + \operatorname{tr}(V_k S_{k+1}) + X_k + \gamma_k - (K_k^1 W_{k+1})^{\mathrm{T}} (2T_k^{\mathrm{T}} + \Gamma_k^{\mathrm{T}} W_{k+1}) - T_k K_k^2 T_k^{\mathrm{T}}] \quad (62i)$$

where $\hat{x}_k$ is again generated by the well-known discrete-time Kalman one step ahead predictor for the discrete-time system (11). The solution matches the one in the deterministic case (Van Willigenburg 1991). If Y_{i-1} is replaced by Y_i then, as in the regulator case, $\hat{x}_i$ is generated by the Kalman filter. Clearly also the digital optimal tracker (62) is certainty equivalent.

6. Conclusions

In this paper we considered 'true' digital control problems, i.e. problems involving continuous-time criteria, which explicitly take into account the inter-sample behaviour, which relaxes the demand for a 'small' sampling time. Using stochastic dynamic programming, we have derived the digital optimal regulator and tracker for linear time-varying systems disturbed by additive white noise, where the state information at the sampling instants is incomplete and corrupted by additive white noise. Both problems appear to be certainty equivalent so the result equals the deterministic digital optimal regulator and tracker (Van Willigenburg 1991) where the state is replaced by its estimate generated by the discrete-time Kalman one step ahead predictor. The derivations in this paper were fundamentally different from the ones presented in the deterministic case. Expressions for the cost of both the digital optimal regulator and tracker have been derived, which are explicit in the system, criterion, and covariance matrices. In the deterministic case only an expression for the regulator cost was presented.

The numerical computation of the digital optimal regulator, which until now has only been considered for time-invariant systems, together with the numerical computation of the digital optimal tracker are treated in a companion paper (Van Willigenburg 1991). It is believed that the digital optimal regulator result permits for the first time the computation of a digital optimal compensator for nonlinear systems that have to track reference trajectories, e.g. a robot performing a prescribed motion or a fermentation batch process. The linearized dynamics about the trajectory in these cases constitute a time-varying system.

The digital optimal tracker for stochastic linear systems has never been considered in the literature before. This is remarkable since it can be applied in all situations where a linear continuous-time system, controlled by a digital computer, has to track a reference trajectory, e.g. a cartesian type robot performing a prescribed motion.

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