Modelling dynamic systems: BRD-20306

Lecture notes of units 1-6:

Unit 1:  People in control. Mathematical preliminaries. Chapters 1 & 2.

Unit 2:  From a sketch and science towards a mathematical model. Chapter 3.1-3.4.

Unit 3:  Writing systems models in an appropriate form: the state-space form. Chapter 3.5-3.10.


Unit 5/6:  System properties and control. Chapter 5.

Important Remark:

Although several exercises are mentioned in these lecture notes many others will be presented and treated during the lectures.

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See also:

http://www.myphysicslab.com/index.html
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1 People in control

1.1 Introduction

Mankind is dominating planet earth. Whether you like it or not, that's how it is. And we are dominating because we have gained control over many things, one important thing being agriculture. Control is all around us and actually most of our lives we are busy controlling! We are driving cars, we are raising children, we are educating, we are learning to gain control, we are managing other people and others are managing us, we are trying to control economies, we are negotiating ...

After reading this chapter the reader should:

(1) Have improved their (intuitive) notion of control, measurement and modelling
(2) Be able to draw the block diagram of a control system
(3) Know the meaning of control system design, system identification, optimal input design and adaptive control
(4) Know the difference between scientific modelling (first principles modelling), black box modelling and grey box modelling. Know the meaning of model validation
(5) Realize the importance of control in life

1.2 Questions

Why are people most successful in control? To answer this question we should ask ourselves: "What is needed to be successful in control?". To successfully drive a car what is needed? People drive cars pretty well but the control of economies often is not very successful. Why is that?

To drive a car you should 'know how it works'. If you mix up the gas, the brakes and the steering wheel ... Now most of us know all this when they drive a car for the first time ... So more seems to be needed ... Knowing perfectly how the car works, there is yet another problem: "How does the traffic work?" Is this a different problem?

Next let us focus on the control of economies. At some stages we have been quite successful at this (Keynes in the thirties). However at several occasions we faced an economical crises! Do we know how economies work?
What actually does it mean when we say: "We know how it works?". And if we know how it works, do we know how to control? Or do we need something else as well?

1.3 Answers
Let us start with "knowing how it works". Roughly what does that mean? In the case of driving a car it means that we have developed a model, usually in our brain, of what happens with the car if we push the gas, turn the steering wheel, or push the brake. To keep the example simple let us forget about the traffic rules and pretend we are alone on the road. Then is our model of the car sufficient to control it? Having this model how precisely are you going to steer? At this stage you may certainly ask: "What are you driving at?"

To know how to control the car you have to know its destination. Knowing its destination we still don't know exactly how to control the car. When for instance should we arrive at our destination, when and where do we depart? Even if we know all this, there are infinitely many ways to control the car! Roughly speaking knowing the model of the car is not enough for control. We also need to know control objectives. Knowing these there may still be several ways to control the system. To make a final decision we need either additional control objectives or a rule to decide which of the possible controls to select.

Now suppose the destination is 10 km away from where we depart and we know all of the above. Then we know precisely how to control the car. Remember we are also alone on the road. Now try to answer the following question: "Can you drive the car in this manner safely to its destination with your eyes closed?"

The answer is: "no". And the reasons are twofold. (1) the model of the car cannot be correct 100% and (2) also the control cannot be realised with an accuracy of 100%. So to drive the car safely to its destination we need to keep our eyes open. Keeping our eyes open we are actually making measurements of where we really are. These measurements must also be used to control the car properly.

1.4 A control system
Now let us generalize our example. The car we call the system that has to be controlled. Then our model of the car becomes the model of the system or the systems model. The driver of the car we call the controller of the car. This controller
uses the systems model as well as measurements (car position measurements) to control (steer, gas, brake) the car. The system (the car) together with its controller (the driver) we call the control system. Fig. 1 represents the diagram of a control system.

![Control System Diagram]

**Figure 1: Diagram of a control system**

### 1.5 The systems and control group: courses and research

Control is a major topic in life, although very often, we are unaware of this. Control system design is what the systems and control group (SCO recently changed into BRD) at Wageningen University is concerned with. Two courses on this subject are toughed by us: (1) Systems and Control Theory (BRD-31306) and (2) Control Engineering and Process Control (BRD-21306). The first course is concerned with optimal control system design (researched by Gerard van Willigenburg) while the second course is less ambitious but closer to current industrial practice (researched by Ton van Boxtel). The systems for which we design controllers range from bioreactors, wastewater treatment plants, dryers, greenhouses and crops to robots and autonomous vehicles.

In science control is not necessarily the ultimate objective. The scientific objective is often "to know and understand how it works". In this case the systems model is the important object. In Figure 1 observe that measurements are used to aid the selection of the control because the model is not accurate 100%. The measurements supply information about what is really happening (the real position of the car) which...
is different from what the model says because the model is not accurate 100%. Therefore the measurements can also be used to improve the model. If the measurements entering the controller are also used to improve the model the controller is called an adaptive controller, because the model is adapted, see Figure 2 (adaptive controllers are researched by Gerard van Willigenburg). When adapting the model we have to keep in mind that also measurements are not exact 100%.

![Figure 2: An adaptive control system](image)

If modelling instead of control is the objective, then improving the model by means of employing the measurements is called system identification, see Figure 3 (researched by Karel Keesman, Hans Stigter and Gerrit van Straten). A course on this topic is called System identification and model predictive control (BRD-31806).

Usually the information content of the measurements depends on the control of the system. For example if we control the car without turning the steering wheel the measurements will contain no information about the influence of steering on the car movement. So an interesting question that arises is: "Which control maximizes the information content of the measurements?" This question is raised and answered by what is called optimal input design, see Figure 3 (researched by Karel Keesman and Hans Stigter).
Having built a mathematical model another important question is: "How good is the model?". Measurements may be used to verify or validate the model by observing how well they correspond with the outcome of the model. This is called model validation which is part of system identification. It is well known that to validate a model properly measurements different from the ones used to build the model have to be used.

The model of the car resides in our brain and is probably not mathematical. However mathematics is the language of science and computing while scientific knowledge, like chemical, physical and mechanical laws, form the basis for building models. Therefore control system design requires a mathematical model of the system and a mathematical description of the control objectives. The chemical, physical and mechanical laws are sometimes referred to as first principles and the associated modelling as first principles modelling. If we build a mathematical model of the car we exploit several known laws from mechanics. Another type of mathematical modelling, called black box modelling ignores totally all scientific knowledge and builds the model entirely from measurements. So called grey box modelling combines first principles and black box modelling. In this first and introductory course toughed by the systems and control group entitled modelling dynamic systems (BRD-20306) we focus on building mathematical models using scientific knowledge (first principles) and measurements (measurement data). However given the importance of control in real life we very much recommend to study the control system design courses.

Figure 3: System identification / optimal input design
(1) Systems and Control Theory (BRD-31306)
(2) Control Engineering and Process Control (BRD-21306)
(3) Parameter Estimation and Model Structure Identification (BRD-31806)

They rely very much on the contents of this course.

Figure 4: Courses of the BRD (formerly Systems & Control) group

1.6 Summary
1) Control is an important part of our lives although we usually do not realise this.
2) Controllers require a model of the system as well as measurements.
3) Control system design requires a mathematical model of the system and a mathematical description of the control objectives.
4) A controller is called adaptive if the measurements are not just used to decide on the control but are also used to improve the model. System identification is not concerned with control but only with improving the model using measurement data. Optimal input design is concerned with finding the best control for system identification.
5) Mathematical models are built from scientific knowledge (laws) and/or measurement data. Scientific modelling (first principles modelling) is highly based on scientific knowledge (first principles like physical, chemical or
mechanical laws). Black box models ignore totally all scientific knowledge and are entirely build from measurement data. Grey box models are in between these two. Building a mathematical model is the subject of system identification. Model validation is an important part of system identification.

2 Mathematical preliminaries: Taylor series expansions, linearisation, complex numbers and matrices

2.1 Introduction
Building mathematical models, using them to compute system behaviour, and interpreting the results, are the main issues of this course. The mathematics needed to perform these tasks is summarized in this section. The four mathematical subjects mentioned in the title are the most important ones needed for this course. They will be reviewed in this chapter through several examples.

After reading this chapter the student

(1) Is able to reproduce and use Taylor series expansions

(2) Is able to understand and make computations with complex numbers

(3) Is able to understand and make calculations with matrices

2.2 Did you know …

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]  \hspace{1cm} (2.1)

\[ j^2 = -1 \]  \hspace{1cm} (2.2)

\[ e^{j\phi} = \cos(\phi) + j\sin(\phi) \]  \hspace{1cm} (2.3)

Equation (2.1) is a special case of what is called a Taylor series expansion. The formula is actually used to calculate \( e^x \) in calculators and computers because the series \( 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \) converges for all values of \( x \). Can you see why?

Equation (2.2) defines the so called imaginary number \( j \). Alternatively we may define,
The imaginary number $j$ appears in equation (2.3). The number on the right hand side of the equality sign is a so called **complex number** which has a **real part** namely $\cos(\phi)$, a real number, and an **imaginary part** $\sin(\phi)$, because $\sin(\phi)$ is multiplied with the imaginary number $j$. To prove that equation (2.3) holds we need to know about Taylor series expansions. Equation (2.3) is important to understand the properties of complex numbers. Complex numbers and Taylor series expansions themselves turn out to be very important to analyse mathematical systems models.

### 2.3 Taylor series expansions

The following equation represents **Taylor series expansions** of the infinitely differentiable function $f(x)$,

$$
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + ... \quad (2.5)
$$

The equation may be interpreted as follows. Knowing the value of the function and all its derivatives for one value $x = x_0$ we can compute the function values for arbitrary values $x$. This is a rather remarkable property of infinitely differentiable functions $f(x)$. We will not prove that equation (2.5) holds. However for a simple example you can check it. Take $f(x) = x^2$. So $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$. Also take $x_0 = 1$, $x = 3$. Then equation (2.5) reads,

$$
3^2 = 1^2 + \frac{2}{1!} + \frac{2}{2!} 2^2 + 0 \quad (2.6)
$$

For any value of $x_0$ and $x$ equation (2.5) must be satisfied. Verify this by taking some other values. Equation (2.5) is satisfied for any infinitely differentiable function. Try another second order polynomial function.
Equation (2.1) is obtained from equation (2.5) by taking \( f(x) = e^x \), \( x_0 = 0 \). Check this. From equation (2.1) taking \( x = j\phi \) we obtain,

\[
e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{j^2\phi^2}{2!} + \frac{j^3\phi^3}{3!} + \ldots \tag{2.7}
\]

Using equation (2.2) this becomes,

\[
e^{j\phi} = 1 + j\phi - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + j\frac{\phi^4}{4!} + j\frac{\phi^5}{5!} - \ldots \tag{2.8}
\]

which after rearrangement of the terms on the right hand side of the equality sign becomes,

\[
e^{j\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \ldots\right) + j\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \ldots\right) \tag{2.9}
\]

The first term in between brackets in equation (2.9) is a Taylor series expansion of \( \cos(\phi) \) while the second term in between the brackets is a Taylor series expansion of \( \sin(\phi) \) (verify this using equation (2.5)). So equation (2.9) is identical to equation (2.3)! The only equations we have used to obtain this result are (2.2) and (2.5)!!

### 2.4 Linearisation and linear approximations of functions

Consider equation (2.5) again. Suppose that \( x - x_0 \) is small. Mathematically speaking, small values of \( x - x_0 \) define a so called neighbourhood of \( x_0 \). When \( x - x_0 \) is small from equations (2.5) \( f(x) \) may be approximated by,

\[
f(x) \approx f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) \tag{2.10}
\]

because the other terms in equation (2.5) are negligible. The approximation (2.10) may be interpreted as follows, see figure 2.
Near \( x = x_0 \) the function \( f(x) \) may be very well approximated by the straight line described by the right hand side of equation (2.10).

What is so interesting about this approximation? Well, the approximation is a linear function of \( x \). And linear functions are much more easy to analyse and calculate than non linear functions. Later in this course we shall see that the same holds for linear systems versus non-linear systems. The straight line described by equation (2.10) is called a linearisation of the function \( f(x) \) at \( x = x_0 \). This linearisation is an accurate approximation of the function \( f(x) \) itself in the neighbourhood of \( x_0 \). Later in this course we will compute linearisations of non linear systems.

### 2.5 Complex numbers

Imaginary and complex numbers are very useful to analyse linear systems, as we shall see later in this course. In paragraph 2.2 we already introduced imaginary numbers and complex numbers which are different from the real numbers that we are familiar with. The set of all complex numbers is denoted by \( C \) and is given by,

\[
C = \{a + jb \mid a \in R, b \in R\}, \quad j^2 = -1 \quad (j = \sqrt{-1})
\]  

(2.11)
Of the complex numbers $a + jb$ in (2.11) $a$ is called the real part of the complex number and $b$ the imaginary part, which is the part that multiplies $j$. The set of all imaginary numbers is denoted by $I$ and is given by,

$$I = \{jb \mid b \in R\}, \quad j^2 = -1 \quad \left( j = \sqrt{-1} \right)$$

(2.12)

Clearly the set of complex numbers $C$ contains the set of imaginary numbers $I$ and also the set of real numbers $R$,

$$C \supset I, \quad C \supset R$$

(2.13)

We will use complex numbers to make computations which are useful to analyse linear systems. The rules of computation are identical to the rules we know for real numbers. The only two additional things we have to be aware of are the following. (1) any complex number always separates into a real part, which may be zero, and an imaginary part, which may also be zero. In other words we cannot and may not mix real parts with imaginary parts. So for instance we cannot add the real part to the imaginary part of a complex number. (2) when making computations we have to realize that $j^2 = -1$. A few examples of computations with complex numbers are listed below.

\[
(3 + 4j) + (1 - 2j) = 3 + 4j - 2j = 4 + 2j
\]

(2.14)

\[
(3 + 4j) - (1 - 2j) = 3 - 1 + 4j + 2j = 2 + 6j
\]

(2.15)

\[
(3 + 4j)(1 - 2j) = 3 - 6j + 4j - 8j^2 = 3 - 2j - 8j^2 = 3 - 2j + 8 = 11 - 2j
\]

(2.16)

When dividing two complex numbers, although the rules do not change, we need a 'trick' to obtain the real and imaginary part of the answer,

\[
\frac{(3 + 4j)}{(1 - 2j)} = \frac{(3 + 4j)(1 + 2j)}{(1 - 2j)(1 + 2j)} = \frac{-5 + 10j}{5} = -1 + 2j
\]

(2.17)
The 'trick' is to multiply the numerator and the denominator in (2.17) by \((1+2j)\). Then after multiplying out the denominator it becomes a complex number with a zero imaginary part. This is needed to specify the real and imaginary part of the answer. Now the question is: "How do we know we have to multiply with \((1+2j)\)?". The complex number \(1+2j\) is the so called **complex conjugate** of the complex number \(1-2j\) in the denominator. The complex conjugate of a complex number is the same number except for the sign of the imaginary part, which is reversed. Multiplying a complex number with its complex conjugate gives a complex number with a zero imaginary part,

\[
(a + jb)(a - jb) = a^2 - ajb + ajb - j^2b^2 = a^2 + b^2
\]  

and that is precisely what we need in (2.17). How can you verify in (2.17) that \(-1+2j\) is the correct answer?

Complex numbers \(a + jb\) can be represented graphically in what is called the **complex plane**, see Figure 6.

![Figure 6: The complex plane with the real axis Re and the imaginary axis Im](image)

Next we turn our attention to the complex numbers, given by equation (2.3),

\[
e^{j\phi} = \cos(\phi) + j\sin(\phi)
\]  

(2.19)
From the right hand side of (2.19) observe that these numbers lie on the unit circle of the complex plane, see Figure 7.

![Figure 7: Complex numbers on the unit circle](image)

Combining Figure 6 and Figure 7 every complex number can be written in two ways,

![Figure 8: Two ways to write complex numbers](image)

The second way to write a complex number according to Figure 8 is $A e^{j\phi} = A \cos(\phi) + jA \sin(\phi)$. This notation determines the complex number through $A$ and $\phi$. $A$ is called the absolute value of the complex number, and equals the distance between the complex number and the origin in the complex plane. $\phi$ is called the argument of the complex number. It is the angle that the line connecting the complex number with the origin makes with the real axis. Knowing the real part $a$
and the imaginary part $b$ of a complex number, according to Figure 8, we can compute its absolute value $A$ and argument $\phi$ as follows,

$$A = \sqrt{a^2 + b^2}, \ \phi = \arctan\left(\frac{b}{a}\right)$$

(2.20)

On the other hand from $A, \phi$ we can compute $a, b$ as follows (again see Figure 8),

$$a = A\cos(\phi), \ b = A\sin(\phi)$$

(2.21)

The multiplication of complex number has an interesting graphical interpretation. This can be seen if we write complex numbers using their absolute value and argument,

$$A_1e^{j\phi_1}A_2e^{j\phi_2} = A_1A_2e^{j(\phi_1+\phi_2)}$$

(2.22)

So multiplication of complex numbers comes down to multiplying their absolute values and adding up their arguments. The graphical representation of this is represented by Figure 9.

![Figure 9: Graphical representation of the multiplication of complex numbers](image)

2.6 Matrices
See module 2.
3 From a sketch and science towards a mathematical model of a dynamic system

3.1 Introduction
In chapter 1 we explained roughly what is a system and a mathematical model of a system. Looking at the title of this chapter there is one questions left. What is a dynamic system? It is a system with "memory" meaning that its future behaviour depends on the current content of its "memory". For instance the current position and velocity of a car determine the future position and velocity of the car so position and velocity are the "memory" of the car. The "memory" may also be called the "state". The current state determines future states. On the other hand the current "state" ("memory" content) depends on past states. So one could say that the current "state" ("memory" content) is what separates the future from the past. From a mathematical modelling point of view a dynamic system is a system described by differential equations. Actually almost any phenomenon that occurs in nature is dynamical; it has a "memory" (a "state") and is mathematically described by differential equations. Therefore building mathematical models of dynamic systems has a large potential.

After reading this chapter the student

1. Is able to make simple drawings of simple systems
2. Is able to derive differential equations from these drawings and first principles, like physical, chemical or mechanical laws
3. Is able to derive differential equations from verbal system descriptions and first principles, like physical, chemical or mechanical laws
4. Is able to put the differential equations in state-space form

3.2 A simple example: the temperature in a greenhouse
Suppose you are an engineer who is asked to select a proper heating system for a greenhouse. In addition the greenhouse climate control system manufacturer asks
you to deliver to him a computer program to predict the temperature in the greenhouse when the heating system is turned on. What do you do?

You build a mathematical model of the greenhouse. The greenhouse is a dynamic system the greenhouse temperature being the memory. The future greenhouse temperature depends on the current greenhouse temperature (the current memory content) which in turn depends on the past greenhouse temperature.

To build a mathematical model it is usually very helpful to start by making a simple drawing of the system, see Figure 10.

![Figure 10: A simple drawing of the system](image-url)

The next step towards building a mathematical model is by putting relevant phenomena, associated quantities and corresponding mathematical symbols in the drawing.

![Figure 11: Adding relevant processes, quantities and mathematical symbols](image-url)

Next applying scientific knowledge (first principles), in this case physical laws describing the heat transport, leads to the following differential equation,
\[ \frac{dT_g}{dt} = c_1 (T_o - T_g) + c_2 H \]  

(3.1)

In equation (3.1) \( c_1, c_2 \) are two constants that depend on several physical properties of the system. The constant \( c_1 \) depends on the volume of the greenhouse, the heat transfer coefficient and surface of the walls of the greenhouse and the heat capacity of the air inside the greenhouse. In the next section we will be more explicit about how to obtain exactly, from physical laws, these constants.

Equation (3.1) is a simple mathematical model of the greenhouse. To interpret the model the meaning and units of all the quantities appearing in the model must be specified. Without this specification the mathematical model is incomplete. The names and units are also determined by scientific knowledge (first principles) and the units, which can be selected in different ways, should be consistent. In the case of equation (3.1) the following names and units apply,

\[
\begin{align*}
T_g & \quad \text{Greenhouse temperature} \quad \left[ ^\circ C \right] \\
T_o & \quad \text{Outside temperature} \quad \left[ ^\circ C \right] \\
\frac{dT_g}{dt} & \quad \text{Greenhouse temperature change} \quad \left[ ^\circ C s^{-1} \right] \\
H & \quad \text{Heating power} \quad \left[ W = Js^{-1} \right] \\
c_1 & \quad \text{Constant related to heat transfer} \quad \left[ s^{-1} \right] \\
c_2 & \quad \text{Constant related to heating} \quad \left[ ^\circ C J^{-1} \right]
\end{align*}
\]

The consistency of the units is checked as follows,

\[
\frac{dT_g}{dt} = c_1 (T_o - T_g) + c_2 H \\
\left[ ^\circ C s^{-1} \right] = \left[ s^{-1} \right] \left[ ^\circ C \right] + \left[ ^\circ C J^{-1} \right] \left[ Js^{-1} \right]
\]

(3.2)
Having the mathematical model (3.1) with its associated description and units we can start to address the problem of selecting an appropriate heating system for the greenhouse and the computation of the greenhouse temperature.

To select the heating system the most important thing is the maximum heating power. To compute it we can use the mathematical model (3.1) with its associated description and units. However we also need to know the worst case. The required heating power is maximum when the outside temperature $T_o$ is the lowest and the greenhouse temperature $T_g$ is the highest. Suppose the lowest outside temperature we want to accommodate for is $-20^\circ C$ and that in this case we want the greenhouse temperature not to go below $+5^\circ C$. Then from the mathematical model (3.1) with its associated description and units we can compute the maximum heating power required. To do this we must consider the situation where the outside temperature and the greenhouse temperature remain constant. This implies,

$$\frac{dT_g}{dt} = 0, \quad T_g = +5^\circ C, \quad T_o = -20^\circ C \Rightarrow H = \frac{c_1}{c_2} 25 \ W \quad (3.3)$$

Of course there are usually other considerations, such as the price, to select the heating system. We will not consider these because our primary objective here is to show the importance of the mathematical model in making the selection.

The second issue, computing the temperature when we know the heating power, also requires the mathematical model (3.1). Our system, described by equation (3.1), is a dynamical system because it is described by a differential equation. Remember that we stated that a dynamical system has a memory and to compute its future behaviour we need to know the current contents of its memory. The memory of our dynamic system is the greenhouse temperature $T_g$. Say we start the computation at time $t_0$ then we need to know $T_g(t_0)$. Also we need to know the heating power $H(t), \ t \geq t_0$ where $t$ denotes time. To compute the greenhouse temperature $T_g(t), \ t \geq t_0$ we furthermore need to know the outside temperature $T_o(t), \ t \geq t_0$. To see how we compute the greenhouse temperature $T_g(t), \ t \geq t_0$ from the mathematical model we need to consider a Taylor series expansion of $T_g(t)$, 

21
\[
T_g(t) = T_g(t_0) + \frac{T'_g(t_0)}{1!}(t-t_0) + \frac{T''_g(t_0)}{2!}(t-t_0)^2 + \ldots 
\]  
(3.4)

Observe that \(T'_g(t_0) = \left. \frac{dT_g}{dt} \right|_{t_0} \) and using the mathematical model (3.1),

\[
\left. \frac{dT_g}{dt} \right|_{t_0} = c_1 \left( T_o(t_0) - T_g(t_0) \right) + c_2 H(t_0) 
\]  
(3.5)

The right hand side of equation (3.5) is entirely known and therefore \(T'_g(t_0) = \left. \frac{dT_g}{dt} \right|_{t_0} \) in equation (3.4) may be computed. Now if \(t - t_0\) in equation (3.4) is small the higher order terms \(\frac{T''_g(t_0)}{2!}(t-t_0)^2 + \ldots\) may be neglected. As we have seen in paragraph 2.4 this comes down to approximating the function locally by a straight line, see Figure 5. Using this approximation we may compute \(T_g(t)\) for some value \(t = t_0 + \Delta t\), where \(\Delta t\) is small, as follows,

\[
T_g(t_0 + \Delta t) \approx T_g(t_0) + \left. \frac{dT_g}{dt} \right|_{t_0} \Delta t 
\]  
(3.6)

Equation (3.6) is graphically represented by Figure 12.

\[\text{Figure 12: Approximating } T_g(t_0 + \Delta t) \text{ by } T_g(t_0) + \left. \frac{dT_g}{dt} \right|_{t_0} \Delta t\]
Next we can compute,

\[ T_g(t_0 + 2\Delta t) = T_g(t_0 + \Delta t) + \left. \frac{dT_g}{dt} \right|_{t_0 + \Delta t} \Delta t \]  

(3.7)

Verify that indeed we can compute the right hand side of equation (3.7) using the mathematical model (3.1). Then it is easy to see that we can compute \( T_g(t_0 + k\Delta t) \), \( k = 1, 2, 3, \ldots \) recursively as follows,

\[ T_g(t_0 + k\Delta t) = T_g(t_0 + (k-1)\Delta t) + \left. \frac{dT_g}{dt} \right|_{t_0 + (k-1)\Delta t} \Delta t, \quad k = 1, 2, 3, \ldots \]  

(3.8)

There is a trade off in the selection of \( \Delta t \). Taking it smaller increases the accuracy of the computation, see Figure 12, but at the expense of computation time because more computational steps are required over a fixed time interval \( t_0 \leq t \leq t_f \). Making \( \Delta t \) smaller increases the accuracy up to a certain point were rounding errors of computer computations start to dominate the computation. Then the accuracy decreases again.

The recursive procedure to compute \( T_g(t) \), \( t = t_0 + k\Delta t \), \( k = 1, 2, 3, \ldots \) is called numerical integration. The reason is that to obtain \( T_g(t) \), \( t \geq t_0 \) which is the solution of the differential equation (3.1) we have to integrate this differential equation. The solution consists of numerical values \( T_g(t_0 + k\Delta t) \) and is called a numerical solution of the differential equation. On the other hand if we can find an analytical function description of \( T_g(t) \), \( t \geq t_0 \), for example,

\[ T_g(t) = c_1 \sin(t - t_0)T_o(t) + c_2 \left( H(t) - H(t_0) \right) + T_g(t_0) \]  

(3.9)

this is called an analytical solution of the differential equation (equation (3.9) is meant as an example it is not actually the solution of the differential equation (3.1)). Analytical solutions are to be preferred because they can be evaluated more easy and quickly and they provide more insight. In equation (3.9), if we change \( T_g(t_0) \) or
\( H(t) \), we can see immediately how this affects the solution. Using numerical integration we can only recompute the solution for different values \( T_g(t_0) \), but the result and changes are not a-priori obvious and have to be recomputed for every value \( T_g(t_0) \). Unfortunately for most differential equations analytical solutions cannot be obtained. Luckily for many applications, such as choosing an appropriate heating system and computing the greenhouse temperature, numerical solutions, which can always be obtained, suffice!

More advanced numerical integration procedures (ode23, ode45 in Matlab) are commonly used in practice. They approximate in some way part of the higher order terms in the Taylor series expansion which we have neglected. As a result of this the time steps \( \Delta t \) can be chosen larger and the accuracy and computational efficiency increases. The numerical integration method presented in this section is called explicit Euler numerical integration. It is well known for its simplicity and numerical stability.

### 3.3 Obtaining mathematical models from first principles systematically

The way in which we obtained the mathematical model (3.1) of the greenhouse from first principles, was not entirely transparent or systematic. In this paragraph we will derive this model in a systematic way and furthermore consider a second example.

A systematic way to obtain a mathematical model from first principles is by means of a so called balance equation. The balance equation is represented below, followed by a simplified version,

\[
\begin{bmatrix}
\text{Rate of mass or energy or momentum or electrical charge accumulation in the system} \\
\text{Rate of mass or energy or momentum or electrical charge entering the system}
\end{bmatrix} = \begin{bmatrix}
\text{Rate of mass or energy or momentum or electrical charge leaving the system}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{Rate of change}
\end{bmatrix} = \begin{bmatrix}
\text{Rate of input}
\end{bmatrix} - \begin{bmatrix}
\text{Rate of output}
\end{bmatrix}
\]  

(3.10) 

(3.11)
The balance equation represents **fundamental general scientific principles** from which we can derive laws and formulas that apply to specific systems, as we shall see. It is important to note that equations (3.10), (3.11) are formulated in terms of **rates**. Rates are changes per unit of time.

**Example 1: The greenhouse**

For the greenhouse equation (3.10) applies where the energy is heat energy. So we formulate our mathematical model in terms of heat energy rates. From first principles the heat energy in the greenhouse equals,

$$\rho c_p VT_g$$  \hspace{1cm} (3.12)

where,

- $\rho$ greenhouse air density $[kgm^{-3}]$
- $c_p$ greenhouse air heat capacity $[J(kg\,^oC)^{-1}] = [J\,kg^{-1}\, ^oC^{-1}]$
- $V$ greenhouse volume $[m^3]$
- $T_g$ greenhouse air temperature $[^oC]$

Therefore,

$$\left[ \text{Rate of heat energy accumulation in the greenhouse} \right] = \frac{d \rho c_p VT_g}{dt}$$  \hspace{1cm} (3.13)

Furthermore,

$$\left[ \text{Rate of heat energy input in the greenhouse} \right] = H$$  \hspace{1cm} (3.14)

$$\left[ \text{Rate of heat energy leaving the greenhouse} \right] = h_w s_w (T_g - T_o)$$  \hspace{1cm} (3.15)

where,
**H** Effective heating power \[ W = Js^{-1} \]

**\( h_w \)** heat transfer coefficient of the walls \[ J \left( \frac{^\circ C}{m^2 \cdot s} \right)^{-1} = J \frac{^\circ C}{m^2 \cdot s^{-1}} \]

**\( s_w \)** total surface of the walls \[ m^2 \]

**\( T_o \)** outside temperature \[ ^\circ C \]

From equations (3.13)-(3.15), after dividing each equation by \( \rho c_p V \) which is a constant number, we obtain the following mathematical model including the associated unit check.

\[
\frac{dT_g}{dt} = \frac{H}{\rho c_p V} - \frac{h_w s_w (T_g - T_o)}{\rho c_p V} \quad (3.16)
\]

In equation (3.16) the unit of \( \frac{1}{\rho c_p V} \) is written directly as \([J^{-1} \cdot ^\circ C]\). This unit is easily obtained from the units of \( \rho, c_p, V \) mentioned previously. Comparing the mathematical models (3.16) and (3.2) we see that indeed they are identical and,

\[
c_1 = \frac{h_w s_w}{\rho c_p V}, \quad c_2 = \frac{1}{\rho c_p V} \quad (3.17)
\]

The model (3.16) which is derived in a systematic manner, is to be preferred over (3.2) because it is build entirely out of known physical quantities. Observe that we require (3.16) to obtain (3.17) which is needed to specify the constants in the model (3.2).

**Example 2: Newton’s law**

Consider a mass with a force acting on it. The force may vary but is always pointing in the same direction. Our aim is to determine the motion of the mass. Again to build a mathematical model we start to make a drawing of the system including relevant processes, quantities and mathematical symbols.
Figure 13: The system; a mass with a force acting on it

Is this a dynamic system? Yes because the future position depends on the current position and so the position is (part?) of the memory of the system. The systematic way to obtain a mathematical model of this system is to use a balance equation and first principles. But actually you should be familiar with the mathematical model of this system. It is Newton’s law,

\[ F = m \cdot a \]

\((3.18)\)

Although we have a dynamic system the mathematical model \((3.18)\) doesn’t look like a differential equation. However it actually is a differential equation. To see this observe the relationship between the position \(s\) in Figure 13 and \(a\) in equation \((3.18)\).

\[ a = \frac{d^2s}{dt^2} \]

\((3.19)\)

Using \((3.19)\) equation \((3.18)\) becomes,

\[ \frac{d^2s}{dt^2} = \frac{F}{m} \]

\((3.20)\)

Next let us derive the mathematical model \((3.20)\) from the balance equation \((3.10)\) and first principles. The energy in equation \((3.10)\) in our system is kinetic energy associated to the motion of the mass. Mind you the derivation is a bit complicated. That’s one of the reasons why Newton became famous! Consider the labour \(A\) associated to the movement of a constant force \(F\) over a distance \(s\),
\[ A = F \quad s \quad \text{[kgm}^2\text{s}^{-2}] = [\text{kgms}^{-2}] [m] \quad (3.21) \]

The labour \( A \) in our system is used to increase the kinetic energy \( E_k \) of the mass,

\[ \Delta E_k = A = Fs \quad (3.22) \]

However in our system the force is not necessarily constant. Let \( \delta s \) denote an increment (that is a very small change) of the position \( s \). During this increment the force is (approximately) constant and the associated kinetic energy increment \( \delta E_k \), according to equation (3.22), equals,

\[ \delta E_k = Fs \delta s \quad (3.23) \]

Let \( \delta t \) denote the time increment during which the increments \( \delta s \) and \( \delta E_k \) occur. Then from equation (3.23) we obtain,

\[ \frac{\delta E_k}{\delta t} = F \frac{\delta s}{\delta t} \quad (3.24) \]

and in the limit \( \delta t \to 0 \) this becomes,

\[ \text{[Rate of kinetic energy change]} = \frac{dE_k}{dt} = F \frac{ds}{dt} = Fv = \text{[Kinetic energy input rate]} \quad (3.25) \]

where \( v = \frac{ds}{dt} \) is the velocity.

On the other hand from first principles we know,

\[ E_k = \frac{1}{2}mv^2 \quad (3.26) \]

Differentiating (3.26) with respect to time by applying the chain rule gives,
Combining equations (3.25) and (3.27) we obtain,

\[
\text{[Rate kin. en. change]} = m\nu \frac{d\nu}{dt} = F = \text{[Kin. en. input rate]} \Rightarrow F = m\nu \frac{d\nu}{dt} = ma \quad (3.28)
\]

which is Newton’s law (3.18) from which we obtain our mathematical model (3.20) again.

### 3.4 Representing models of dynamic systems in the state space form

**Example 1**

Reconsider the system represented by Figure 13 and its mathematical model (3.20),

\[
\frac{d^2 s}{dt^2} = \frac{F}{m} \quad \left[ ms^{-2} \right] = \left[ kg \, ms^{-2} \right] \left[ kg^{-1} \right] \quad (3.29)
\]

Now suppose we want to compute the position \( s(t), t \geq t_0 \) from our mathematical model. What do we need to know? Of course we need to know "where we start" that is the initial position \( s(t_0) \). Clearly we also need to know \( F(t), t \geq t_0 \) the force acting on the mass. Is this enough? No. Hopefully you remember from your physics lessons that to compute the motion of a mass, knowing all the forces acting on it, you need to know both the initial position \( s(t_0) \) and the initial velocity \( s'(t_0) = \frac{ds}{dt} \bigg|_{t=t_0} \). What you also learned is an analytical solution of the differential equation (3.29) which holds if the force is constant: \( F(t) = F \). Then \( a = \frac{F}{m} \) is also constant and,

\[
s(t) = s(t_0) + s'(t_0)(t-t_0) + \frac{1}{2} a(t-t_0)^2, \quad t \geq t_0 \quad (3.30)
\]

If the force is not constant we cannot find an analytical solution and we can only calculate numerical solutions. In paragraph 3.2 we learned how to numerically
integrate the differential equation (3.2). This differential equation is a so called first-order differential equation because the first derivative is the highest one appearing in this differential equation. This first derivative could be used to approximate locally the function we want to calculate by a straight line, see Figure 5. This local approximation by a straight line forms the basis for numerical integration. However, now we have the second-order differential equation (3.29) from which we cannot obtain first derivatives to approximate the function \( s(t) \). To resolve this problem we will write higher-order differential equations as a set of first-order differential equations. This, we shall see, is always possible. After we have done this we can numerically integrate all the first-order differential equations in the same manner as equation (3.2) in paragraph 3.2.

The procedure to obtain the first-order differential equations is as follows. This procedure also applies to sets of coupled higher order differential equations.

(1) Introduce so called state variables. A state variable is a variable of which derivatives appear in the differential equation(s). In the case of (3.29) \( s \) is a state variable. However all time derivatives of \( s \) up to the highest one appearing in the differential equation(s) are also state variables. So in the case of (3.29) \( \dot{s} = \frac{ds}{dt} \) is also a state variable because the highest derivative appearing in the differential equation(s) is the second derivative \( s'' = \frac{d^2s}{dt^2} \).

State variables we will denote by \( x_1, x_2, \ldots \) so in the case of (3.29), \( x_1 = s, x_2 = \frac{ds}{dt} \).

(2) Using the state variables write each original \( n^{th} \)-order differential equation as a set of \( n \) first-order differential equations with the first-derivative of each equation appearing solemnly and explicitly on the left hand side. In the case of our example these two differential equations are,

\[
\frac{dx_1}{dt} = x_2
\]

(3.31)

\[
\frac{dx_2}{dt} = \frac{F}{m}
\]

(3.32)
From equations (3.31), (3.32) it is immediately obvious now that to compute \(x_1(t), x_2(t), t \geq t_0\) we require both \(x_1(t_0)\) and \(x_2(t_0)\) the initial value of both position and speed. State variables of dynamic systems models constitute precisely the "memory" or "state" of the dynamic system. Furthermore the system behaviour equals the memory content as a function of time (the state response), that is \(x_1(t), x_2(t), t \geq t_0\) for the systems model (3.31), (3.32). Apart from state variables, systems models written in state space form contain input variables (inputs) denoted by \(u_1, u_2, \ldots\) and parameters denoted by \(p_1, p_2, \ldots\).

Parameters in a systems model are all the constants in the model. For the systems model (3.29) \(p_1 = m\). Input variables are the variables that are left over. For the systems model (3.29) we obtain \(u_1 = F\). Input variables influence the system behaviour, but they are not influenced by, or part of, the system behaviour. The state-space form of the systems model (3.29) equals,

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= \frac{u_1}{p_1}
\end{align*}
\]  

(3.33)

(3.34)

**Example 2: A set of higher-order differential equations written in state space form**

Consider the following set of coupled higher-order differential equations,

\[
\begin{align*}
\frac{d^3 p}{dt^3} + c_1 \frac{d^2 q}{dt^2} + r + s &= 0 \\
\frac{d^2 q}{dt^2} &= c_2 \frac{d^2 p}{dt^2} + q + c_3 s
\end{align*}
\]

where \(c_1, c_2, c_3\) are constants. To write the set of differential equations (3.35), (3.36) in state space form we first have to recognise, respectively, the state variables, the
parameters and the inputs. To determine the state variables note that the highest
derivative of \( p \) that appears is \( \frac{d^3 p}{dt^3} \) so,

\[
x_1 = p, \quad x_2 = \frac{dp}{dt}, \quad x_3 = \frac{d^2 p}{dt^2}
\]

are state variables. Similarly, since the highest derivative of \( q \) that appears is \( \frac{d^2 q}{dt^2} \),

\[
x_4 = q, \quad x_5 = \frac{dq}{dt}
\]

are also state variables. To decide about the parameters and inputs in equations
(3.35), (3.36) the constants in the model must be known. The constants are \( c_1, c_2, c_3 \).
Therefore the parameters are,

\[
p_1 = c_1, \quad p_2 = c_2, \quad p_3 = c_3
\]

and the input variables (inputs) are,

\[
u_1 = r, \quad u_2 = s.
\]

Knowing the state variables, the parameters and the inputs the systems model (3.35)
(3.36) can be written in state space form,

\[
\frac{dx_1}{dt} = x_2 \tag{3.37}
\]

\[
\frac{dx_2}{dt} = x_3 \tag{3.38}
\]

\[
\frac{dx_3}{dt} = -p_1 \left( p_2 x_3 + x_4 + p_3 u_2 \right) - u_1 - u_2 \tag{3.39}
\]
\[
\frac{dx_4}{dt} = x_3 \tag{3.40}
\]
\[
\frac{dx_5}{dt} = p_2 x_3 + x_4 + p_3 u_2 \tag{3.41}
\]

Equations (3.37), (3.38), (3.40) are obtained almost trivially. Equation (3.41) is obtained directly from (3.36). Equation (3.39) is obtained from (3.35) using (3.41). To see this observe that we may rewrite equation (3.35),

\[
\frac{d^3 p}{dt^3} = -c_1 \frac{d^2 q}{dt^2} - r - s \tag{3.42}
\]

In equation (3.42) the problem is that \( \frac{d^2 q}{dt^2} \) is not a state variable but the first derivative of the state variable \( \frac{dq}{dt} = x_3 \). Therefore, to obtain a first-order differential equation with the first derivative appearing solemnly and explicitly on the left hand side, we need to substitute the right hand side of equation (3.41) for \( \frac{d^2 q}{dt^2} \) in (3.42). In this way we obtain (3.39).

The procedure outlined in this paragraph to write systems described by differential equations in state-space form cannot be applied to any set of coupled differential equations. The procedure may only be applied if the differential equations do not contain any derivatives of inputs to the system. To investigate this we first have to determine the input variables using the underlined definition at the bottom of page 31. Fortunately the differential equations obtained from first principles usually do not contain such derivatives. To help you recognise differential equations which do contain such derivatives here is an example, where \( r \) is an input to the system.

\[
\frac{d^2 s}{dt^2} + s = r + \frac{dr}{dt} \tag{3.43}
\]
For a thorough mathematical treatment of the representation of systems in state-space form and by differential equations we refer to the book: "Introduction to mathematical systems theory" by Jan Willem Polderman and Jan C. Willems.

3.5 The importance of writing dynamic systems models in state-space form

In the previous section we have actually witnessed several reasons why it is important to write systems in state-space form.

(1) Only models represented in state-space form are suitable for numerical integration for which standard software is available.

(2) The systems memory may be identified with the systems state variables.

(3) The system behaviour is identical to the contents of the systems memory as a function of time which is identical to the time response of the state variables. System behaviour is computed by numerical integration of the state-space model.

(4) Cause and effect are clearly distinguished in systems models written in state-space form. The cause being the inputs and the initial values of the state variables, the effect being the systems behaviour.

(5) Inputs, state variables and parameters, which are fundamentally different, are distinguished in the state-space form.

3.6 Writing systems models in state-space form using vector notation: the state equation

To simplify the state-space notation the state variables $x_1, x_2, \ldots, x_n$ are collected in the column vector $x$,

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,
$$

(3.44)

The vector $x$ is called the state vector or state of the systems model. The number of state variables, denoted by $n_x$, is called the dimension of the systems model.

Similarly the inputs are collected in the column vector,
The vector $u$ is called the input vector or input of the systems model. The parameters are collected in the column vector,

$$ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n. \quad (3.45) $$

The vector $p$ is called the parameter vector of the systems model. Then the general state-space form of the systems model becomes,

$$ \frac{dx}{dt} = f(x,u,p) \quad (3.47) $$

and is called the state equation where,

$$ \frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} \in \mathbb{R}^n, \quad f(x,u,p) = \begin{bmatrix} f_1(x,u,p) \\ f_2(x,u,p) \\ \vdots \\ f_n(x,u,p) \end{bmatrix} \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^n, \quad p \in \mathbb{R}^p. \quad (3.48) $$

Because the outcome of $f(x,u,p)$ in (3.48) is a vector, $f(x,u,p)$ is called a vector function.
Example 1
Consider the state-space model (3.33), (3.34). In vector form this model is described by:

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} = \begin{bmatrix}
x_2 \\
\frac{u_1}{p_i}
\end{bmatrix} = \begin{bmatrix}
f_1(x,u,p) \\
f_2(x,u,p)
\end{bmatrix} = f(x,u,p)
\]

Exercise 1
For the state-space systems model (3.37)-(4.1) determine the system dimension \(n_x\) and the vector function \(f(x,u,p)\). Also determine \(n_u, n_p\).

3.7 Obtaining state-space models from verbal system descriptions

Example 1
Consider the following verbal system description. A biochemical reactor contains fluid with two components: biomass and substrate. The biomass increases because it consumes (eats) the substrate. An example is a waste water treatment plant where the biomass is used to eat waste chemicals which form the substrate.

Consider a perfectly mixed biochemical reactor where the fluid flowing in, which contains variable concentrations of substrate and biomass, can be regulated with a valve. The fluid flowing out of the biochemical reactor can also be regulated with a valve. Furthermore the consumption of substrate and the associated biomass generation per m\(^3\) in the reactor are both proportional to the concentration of biomass.

To build a mathematical model in state-space form from this description we will follow the procedure described below.

(1) Using the description above draw a picture (sketch) of the biochemical reactor.
(2) Put relevant processes, quantities and mathematical symbols in the picture (sketch) to help you build a mathematical model of this system.
(3) Using balance equations (scientific knowledge) derive differential equations which describe this system.
(4) Attach units to all the variables appearing in the differential equations.
(5) Determine the state variables, parameters and inputs of this systems model
(6) Write the systems model in state-space form
Below the results of these six steps are listed.

(1), (2)

![Diagram of a process flow diagram with labels: Flow in $F_{in}$, Flow out $F_{out}$, Valve, Stirrer for mixing, Biomass Conc. in $B_{in}$, Substrate Conc. in $S_{in}$, Biomass Conc. $B$, Substrate Conc. $S$, Volume $V$.]

(3) The consumption of substrate $C_s$ per m$^3$ is proportional to the biomass concentration $B$: $C_s = c_1 B$

The generation of biomass $B_g$ per m$^3$ is proportional to the biomass concentration $B$: $B_g = c_2 B$

Balance equations:

[rate of change of biomass] =

[generation rate biomass] + [input rate biomass] - [output rate biomass]

$$\frac{d(BV)}{dt} = c_2 BV + F_{in} B_{in} - F_{out} B$$

(3.49)

[rate of change of substrate] =

[generation substrate] + [input substrate] - [output substrate]

$$\frac{d(SV)}{dt} = c_1 BV + F_{in} S_{in} - F_{out} S$$

(3.50)

[rate of change of mass/volume] =

[input rate mass/volume] - [output rate mass/volume]

$$\frac{dV}{dt} = F_{in} - F_{out}$$

(3.51)
\[ B, B_{in}, S, S_{in} : [\text{mol m}^3], \quad F_{in}, F_{out} : [\text{m}^3 \text{s}^{-1}], \quad \frac{d(BV)}{dt}, \frac{d(SV)}{dt} : [\text{mol s}^{-1}] \]
\[ c_1, c_2 : [\text{s}^{-1}] \]

\[ \frac{d(BV)}{dt} = B \frac{dV}{dt} + V \frac{dB}{dt} \Rightarrow \frac{dB}{dt} = \frac{1}{V} \left( \frac{d(BV)}{dt} - B \frac{dV}{dt} \right) = c_2 B + \frac{F_{in} (B_{in} - B)}{V} \]
\[ \frac{d(SV)}{dt} = S \frac{dV}{dt} + V \frac{dS}{dt} \Rightarrow \frac{dS}{dt} = \frac{1}{V} \left( \frac{d(SV)}{dt} - S \frac{dV}{dt} \right) = c_1 B + \frac{F_{in} (S_{in} - S)}{V} \]

State variables \( x_1 = B, \quad x_2 = S, \quad x_3 = V \). Parameters \( p_1 = c_1, p_2 = c_2 \). Input variables (inputs) \( u_1 = F_{in}, u_2 = F_{out}, u_3 = B_{in}, u_4 = S_{in} \). Then the state-space model becomes,

\[ \frac{dx_1}{dt} = c_2 x_1 + \frac{u_1 (u_3 - x_1)}{x_3} \quad (3.52) \]
\[ \frac{dx_2}{dt} = c_1 x_1 + \frac{u_1 (u_4 - x_2)}{x_3} \quad (3.53) \]
\[ \frac{dx_3}{dt} = u_1 - u_2 \quad (3.54) \]

3.8 Control inputs, external inputs, measurements, output variables and the output equation

So far mathematical systems models in state-space form are represented by the state equation,

\[ \frac{dx}{dt} = f(x, u, p), \quad x, f \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad p \in \mathbb{R}^r \quad (3.55) \]

The state equation describes the dynamic phenomena responsible for the system behaviour. Knowing the initial state \( x(t_0) \) and the inputs \( u(t), t \geq t_0 \) the system behaviour \( x(t), t \geq t_0 \) is computed by means of numerical integration. The inputs of
the system may be classified into control inputs and external inputs. Control inputs can be manipulated (changed) by the control engineer at each time $t \geq t_0$ and therefore are suitable to control the system. The heat supplied by the heating system in the greenhouse is an example of a control input. Often control inputs have an upper and lower bound, like the heat supplied by the greenhouse heating system,

$$u_i^{\text{min}} \leq u_i \leq u_i^{\text{max}}, \quad i = 1, 2, \ldots, n_u$$  \hspace{1cm} (3.56)

External inputs on the other hand are fully determined by external conditions and therefore cannot be used for control purposes at all. The outside temperature of the greenhouse is an example of an external input. To distinguish external inputs from control inputs external inputs are denoted by $d$ instead of $u$. With this distinction the state equation (3.55) becomes,

$$\frac{dx}{dt} = f(x, u, d, p), \quad x, f \in R^n, \quad u \in R^{n_u}, \quad d \in R^{n_d}, \quad p \in R^r$$  \hspace{1cm} (3.57)

In chapter 1 we explained that for control purposes and to improve the model (system identification) measurements are needed. So far our mathematical systems model in state-space form (3.57) does not describe measurements. Variables that are measured will be denoted by $y_1, y_2, \ldots, y_n$, and are collected in the column vector,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in R^n$$  \hspace{1cm} (3.58)

Let us reconsider the mass on which a force is acting, see Figure 13 and the associated state-space model (3.33), (3.34). Suppose that we measure the position of the mass. This is described by,

$$y_1 = x_1$$  \hspace{1cm} (3.59)

If we also measure the speed this is described by,
\[ y_2 = x_2 \]  

(3.60)

Using the vector notation (3.58) equations (3.59), (3.60) may be replaced with,

\[ y = x \]  

(3.61)

Equation (3.61) states that all the state variables are measured. For control purposes this is the most favourable situation because we measure (observe) the complete behaviour of the system. If equation (3.61) applies control engineers call this 'having complete state information'. Usually however not all the state variables can be measured. In general measured variables \( y_1, y_2, \ldots, y_n \) are functions of the state variables, the inputs and the parameters,

\[ y = g(x,u,d,p), \quad y,g \in \mathbb{R}^n \]  

(3.62)

where \( g \in \mathbb{R}^n \) is a vector function,

\[
\begin{bmatrix}
  g_1(x,u,d,p) \\
  g_2(x,u,d,p) \\
  \vdots \\
  g_n(x,u,d,p)
\end{bmatrix} \in \mathbb{R}^n
\]  

(3.63)

When analysing systems we may not just be interested in the system behaviour \( x(t), t \geq t_0 \) but also in the behaviour of variables that are functions of the state variables, the parameters and possibly the inputs. Such variables are also described by equation (3.62) and are called output variables or outputs of the system. Therefore equation (3.62) is called the output equation of the system. Recall that equation (3.57) is called the state equation of the system. Together the state and output equation (3.57), (3.62) determine the state-space model. This model is fully determined by the two vector functions \( f, g \). Mathematicians therefore sometimes just speak about \( f \) and \( g \) when they talk about a mathematical systems model in state-space form. If \( n_u > 1, n_y > 1 \) the system is called a multi-input multi-output or MIMO system. If \( n_u = 1, n_y = 1 \) the system is called a single-input single-output or SISO system. Similarly we have SIMO and MISO systems. Also systems may just be called single-input or multi-input or single-output or multi-output.
As an example of an output variable consider the kinetic energy associated to the motion of the mass. According to equations (3.31), (3.32), (3.26) the associated output equation (3.62) is,

\[
y_1 = \frac{1}{2} p_i x_2^2
\]  

(3.64)

If we also consider the impulse of the mass then,

\[
y_2 = p_i x_2
\]  

(3.65)

becomes another model output.

### 3.9 State-space models: complete but not unique

The general state-space model of systems including outputs is described by the state equation (3.57) and the output equation (3.62). Representing a systems model in the state-space form (3.57), (3.62) can be done in different ways. To see this observe that the numbering of the state variables, the input variables and the parameters can be performed in different ways. Renumbering is actually a special case of changing the basis used to represent the variables (Mathematics at work Part 1, Chapter 1: Vectors). Changing the basis used to represent variables is actually not the only way in which different state-space models of the same system can be obtained. The next example demonstrates this.

**Example 1**

Reconsider the bioreactor described in paragraph 3.7 on page 36. Assume that the cross sectional area of the reactor is everywhere constant. Then in the differential equations (3.49), (3.50) we may replace the volume \( V \left[ m^3 \right] \) by \( hA \left[ m^3 \right] \) where \( h \left[ m \right] \) is the height of the liquid in the tank and \( A \left[ m^2 \right] \) is the constant cross sectional area. Doing so \( A \) becomes a parameter and instead of the state variable \( V \) we now obtain the state variable \( h \).
Exercise 1
Determine the state-space model when the substitution suggested above is made. Check that the interpretation of this state-space model is the same as that of the original state-space model.

Although there are different ways to represent the system in state-space form, each representation is complete, meaning that it completely describes the system behaviour. So if you and someone else build a state-space model of a system from the same first principles your results may be different. However the interpretation of all models in terms of the original system is unique.

Exercise 2
Suppose we measure the height of the liquid in the tank and the substrate concentration. For each of the two state-space models write the output equation.

3.10 Summary
From verbal or other types of system descriptions we can usually make a drawing of a system. Then after adding the relevant processes, quantities and mathematical symbols, from balance equations and first principles we may derive differential equations including the units of each quantity appearing in these differential equations. These differential equations describe the dynamic behaviour of the system.

Next we can write the differential equations in state-space form as follows:

(1) Identify the state variables (states) $x_1, x_2, \ldots, x_n$. State variables are the variables of which time derivatives appear in the differential equations. Also all time derivatives of the state variables are state variables up to (so not including) the highest time-derivative appearing in the differential equations.

(2) Identify the parameters $p_1, p_2, \ldots, p_n$. Parameters are all the constants appearing in the differential equations.

(3) Identify the input variables (inputs) $u_1, u_2, \ldots, u_n$. Usually these are the variables that are left over. In exceptional cases derivatives of inputs may appear in the differential equations. To recognise this we first have to identify the inputs instead of the state variables. Inputs are variables that are not
constant and which influence the system, while they are not influenced by the system.

(4) Identify the output variables (outputs) of the system \( y_1, y_2, \ldots, y_n \). Outputs are either measured variables or variables of interest. These variables in general depend on the state variables, the inputs and the parameters.

(5) Having determined the state variables, parameters, inputs and outputs the state space form of systems can be written in vector form as follows,

State equation:

\[
\frac{dx}{dt} = f(x, u, p), \quad \frac{dx}{dt}, x, f \in \mathbb{R}^n, \quad u \in \mathbb{R}^n, \quad p \in \mathbb{R}^r
\]  

(3.66)

Output equation:

\[
y = g(x, u, p), \quad y, g \in \mathbb{R}^r
\]  

(3.67)

(6) Inputs may be distinguished into control inputs \( u_1, u_2, \ldots, u_n \) and external inputs \( d_1, d_2, \ldots, d_n \). In this case the state space form becomes,

State equation:

\[
\frac{dx}{dt} = f(x, u, d, p), \quad \frac{dx}{dt}, x, f \in \mathbb{R}^n, \quad u \in \mathbb{R}^n, \quad d \in \mathbb{R}^d, \quad p \in \mathbb{R}^r
\]  

(3.68)

Output equation:

\[
y = g(x, u, d, p), \quad y, g \in \mathbb{R}^r
\]  

(3.69)

From a mathematical point of view the system is fully described by the two vector functions \( f, g \). So all the properties of the system are determined by these two vector functions. If \( n_u = 1, n_y = 1 \) the system is called a single-input single-output or SISO system. If \( n_u > 1, n_y > 1 \) the system is called a multi-input multi-output or MIMO system. Similarly we have SIMO and MISO systems. Also systems may just be called single-input or multi-input or single-output or multi-output. A systems model represented in the state-space form (3.66), (3.67) or (3.68), (3.69) is complete but not unique. The interpretation of different state-space models in terms of the original system is unique.
The main reasons to represent systems models in the state-space form (3.66), (3.67) or (3.68), (3.69) are the following.

(1) Only models represented in state-space form are suitable for numerical integration for which standard software is available.

(2) The systems memory may be identified with the systems state variables collected in the state vector $x$.

(3) The system behaviour is identical to the contents of the systems memory as a function of time $x(t), t \geq t_0$ or in other words the time response of the system state. System behaviour is computed by numerical integration of the state-space model.

(4) Cause and effect are clearly distinguished in systems models written in state-space form. The cause being the inputs $u(t), t \geq t_0$ and the initial values of the state variables: the initial state $x(t_0)$. The effect being the systems behaviour $x(t), t \geq t_0$ and the model output response $y(t), t \geq t_0$.

(5) Input variables, state variables, output variables and parameters, which are fundamentally different, are distinguished in the state-space form.

The state space model is entirely determined by the two vector functions $f, g$ in equations (3.66), (3.67) or (3.68), (3.69). Note that system outputs need not necessarily be specified. In that case the system is entirely determined by the vector function $f$.

4 A simple control problem: steady states

4.1 Introduction

How can we maintain the concentrations in a chemical reactor? How can we maintain the temperature in a greenhouse? How can we maintain the speed of a car? For many systems the control objective is to keep several or all of the state variables at a fixed value. If all of the state variables remain constant this constant state is called a steady state.
After reading this chapter the student should

(1) Recognise that realising a steady state is a control problem
(2) Be able to compute the control that realises a steady state from the state equation

Keep the state constant → Steady state → Compute the associated control

4.2 Computation of steady states

Example 1
Consider the greenhouse systems model (3.16), (3.17). Assume the outside temperature $T_o$ to be constant. Then the associated state-space model is,

$$\frac{dx_1}{dt} = p_1(p_3 - x_1) + p_2 u_1$$

$$\left[\begin{array}{c}
-oCs^{-1} \\
\end{array}\right] = \left[\begin{array}{c}
s^{-1} \left[ oC \right] \\
\end{array}\right] + \left[\begin{array}{c}
Js^{-1} \\
\end{array}\right] \left[ J^{-1} oC \right]$$

(4.1)

with,

$$x_1 = T_g, \ u_1 = H, \ p_1 = \frac{h_w s_w}{\rho c_p V}, \ p_2 = \frac{1}{\rho c_p V}, \ p_3 = T_o$$

(4.2)

Suppose we want to maintain a fixed greenhouse temperature. Is this possible? Observe that actually this is a simple control problem where we desire,

$$x(t) = x_1(t) = x^{ss}$$

(4.3)

by selecting appropriately the control inputs, where $x^{ss}$ is the desired steady state. Observe that (4.3) implies,

$$\frac{dx_1}{dt} = 0$$

(4.4)
Substituting (4.4) and (4.3) into the state-space model (4.1) to obtain,

\[ 0 = p_1 \left( p_3 - x^{ss} \right) + p_2 u_1 \Rightarrow u_1 = \frac{p_1 \left( x^{ss} - p_3 \right)}{p_2} \quad (4.5) \]

So the control (4.5) realises the steady state (4.3). Actually the control (4.5) also steers the state from any initial value \( x(t_0) \) to the value \( x^{ss} \) ultimately. Try to explain this using equations (4.1), (4.2). The next example is similar but applies to all systems that can be represented in state-space form.

**Example 2**
Consider the state equation,

\[ \frac{dx}{dt} = f(x, u, p), \quad x, f \in \mathbb{R}^n, \ u \in \mathbb{R}^u, \ p \in \mathbb{R}^p \quad (4.6) \]

Suppose we want,

\[ x(t) = x^{ss} \quad (4.7) \]

Again this is a control problem because we may use the control input \( u(t) \) to achieve (4.7). Like (4.3), equation (4.7) implies,

\[ \frac{dx}{dt} = 0 \quad (4.8) \]

Equations (4.6), (4.7), (4.8) together imply,

\[ f \left( x^{ss}, u, p \right) = 0, \ x^{ss} \in \mathbb{R}^n, \ u \in \mathbb{R}^u, \ p \in \mathbb{R}^p \quad (4.9) \]

So we should select the control input \( u \) to satisfy (4.9). Therefore (4.9) represents a set of \( n_x \) non-linear algebraic equation with \( n_u \) unknown control inputs. In general, a solution only exists if \( n_u = n_x \), meaning that the number of control inputs must be
equal to the number of states. In the previous example 1 both were equal to 1. If \( n_u = n_x \) is not satisfied, in general, arbitrary steady states cannot be realised. However there may still exist some steady states for certain constant control inputs.

### 4.3 Solving sets of non linear algebraic equations

To realise an arbitrary steady state, from example 2 in the previous paragraph we learned that we have to solve a set of \( n_x \) non linear algebraic equations in \( n_x \) unknowns. In this section we will present and explain a numerical method to solve this problem. The method is called Newton’s method.

Because the parameters and the steady state in (4.9) are fixed we will write \( f(u) \) instead of \( f(x^n, u, p) \). We solve,

\[
f(u) = 0, \quad f, u \in R^n
\]  

(4.10)

The recursive numerical procedure to obtain the solution of (4.10) is described by,

\[
u(k + 1) = u(k) - \left(f'(u(k))\right)^{-1} f(u(k)), \quad k = 1, 2, ...
\]  

(4.11)

where,

\[
f'(u(k)) = \left. \frac{df}{du} \right|_{u=u(k)}
\]  

(4.12)

The recursion (4.11) is started with the value \( u(1) \) which should preferably, but not necessarily, be close to the solution. From (4.11) and \( u(1) \) we obtain \( u(2) \). This computation is graphically represented in Figure 14 when we solve one non linear equation with one unknown.
At \( u(k) \) the function is approximated by the straight line \( l(u,k) \) described by,

\[
l(u,k) = f(u(k)) + f'(u(k))(u - u(k))
\]  \hspace{1cm} (4.13)

For \( k = 1, 2 \) these lines (---) are plotted in Figure 14. Now \( u(k+1) \) is computed to be the zero of the line \( l(u,k) \),

\[
\begin{align*}
    f(u(k)) + f'(u(k))(u - u(k)) &= 0 \Rightarrow \\
    f(u(k)) + f'(u(k))u - f'(u(k))u(k) &= 0 \Rightarrow \\
    f'(u(k))u &= f'(u(k))u(k) - f(u(k)) \Rightarrow \\
    u &= u(k) - \left(f'(u(k))\right)^{-1}f(u(k))
\end{align*}
\]  \hspace{1cm} (4.14)

The final result of (4.14) is identical to (4.11) when \( f, u \in \mathbb{R}^1 \) (when \( f \) and \( u \) are both scalars) as they are in Figure 14. As you can see from Figure 14 the zeros \( u(k) \) of the lines \( l(u,k) \), \( k = 1, 2, \ldots \) quickly converge to the zero of the function.
Observe that equation (4.11) only requires the computation of \( f(u(k)) \) and \( f'(u(k)) \) the value of the function and its first derivative at \( u(k) \).

For complicated functions \( f(u) \), an analytical description of the derivative function \( f'(u) \) may be hard to obtain. Without such an analytical description we can approximate the function value \( f'(u(k)) \) by,

\[
f'(u(k)) \approx \frac{f(u(k) + \Delta u) - f(u(k))}{\Delta u}
\]

(4.15)

where \( \Delta u \) is an increment of \( u(k) \) which should be sufficiently small, but not too small, since then rounding errors destroy the accuracy of the computation. The approximation is based on the definition of the derivative function. This definition is equal to the right hand side of equation (4.15) when we take the limit \( \Delta u \to 0 \). Computing the derivative according to equation (4.15) is called numerical differentiation by means of computing finite differences.

Now without proof we state that equation (4.11) also applies, and converges quickly, when \( f, u \in \mathbb{R}^n, n_i \geq 1 \). To interpret equation (4.11) in this case we need to introduce the derivative of a vector function of a vector. The derivative of a vector function of a vector \( f(u), f \in \mathbb{R}^n, u \in \mathbb{R}^m \), is the following \( n \times m \) matrix,

\[
f'(u) = \frac{df}{du} = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m}
\end{bmatrix} \in \mathbb{R}^{n \times m}
\]

(4.16)

This matrix is a generalised derivative and is called the Jacobian. A compact way to write the Jacobian is,
\[
\frac{df}{du}_{ij} = \frac{\partial f_i}{\partial u_j} \quad (4.17)
\]

where the indices \( ij \) refer to matrix element \( i, j \). Because in equations (4.10), (4.11) both \( f \) and \( u \) have dimension \( n \), the Jacobian is a square matrix (a square matrix has as many rows as columns and has an inverse). Therefore \( (f'(u(k)))^{-1} \) in (4.11) is well defined; it is the inverse of the square Jacobian matrix \( f'(u(k)) \).

**Exercise 1**
Consider the following state-space model of a system,

\[
\begin{align*}
\frac{dx_1}{dt} &= 2u_1^2 - u_2 + x_1, \\
\frac{dx_2}{dt} &= -u_1 - u_2^2 + x_2
\end{align*}
\quad (4.18)
\]

Compute the control \( u \) that realises the steady state \( x^{ss} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} \). To do this determine the set of two non linear algebraic equations that have to be solved. To solve these equations use the recursive numerical procedure (4.11). Start with \( u(1) = \begin{bmatrix} 2 \\ 1.7 \end{bmatrix} \) and compute \( u(2), u(3) \). If you make no mistakes \( u(3) \) is very close to the solution \( \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix} \).

**4.4 Summary**

(1) Steady states have the property \( f(x,u) = 0, f, x \in \mathbb{R}^n, u \in \mathbb{R}^m \). To keep a system at an arbitrary desired steady state \( x(t) = x^{ss} \) we must solve \( u^{ss} \) such that \( f(x^{ss}, u^{ss}) = 0 \) which constitute \( n \) equations in \( m \) unknowns. In general a solution only exists if \( m = n \). If this is the case we can use Newton's method to find a solution.
Newton’s method to solve \( f(u) = 0, f, u \in R^n \) is based on the following iteration. \( u(k + 1) = u(k) - \left( f'(u(k)) \right)^{-1} f(u(k)), k = 1, 2, \ldots \) The iteration \( u(k), k = 1, 2, \ldots \) converges to a solution. Several solutions may exist. The more close \( u(1) \) is to the solution the more quick the iteration converges.

\[
f'(u(k)) = \frac{\partial f}{\partial u} \bigg|_{u=u(k)} \in R^{n \times n}, \quad \left[ \frac{\partial f}{\partial u} \right]_{ij} = \frac{\partial f_i}{\partial u_j}
\]

5 System properties and control

5.1 Introduction
Chemical reactors sometimes explode. Ill understood motorbike accidents are investigated. The steer of a motorbike always shakes a little. Sometimes however, for no apparent reason, this shaking increases and keeps increasing and cannot be stopped.

Dynamic systems may exhibit unexpected behaviours. Roughly speaking unexpected behaviour means that if you change the inputs or the initial state gradually the behaviour does not change gradually. A chain reaction is another example of unexpected behaviour. Linear systems do not exhibit unexpected behaviour, non linear systems do. In terms of their behaviour non linear systems are therefore much more interesting (and sometimes dangerous). Almost any biological system is non linear. Because of their unexpected behaviour, in general, non linear systems are more difficult to analyse and control than linear systems. This explains why control engineers and people performing system analysis prefer to work with linear systems models. Under certain conditions, which we will explore in this chapter, a linear systems model is an acceptable approximation of a non linear systems model. For this reason linearity is probably the most important system property. This property will be introduced and investigated in several ways in this chapter.

Often we want to realise a steady state. For example in a chemical reactor one wants to bring the concentrations, which are the state variables, at a certain level and then keep them at that level,
In equation (5.1) \( x^{ss} \) represents the desired steady state and \( t_i \) the time this state should be reached. An important question that arises here is whether small deviations (perturbations) of the state from the steady state, due to internal or external disturbances, tend to zero after a while, or whether they increase. In the latter case our chemical reactor might explode! If the small perturbations tend to zero after a while the steady state \( x^{ss} \) is called a stable steady state of the system. If the steady state is unstable we may use the control inputs to control the state deviations to zero. If, using the control inputs, it is possible to control the state perturbations to zero, the steady state \( x^{ss} \) is called stabilisable. Clearly exploding chemical reactors and the motorbike accidents are due to instability of some kind. In this chapter to determine whether or not a steady state is stable or stabilisable we will analyse a linear systems model. This linear systems model is obtained by linearising the original, possibly non linear systems model, around the steady state \( x^{ss} \).

From what we have just mentioned it should be clear that linearity, non-linearity, stability and stabilisability are important system properties. Furthermore the linearisation of systems models turns out to be important to analyse the system. The most important system properties are introduced in this chapter. To simplify the notation in this chapter we will not distinguish external from control inputs. In this section all inputs are assumed to be collected in the input vector \( u \).

After reading this section the student should,

1. Be able to determine from a state space model whether a system is linear, linear in the state, linear in the control inputs, linear in the parameters, linear in the external inputs.
2. Be able for a linear system to calculate from \( f, g \) the matrices \( A, B, C, D \) which determine the general state-space form of a linear system, see the figure below.
3. Be able to determine whether a steady state is stable or not by computing the linearised model around the steady state and determining whether this model is stable, see the figure below.
4. Know the interpretation of stable and unstable steady states from the behaviour and meaning of the associated deviation (perturbation) variables.
(5) Determine the linearised model around optimal state and control trajectories. Know the meaning, interpretation and advantage of this model for control system design.

(6) Be able to determine from a state space model whether a system is time invariant or time-varying.

5.2 Taylor series expansion and linearisation of systems

From the introduction it has become clear that the linearisation of systems around steady states is needed to investigate stability and stabilisability of the steady state. To understand the linearisation of systems we have to look into Taylor series expansions of the vector functions \( f, g \) which fully determine our state-space model.

Because the constant parameters do not influence the results in this paragraph, to simplify the notation, the parameter vector is dropped from our notation. This implies that we write \( f(x, u) \) instead of \( f(x, u, p) \) which we have written so far to stress the dependence on the constant parameters.

Consider the general state equation (3.66) with \( n_x = n_u = 1 \). The Taylor series expansion of \( f(x, u) \) around the values \( x = x^s, u = u^s \) is,

\[
f(x, u) = f(x^s, u^s) + \frac{\partial f}{\partial x} \bigg|_{x = x^s, u = u^s} (x - x^s) + \frac{\partial f}{\partial u} \bigg|_{x = x^s, u = u^s} (u - u^s) + O(2) \tag{5.2}
\]

In equation (5.2) \( O(2) \) represents all the remaining terms in the Taylor series expansion which contain second and higher order terms in \( x - x^s \) and \( u - u^s \). If
$x - x^{ss}$ and $u - u^{ss}$ are small, that is in a neighbourhood of $x^{ss}, u^{ss}$, neglecting $O(2)$ is acceptable. Neglecting $O(2)$ means approximating the function by a straight line as in Figure 5 and Figure 14. If the system is linear, as we shall see later in paragraph 5.4, the function is this straight line and $O(2)$ is equal to zero.

Next consider equation (3.66) with $n_x = n_u = 2$. Then we obtain,

$$f(x,u) = \left[ \begin{array}{c} f_1(x,u) \\ f_2(x,u) \end{array} \right] = \left[ \begin{array}{c} f_1(x^{ss},u^{ss}) \\ f_2(x^{ss},u^{ss}) \end{array} \right] + \left[ \begin{array}{c} \frac{\partial f_1}{\partial x_1} \bigg|_{x=x^{ss},u=u^{ss}} (x_1-x_1^{ss}) + \frac{\partial f_1}{\partial x_2} \bigg|_{x=x^{ss},u=u^{ss}} (x_2-x_2^{ss}) \\ \frac{\partial f_2}{\partial x_1} \bigg|_{x=x^{ss},u=u^{ss}} (x_1-x_1^{ss}) + \frac{\partial f_2}{\partial x_2} \bigg|_{x=x^{ss},u=u^{ss}} (x_2-x_2^{ss}) \\ \frac{\partial f_1}{\partial u_1} \bigg|_{x=x^{ss},u=u^{ss}} (u_1-u_1^{ss}) + \frac{\partial f_1}{\partial u_2} \bigg|_{x=x^{ss},u=u^{ss}} (u_2-u_2^{ss}) \\ \frac{\partial f_2}{\partial u_1} \bigg|_{x=x^{ss},u=u^{ss}} (u_1-u_1^{ss}) + \frac{\partial f_2}{\partial u_2} \bigg|_{x=x^{ss},u=u^{ss}} (u_2-u_2^{ss}) \end{array} \right] + O(2) \quad \text{(5.3)}$$

Observe that,

$$\left[ \begin{array}{c} f_1(x^{ss},u^{ss}) \\ f_2(x^{ss},u^{ss}) \end{array} \right] = f(x^{ss},u^{ss}) \quad \text{(5.4)}$$

$$\begin{array}{c} \frac{\partial f_1}{\partial x_1} \bigg|_{x=x^{ss},u=u^{ss}} (x_1-x_1^{ss}) + \frac{\partial f_1}{\partial x_2} \bigg|_{x=x^{ss},u=u^{ss}} (x_2-x_2^{ss}) \\ \frac{\partial f_2}{\partial x_1} \bigg|_{x=x^{ss},u=u^{ss}} (x_1-x_1^{ss}) + \frac{\partial f_2}{\partial x_2} \bigg|_{x=x^{ss},u=u^{ss}} (x_2-x_2^{ss}) \end{array} = \begin{array}{c} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \end{array} \bigg|_{x=x^{ss},u=u^{ss}} (x-x^{ss}) \quad \text{(5.5)}$$

and similarly,
Using (5.4), (5.5), (5.6) and the definition of the Jacobian (4.16) equation (5.3) becomes,

\[
\frac{\partial f_1}{\partial u_1} \bigg|_{x=x^*, u=u^*} (u_1-u_1^*) + \frac{\partial f_1}{\partial x_2} \bigg|_{x=x^*, u=u^*} (u_2-u_2^*) = \left[ \begin{array}{cc} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{array} \right]_{x=x^*, u=u^*} (u-u^*) \tag{5.6}
\]

Equation (5.7) is completely identical to equation (5.2) ! So equation (5.2) also applies to vector functions of vectors when we define the derivative of a vector function according to (4.16). We might also say that the definition (4.16) is a direct consequence of the generalisation of Taylor series expansion to vector functions of vectors. Although we have only considered the case \(n_x = n_u = 2\) for other values \(n_x, n_u\) which need not necessarily be the same, the derivation is along exactly the same line.

Equation (5.7) without \(O(2)\) is a linear approximation of the vector function \(f(x,u)\) around \(x = x^*, u = u^*\). Without \(O(2)\) all components of the vector function \(f(x,u)\) are approximated by straight lines as in Figure 5 and Figure 14. This approximation is accurate in a neighbourhood of \(x = x^*, u = u^*\) where \(O(2)\) in negligible.

Similarly,

\[
\frac{dx}{dt} \approx f(x^*, u^*) + \frac{\partial f}{\partial x} \bigg|_{x=x^*, u=u^*} (x-x^*) + \frac{\partial f}{\partial u} \bigg|_{x=x^*, u=u^*} (u-u^*) \tag{5.8}
\]

is a linear approximation of the state equation,

\[
\frac{dx}{dt} = f(x,u) \tag{5.9}
\]
Also,

\[ y \approx g(x^{ss}, u^{ss}) + \left. \frac{\partial g}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} (x - x^{ss}) + \left. \frac{\partial g}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} (u - u^{ss}) \]  

(5.10)

is a linear approximation of the output equation,

\[ y = g(x, u) \]  

(5.11)

Both (5.8) and (5.10) are accurate approximations in the neighbourhood of the linearisation point \( x = x^{ss}, u = u^{ss} \).

Remember that in the introduction an important question was whether deviations from the steady state would tend to zero or grow. To verify this introduce so called deviation (or perturbation) variables,

\[ \Delta x = x - x^{ss}, \Delta u = u - u^{ss}, \Delta y = y - y^{ss}, y^{ss} = g(x^{ss}, u^{ss}) \]  

(5.12)

Observe that,

\[ \frac{d\Delta x}{dt} = \frac{d(x - x^{ss})}{dt} = \frac{dx}{dt} - \frac{dx^{ss}}{dt} = f(x, u) - f(x^{ss}, u^{ss}) \]  

(5.13)

From (5.12), (5.7), (5.13) we obtain,

\[ \frac{d\Delta x}{dt} \approx \left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} \Delta u \]  

(5.14)

Equation (5.14) is called the linearised state equation around \( x = x^{ss}, u = u^{ss} \) or a linearisation of the state equation (5.9) around \( x = x^{ss}, u = u^{ss} \). This linearisation describes approximately the dynamic behaviour of small state deviations \( \Delta x \) as a result of small input deviations \( \Delta u \). As we shall see, equation (5.14) is a linear systems model which enables us to analyse stability and stabilisability properties. Similarly from (5.12),(5.10) we obtain,
\[
\Delta y \approx \left. \frac{\partial g}{\partial x} \right|_{x=x^*, u=u^*} \Delta x + \left. \frac{\partial g}{\partial u} \right|_{x=x^*, u=u^*} \Delta u
\]  

(5.15)

which is called the linearised output equation around \( x = x^* \), \( u = u^* \), \( y = y^* = g(x^*, u^*) \) or a linearisation of the output equation (5.11) around \( x = x^* \), \( u = u^* \), \( y = y^* = g(x^*, u^*) \).

Example 1
Recall the state-space model,

\[
\begin{align*}
\frac{dx_1}{dt} &= 2u_1^2 - u_2 + x_1, \\
\frac{dx_2}{dt} &= -u_1 - u_2^2 + x_2
\end{align*}
\]  

(5.16)

In the previous paragraph we realised the steady state \( x^* = \begin{bmatrix} -5 \\ 4 \end{bmatrix} \) by computing the associated control \( u^* = \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix} \). To compute the linearisation (5.14) of the state equation (5.16) around this steady state we must compute \( \left. \frac{\partial f}{\partial x} \right|_{x=x^*, u=u^*} \) and \( \left. \frac{\partial f}{\partial u} \right|_{x=x^*, u=u^*} \) where \( f = \begin{bmatrix} 2u_1^2 - u_2 + x_1 \\ -u_1 - u_2^2 + x_2 \end{bmatrix} \). Using the definition (4.16) we compute,

\[
\left. \frac{\partial f}{\partial x} \right|_{x=x^*, u=u^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \left. \frac{\partial f}{\partial u} \right|_{x=x^*, u=u^*} = \begin{bmatrix} -5 & 1.8004 \\ 4 & 1.4831 \end{bmatrix}
\]

(5.17)

Therefore the linear state equation,

\[
\frac{d\Delta x}{dt} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta x + \begin{bmatrix} 7.2016 & -1 \\ -1 & -2.9662 \end{bmatrix} \Delta u
\]  

(5.18)

describes approximately the behaviour of small deviation variables,
\[
\Delta x = x - x'' = x - \begin{bmatrix} -5 \\ 4 \end{bmatrix}, \Delta u = u - u'' = u - \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}
\] (5.19)

As we shall learn in paragraph 5.5 from the two matrices

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7.2016 & -1 \\ -1 & -2.9662 \end{bmatrix}
\]

that fully determine the linear state equation (5.18) we can compute whether the steady state is stable and whether it is stabilisable.

**Exercise 1**

Recall the state space model (3.52), (3.53), (3.54). Write \( f(x,u,p) \) of this state space model. Let \( x = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \). Show that \( x \) is not a steady state. Although it is not a steady state, we may still obtain the linearised state equation around this temporarily state (as opposed to steady state) in exactly the same manner. Compute this linearised state equation. You may wonder about the use and meaning of this linearised state equation. Both will be explained later in paragraph 5.6.

### 5.3 Linearity

In the previous section we learned how to compute linearised state equations which are important to analyse the stability and stabilisability of steady states of (non linear) systems. The term linearised relates to the fact that we are approximating functions by straight lines as in Figure 5 and Figure 14.

Actually two definitions of linearity, which are slightly different, are used. These two definitions are visualised in Figure 15.
The strongest definition of linearity states that a scalar function of one variable \( f(x) \), \( f, x \in R^1 \) is linear if,

\[
f(\alpha x) = \alpha f(x), \ \forall x \in R, \ \forall \alpha \in R
\]  

(5.20)

This definition demands that the function is represented by a straight line that runs through the origin. The other definition of linearity states that,

\[
\frac{\partial f}{\partial x} = c, \ \forall x \in R
\]

(5.21)

where \( c \) does not depend on \( x \) and therefore is constant. This definition only demands that the function is described by a straight line. The line does not necessarily run through the origin and therefore (5.20) is not necessarily satisfied. Mathematicians call the function \( f \) affine in this case. Mathematically speaking \( (5.20) \Rightarrow (5.21) \) while the reverse is not true. Therefore (5.20) is a stronger definition than (5.21). Also observe,

\[
(5.21) \text{ and } f(0) = 0 \iff (5.20)
\]

(5.22)

The simplest way to check whether (5.20) is satisfied is usually to check (5.21) and \( f(0) = 0 \).
**Example 1**

The function \( f(x) = x + p, \quad f, x, p \in \mathbb{R}^1 \) with \( p \) a constant parameter is a linear function when we apply definition (5.21) and is not a linear function if we apply definition (5.20) unless \( p = 0 \).

For functions of several variables \( f(x), \quad f \in \mathbb{R}^1, \quad x \in \mathbb{R}^{n_x}, \quad n_x \geq 1 \) the definitions are identical. Only now \( c \) in (5.21) becomes a constant row vector. In this case the function is represented by a straight line in the \( n_x \) dimensional space. In the case of definition (5.20) this line runs through the origin.

**Example 2**

\[ f(x) = x_1^2 + x_2, \quad f \in \mathbb{R}^1, \quad x \in \mathbb{R}^2 \]

is not a linear function because \( \frac{\partial f}{\partial x} = [2x_1 \ 1] \)

depends on \( x \).

For vector functions \( f(x), \quad f \in \mathbb{R}^n, \quad x \in \mathbb{R}^{n_x}, \quad n_f \geq 1, \quad n_x \geq 1 \) again the definitions are the same while \( c \) now is a matrix (a Jacobian). Each component of the vector function \( f \) is now represented by a straight line in the \( n_x \) dimensional space. In the case of definition (5.20) all these lines run through the origin.

**Example 3**

\[ f(x) = \begin{bmatrix} 2x_1 - x_2 + p \\ -x_1 \end{bmatrix}, \quad f, x \in \mathbb{R}^2 \]

with \( p \) a constant parameter is a linear function when we apply definition (5.21) and is not a linear function if we apply definition (5.20) unless \( p = 0 \). The reason is that \( \frac{\partial f}{\partial x} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \)

does not depend on \( x \) and the line \( 2x_1 - x_2 + p \) only runs through the origin if \( p = 0 \).

### 5.4 Linear systems

Within the state-space description of systems,

\[ \frac{dx}{dt} = f(x,u), \quad y = g(x,u) \]  

(5.23)
we are dealing with the vector functions,

\[ f(x,u), g(x,u), f \in R^n, g \in R^r, x \in R^n, u \in R^u \]  

(5.24)

A system is called linear if its state-space model (5.23) satisfies,

\[
\begin{align*}
    f(\alpha x, \alpha u) &= \alpha f(x, u) \\
    &\forall x \in R^n, \forall u \in R^u, \forall \alpha \in R \\
    g(\alpha x, \alpha u) &= \alpha g(x, u) \\
    &\forall x \in R^n, \forall u \in R^u, \forall \alpha \in R
\end{align*}
\]  

(5.25)

Observe that definition (5.25) corresponds to the strong definition of linearity (5.20) when we add all the components of the vector \( u \) to the vector \( x \). Definition (5.25) implies that for a linear system all the components of both \( f \) and \( g \) are represented by straight lines in the \( n_x + n_u \) dimensional space that run through the origin. Therefore,

\[ f(0, 0) = 0, g(0, 0) = 0 \]  

(5.26)

must be satisfied. Similar to equation (5.22) the simplest way to check whether (5.25) is satisfied is to compute,

\[ f(0, 0), g(0, 0), A = \frac{\partial f}{\partial x}, B = \frac{\partial f}{\partial u}, C = \frac{\partial g}{\partial x}, D = \frac{\partial g}{\partial u} \]  

(5.27)

and to check whether \( f(0, 0) = 0 \) and \( g(0, 0) = 0 \) and if the matrices \( A, B, C, D \) are not functions of \( x, u \).

**Example 1**

Recall the state-space model of the mass with a force acting on it and with the kinetic energy and the impulse as outputs.

\[
\begin{align*}
    \frac{dx_1}{dt} &= x_2, \\
    \frac{dx_2}{dt} &= \frac{u_i}{p_i}, \\
    y_1 &= \frac{1}{2} p_i x_2^2, \\
    y_2 &= p_i x_2
\end{align*}
\]  

(5.28)
To check whether this system is linear we compute,

\[
\begin{bmatrix}
  x_1 \\
  u \\
  p_1
\end{bmatrix},
\begin{bmatrix}
  \frac{1}{2} p_1 x_2^2 \\
  p_1 x_2
\end{bmatrix}
\]

and,

\[
\frac{\partial f}{\partial x} = A = \begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}, \quad \frac{\partial f}{\partial u} = B = \begin{bmatrix}
  0 \\
  \frac{1}{p_1}
\end{bmatrix}, \quad f(0,0) = 0
\]

\[
\frac{\partial g}{\partial x} = C = \begin{bmatrix}
  0 & p_1 x_2 \\
  0 & p_1
\end{bmatrix}, \quad \frac{\partial g}{\partial u} = D = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}, \quad g(0,0) = 0
\]

Because element \( C_{12} \) of matrix \( C \) is a function of \( x \) the system is not linear.

**Exercise 1**

Change the output equation \( g(x,u) \) so that element \( C_{12} \) of matrix \( C \) becomes independent of \( x,u \).

If a system is linear it can be written as,

\[
\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du
\]

where \( A = \frac{\partial f}{\partial x} \), \( B = \frac{\partial f}{\partial u} \), \( C = \frac{\partial g}{\partial x} \), \( D = \frac{\partial g}{\partial u} \) are all independent of \( x,u \). The fact that these matrices are constant does not necessarily imply that the system is linear. We also require \( f(0,0) = 0 \) and \( g(0,0) = 0 \). Only then the system is represented by equation (5.31).

Summarising, to check whether a system is linear, and if it is, write it in the form (5.31), proceed as follows.
(1) Write the system in state space form. That means determine $f(x,u)$, $g(x,u)$.

(2) Check whether $f(0,0) = 0$, $g(0,0) = 0$. If not the system is not linear.

(3) Compute $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ and check whether these are not functions of $x,u$. If so (5.31) represents this linear system. As soon as one of the elements of the matrices $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ does depend on $x$ or $u$ the system is not linear and is not represented by (5.31).

**Exercise 2**

Modify the linear systems model obtained from exercise 1 such that it is no longer linear while $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ are still independent of $x,u$.

A system is called linear in the state $x$ if,

$$A = \frac{\partial f}{\partial x}, C = \frac{\partial g}{\partial x}$$  \hspace{1cm} (5.32)

do not depend on the state $x$.

A system is called linear in the control $u$ if,

$$B = \frac{\partial f}{\partial u}, D = \frac{\partial g}{\partial u}$$  \hspace{1cm} (5.33)

do not depend on the control $u$. The control system design for systems that are linear in the control or linear in the state in general is more easy than the control system design for non linear systems that are not. Note that systems that are both linear in the state and linear in the control need not necessarily be linear systems.
Exercise 3
Write down a state space model of an arbitrary system that is linear in the state, linear in the control, has the property \( f(0, 0) = 0, \ g(0, 0) = 0 \), but still is not linear.

A system is called linear in the parameters \( p \) if,

\[
E = \frac{\partial f}{\partial p}, \quad F = \frac{\partial g}{\partial p}
\]  

(5.34)

do not depend on \( p \). The property linear in the parameters is important for system identification. For systems that are linear in the parameters many well established techniques exist to improve the model parameters through the use of measurement data.

Exercise 4
Write down a state space model of an arbitrary system that is not linear in the state, not linear in the control, and linear in the parameters.

Linear systems have an important property which is sometimes used to define linear systems. This property is called superposition and can be represented mathematically as follows,

\[
\begin{align*}
&u^1(t) \rightarrow x^1(t), \ y^1(t), \ t \geq t_0, \ x(t_0) = 0 \\
&u^2(t) \rightarrow x^2(t), \ y^2(t), \ t \geq t_0, \ x(t_0) = 0 \\
&\alpha u^1(t) + \beta u^2(t) \rightarrow \\
&\alpha x^1(t) + \beta x^2(t), \ \alpha y^1(t) + \beta y^2(t), \ t \geq t_0, \ x(t_0) = 0, \ \forall \alpha, \beta \in R
\end{align*}
\]

(5.35)

The meaning of the notation in (5.35) is as follows. To the input on the left of the arrow the state and output response of the system on the right correspond if the initial state is zero. Equation (5.35) states that if the input is an arbitrary linear combination of two arbitrary inputs the state and output response is the same linear combination of the two associated individual state and output responses if the initial state is zero.
5.5 Stability of steady states and linear systems

The linearised system about a steady state (5.18), (5.12) is a linear system with a state $\Delta x = x - x^s$ and input $\Delta u = u - u^s$ which represent state and input deviation (perturbation) variables. The steady state $x^s, u^s$ is called stable if for any small initial perturbation $\Delta x(t_0)$, the state perturbation $\Delta x(t)$ tends to zero as $t$ tends to infinity if the input perturbation $\Delta u(t) = 0$. Mathematically this is represented as follows,

$$\Delta u(t) = 0, \forall \Delta x(t_0), \|\Delta x(t_0)\|_2 < \varepsilon \Rightarrow \Delta x(t) \to 0 \text{ as } t \to \infty \quad (5.36)$$

where $\varepsilon$ is a small positive number and $\|\Delta x(t_0)\|_2$ is the length (2-norm) of the vector $\Delta x(t_0)$ which together define a neighbourhood around $\Delta x(t_0) = 0$.

If the inputs are control inputs note that we might use $\Delta u(t)$ to control $\Delta x(t)$ to zero as $t$ tends to infinity. Such a $\Delta u(t)$, instead of a control perturbation, is called a control correction. So a steady state is stable if without the use of control corrections $\Delta u(t)$, the state $\Delta x(t)$ tends to zero as $t$ tends to infinity for arbitrary small initial state perturbations $\Delta x(t_0)$.

A steady state is called stabilisable if by making use of control corrections $\Delta u(t)$ we can control the state deviations to zero ultimately,

$$\forall \Delta x(t_0), \|\Delta x(t_0)\|_2 < \varepsilon, \exists \Delta u(t) \Rightarrow \Delta x(t) \to 0 \text{ as } t \to \infty \quad (5.37)$$

In this course we will only consider the stability of steady states because stabilisability is actually a kind of control problem because we have to find a suitable control correction. Stabilisability is closely related to controllability. These properties are treated in the system and control theory course (BRD 31306).

To investigate the stability of steady states we investigate the stability of the linearised system about the steady state which is a linear system. This linear system only describes state and input perturbations accurately as long as these remain small. Note that in the definition of stability and stabilisability small initial state
perturbations are mentioned. Therefore we may investigate the stability of the linearised system to decide about the stability of steady states. Observe that to analyse the stability (of steady states) output equations are irrelevant.

Because no control corrections are applied we will investigate the stability of linear systems if the input is zero. If a linear system,

$$\frac{dx}{dt} = Ax + Bu, \ x \in R^n, \ u \in R^n$$

(5.38)

has zero input then it can be described by,

$$\frac{dx}{dt} = Ax, \ x \in R^n$$

(5.39)

Consider the particularly simple case where the state $x$ is a scalar. Then the matrix $A$ also reduces to a scalar (a 1x1 matrix). In this case the scalar differential equation (5.39) has the following analytical solution,

$$x(t) = e^{At}x(t_0), \ t \geq t_0$$

(5.40)

Now without proof we state that the solution (5.40) to (5.39) also applies if $n_x \geq 1$. In this case $A$ is a square matrix and the question in (5.40) arises as to the meaning of $e^{At}$. To answer this question recall the series (2.1),

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \ x \in R^1$$

(5.41)

Now similar to (5.41) the so called matrix exponential is defined,

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots, \ I, A \in R^{n_x \times n_x}$$

(5.42)

where $I$ denotes the identity matrix. So $e^A$ and $e^{At}$ are both $n_x \times n_x$ dimensional matrices which complies with equation (5.40).
To compute the matrix exponential and to answer the question regarding the stability of linear systems recall the definition of the eigenvalues and eigenvectors (see module 2 and Mathematics at work, Part 1, Chapter 4) of the square matrix $A$ in (5.39), (5.40). The $n_x$ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n_x}$ of the square matrix $A \in R^{n_x \times n_x}$ and the associated $n_x$ eigenvectors $v_1, v_2, \ldots, v_{n_x}$ satisfy,

$$AV_i = \lambda_i v_i, \quad A \in R^{n_x \times n_x}, \quad v_i \in R^{n_x}, \quad \lambda_i \in R^1, \quad i = 1, 2, \ldots, n_x$$

(5.43)

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n_x}$ are either real or complex numbers. Complex eigenvalues always appear in pairs of complex conjugate numbers. That is if $a + jb$ is an eigenvalue so is $a - jb$. Also the components of the associated eigenvectors are each other’s complex conjugate.

Observe that,

$$\begin{align*}
AV_1 &= \lambda_1 v_1 \\
AV_2 &= \lambda_2 v_2 \\
& \vdots \\
AV_{n_x} &= \lambda_{n_x} v_{n_x}
\end{align*}$$

(5.44)

Let,

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n_x} \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \lambda_{n_x} \end{bmatrix} \in R^{n_x \times n_x}$$

(5.45)

then (5.44) becomes,

$$AV = VD$$

(5.46)

The square $n_x \times n_x$ matrix $V$ that has as columns the eigenvectors of $A$. The matrix $V$ is invertible if all eigenvalues of $A$ are distinct. Then through post multiplication by $V^{-1}$ we obtain from (5.46),
\[ A = VDV^{-1} \]  

(5.47)

Now substitute (5.47) in,

\[ e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + .. \]  

(5.48)

to obtain,

\[ e^{At} = VIV^{-1} + \frac{VDV^{-1}t}{1!} + \frac{VDV^{-1}VDV^{-1}t^2}{2!} + .. = \]

\[ V\left(I + \frac{Dt}{1!} + \frac{D^2t^2}{2!} + .. \right)V^{-1} = Ve^{Dt}V^{-1} \]  

(5.49)

Now from (5.42) and (5.45) verify that,

\[ e^{Dt} = \begin{bmatrix} e^{\lambda t} & 0 & 0 & 0 \\
0 & e^{\lambda t} & 0 & 0 \\
0 & 0 & .. & 0 \\
0 & 0 & 0 & e^{\lambda_{n}t} \end{bmatrix} \]  

(5.50)

Observe that to compute \( e^{Dt} \) we only need to compute the scalar exponentials on the diagonal which is no problem. Then if we are able to compute the eigenvalues and eigenvectors of the matrix \( A \) which we can do using standard software (the function eig within Matlab) then we can compute \( e^{At} \) according to (5.49), (5.50).

Now from (5.40) and (5.49) observe that the solution to (5.39) may be written as,

\[ x(t) = Ve^{Dt}V^{-1}x(t_0), \quad t \geq t_0 \]  

(5.51)

Then from (5.50) observe that if all eigenvalues \( \lambda_i, i = 1, 2,.., n \) are real and smaller than zero \( e^{\lambda_{n}t} \rightarrow 0 \) as \( t \rightarrow \infty \) and as a result \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \) in (5.51) for any \( x(t_0) \). Otherwise this does not hold. Note that the eigenvalues may also appear in
complex conjugate pairs. We will not consider this case. Without proof we summarise
the results which we have only proved for real eigenvalues.

(1) The systems (5.38) and (5.39) are stable if the eigenvalues of the matrix $A$
all have a real part which is negative. Then for the system (5.39) and for any
initial state $x(t_0)$ the system state $x(t) \to 0$ as $t \to \infty$. The same thing
applies to the system (5.38) if $u(t) = 0, t \geq t_0$. Knowing all the eigenvalues
and the associated eigenvectors of the matrix $A$ the matrix exponential $e^{\lambda t}$ is
easily computed using equations (5.45), (5.49), (5.50).

(2) To check whether a steady state is stable we check the stability of the
linearised system (5.14) about the steady state $u(t) = u^s, x(t) = x^s$. If this
linear system is stable so is the steady state and small arbitrary (initial) state
perturbations tend to zero. If not then the steady state is unstable and small
(initial) state perturbations do not tend to zero in general.

Exercise 1
Recall the state-space model (5.16) and its linearised model (5.18) about the steady
state $x^s = \begin{bmatrix} -5 \\ 4 \end{bmatrix}, u^s = \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}$. Determine from the linearised model (5.18) whether
this steady state is stable.

5.6 State-space models of interconnected dynamic systems
Consider two dynamic systems, both represented in state-space form. The
superscripts $a, b$ are used to distinguish between the two state-space models,

$$\frac{dx^a}{dt} = f^a \left(x^a, u^a, p^a \right)$$

(5.52)

$$y^a = g^a \left(x^a, p^a \right)$$

(5.53)

$$\frac{dx^b}{dt} = f^b \left(x^b, u^b, p^b \right)$$

(5.54)

$$y^b = g^b \left(x^b, p^b \right)$$

(5.55)

Suppose the outputs of system $a$ are connected with the inputs of system $b$,
\[ u^b = y^a \]  
\( (5.56) \)

and the outputs of system \( b \) are connected with inputs of system \( a \),

\[ u^a = y^b \]  
\( (5.57) \)

The interconnected system has no inputs and outputs anymore and therefore is an autonomous system. Given the importance of representing systems models in state-space form a natural question to ask is: “What is the state-space model of this interconnected autonomous system?” This state-space model is found as follows:

1. State variables: State variables of both systems \( a \) and \( b \):

   \[ x = \begin{bmatrix} x^a \\ x^b \end{bmatrix}. \]  
   \( (5.58) \)

2. No inputs, so no \( u \).

3. No outputs, so no \( y \) and \( g(x) \).

4. Parameters: parameters of both systems \( a \) and \( b \):

   \[ p = \begin{bmatrix} p^a \\ p^b \end{bmatrix}. \]  
   \( (5.59) \)

To find \( f(x, p) \) of the interconnected autonomous system observe from equations (5.58), (5.52), (5.54) that,

\[ \frac{dx}{dt} = \begin{bmatrix} \frac{dx^a}{dt} \\ \frac{dx^b}{dt} \end{bmatrix} = \begin{bmatrix} f^a(x^a, u^a, p^a) \\ f^b(x^b, u^b, p^b) \end{bmatrix} = f. \]  
\( (5.60) \)

Note that \( f = \begin{bmatrix} f^a(x^a, u^a, p^a) \\ f^b(x^b, u^b, p^b) \end{bmatrix} \) in equation (5.60) still contains the inputs \( u^a, u^b \) of the original separate systems. They have to be eliminated because they are no longer inputs of the interconnected autonomous system. The elimination is achieved
using equations (5.53), (5.55), (5.56), (5.57). Using (5.53), (5.56) $u^b_i$ may be replaced by $g^a_i(x^a, p^a)$, $i = 1, 2, ..., n^b_u$, an expression containing only states and parameters of system $a$. According to equations (5.58), (5.59) these are also states and parameters of the autonomous system. Similarly $u^a_i$ may be replaced by $g^b_i(x^b, p^b)$, $i = 1, 2, ..., n^a_u$, also an expression containing only states and parameters of the autonomous system. Substituting these expressions into (5.60) provides $f(x, p)$ of the interconnected autonomous system.

**Exercise 1**
Write down two arbitrary state-space models, each one having two inputs and two outputs. Interconnect these two systems as mentioned above and write down the state-space model of the interconnected system.

**Exercise 2**
Consider the two systems described by equations (5.52)-(5.55). Assume the only interconnection is described by equation (5.56). This is called a series interconnection (as opposed to parallel interconnection) of the two systems. Explain the name series interconnection. Does the interconnected system have inputs? Explain. Does it have outputs? Explain. How do we obtain the state-space model of the interconnected system?

**Exercise 3**
Consider again the two systems described by equations (5.52)-(5.55). Write down equation(s) that describe a parallel interconnection of the two systems. Also describe how to obtain the state-space model of the parallel interconnected system.

**5.7 Linearisation about optimal state, input and output trajectories**
What is the optimal trajectory of a rocket flying to the moon? What is the cheapest way to start up a chemical or biochemical reactor? What is the best trajectory of a racing car on the race circuit? In all these cases the control objective is not to keep the system in a steady state. In the second part of the system and control theory course (BRD31306), using state-space models of systems we will compute such optimal state trajectories $x^*(t)$ and the associated optimal control $u^*(t)$ and output trajectories $y^*(t)$. Furthermore we will design controllers to compensate for small
modelling errors and disturbances. The control system obtained in this way is represented in Figure 16.

![Control System Diagram](image)

**Figure 16: An optimal control system**

Apart from the computation of the optimal state, control and output trajectories, $x^*(t)$, $u^*(t)$, $y^*(t)$ another important issue is the design of the feedback controller to compensate for small disturbances and modelling errors. The feedback controller design is concerned with the problem of computing control corrections $\Delta u(t)$ from the measured state or output deviations $\Delta x(t) / \Delta y(t)$ with the objective to keep them small. Observe that,

$$\Delta x(t) = x(t) - x^*(t), \Delta u(t) = u(t) - u^*(t), \Delta y(t) = y(t) - y^*(t) \quad (5.61)$$

Since the deviations and corrections are small there dynamic behaviour can be very well approximated by a linearised model about the optimal trajectories $x^*(t)$, $u^*(t)$, $y^*(t)$. This linearised model is used for the feedback controller design because the design for linear systems is far more easy. Similar to (5.14), (5.15) this linearised model is mathematically described by,

$$\frac{d \Delta x}{dt} \approx \frac{\partial f}{\partial x}_{x=x^*(t),u=u^*(t)} \Delta x + \frac{\partial f}{\partial u}_{x=x^*(t),u=u^*(t)} \Delta u \quad (5.62)$$
\[
\Delta y \approx \frac{\partial g}{\partial x} \bigg|_{x=x'(t),u=u'(t)} \Delta x + \frac{\partial g}{\partial u} \bigg|_{x=x'(t),u=u'(t)} \Delta u \tag{5.63}
\]

Although the mathematical notation is very similar there is an important difference between equations (5.62), (5.63) and (5.14), (5.15). In equation (5.14), (5.15) at each time \( t \) the state and control around which we linearise the model is the same, namely, \( x(t) = x^{*}(t) \), \( u(t) = u^{*}(t) \) and so
\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=x'(t),u=u'(t)} \quad B = \left. \frac{\partial f}{\partial u} \right|_{x=x'(t),u=u'(t)},
\]
\[
C = \left. \frac{\partial g}{\partial x} \right|_{x=x'(t),u=u'(t)} \quad D = \left. \frac{\partial g}{\partial u} \right|_{x=x'(t),u=u'(t)}
\]
are the same matrices at each time \( t \). In (5.62), (5.63) at each time \( t \) the point of linearisation is different, namely \( x(t) = x^{*}(t) \), \( u(t) = u^{*}(t) \) and therefore at each time \( t \) the matrices
\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=x'(t),u=u'(t)}, \quad B = \left. \frac{\partial f}{\partial u} \right|_{x=x'(t),u=u'(t)} \quad C = \left. \frac{\partial g}{\partial x} \right|_{x=x'(t),u=u'(t)} \quad D = \left. \frac{\partial g}{\partial u} \right|_{x=x'(t),u=u'(t)}
\]
are different. To mathematically denote this difference the linearisation around the optimal state, control and output trajectory is described by,
\[
\frac{d\Delta x}{dt} = A(t)\Delta x(t) + B(t)\Delta u(t) \tag{5.64}
\]
\[
\Delta y(t) = C(t)\Delta x(t) + D(t)\Delta u(t) \tag{5.65}
\]

Equations (5.64), (5.65) represent a linear system which is time-varying. Time varying systems are the subject of the next paragraph. In summary time-varying linear systems are important for the feedback controller design which is part of the optimal control system design in Figure 16.

**Example 1**

Consider the system described by,
\[
\frac{dx}{dt} = -x^2 + u, \quad y = xu, \quad x, u, y \in \mathbb{R}^1 \tag{5.66}
\]
Suppose the optimal state control and output trajectories are,

\[
\begin{align*}
    x^*(t) &= \sin(t), \\
    u^*(t) &= \cos(t) + \sin^2(t), \\
    y^*(t) &= \sin(t)(\cos(t) + \sin^2(t))
\end{align*}
\]  \tag{5.67}

By substituting (5.67) in (5.66) we may verify that the optimal input, state and output response satisfy the systems model (5.66), as they should. The time-varying linearised state equation (5.64) about these trajectories is completely determined by the two matrices,

\[
\begin{align*}
    A(t) &= \frac{\partial f}{\partial x} \bigg|_{(x(t), u(t), \cos(t) + \sin^2(t))} = -2x(t) = -2\sin(t) \\
    B(t) &= \frac{\partial f}{\partial u} \bigg|_{(x(t), u(t), \cos(t) + \sin^2(t))} = 1
\end{align*}
\]  \tag{5.68}

**Exercise 1**

Determine the two matrices \( C(t), D(t) \) which determine the linearised output equation associated to example 1.

**5.8 Time varying and time-invariant systems**

In the previous section time-varying systems have been introduced. From the state space description of systems models we can immediately see whether a system is time-varying or not. If the time \( t \) appears explicitly in the function description of \( f(x,u) \) or \( g(x,u) \) then the system is time-varying, otherwise it is time-invariant.

**Example 1**

Consider the linearised state equation (5.64), (5.68) from which we obtain

\[
f(\Delta x, \Delta u) = -2\sin(t)\Delta x + \Delta u.
\]  

Because \( t \) explicitly appears in the description of \( f \), due to the term \( \sin(t) \), the linearised state equation, and so the associated system, is time-varying. All the other systems considered so far, except for those in the previous paragraph, are time-invariant.
Observe that in general \( x, u \) are time-functions. So instead of \( f(x,u), g(x,u) \) we may write \( f(x(t), u(t)), g(x(t), u(t)) \). From this notation it is clear that \( f,g \) are also time-functions. The vector functions \( f,g \) are said to depend implicitly on time and explicitly on \( x,u \). The function \( f(\Delta x, \Delta u) = -2\sin(t)\Delta x + \Delta u \) in example 1 depends explicitly on \( \Delta x, \Delta u \) and also explicitly on \( t \). To properly indicate this we should now actually write \( f(\Delta x, \Delta u, t) = -2\sin(t)\Delta x + \Delta u \). Similarly the general state-space notation of time-varying systems becomes,

\[
\frac{dx}{dt} = f(x,u,t), \quad y = g(x,u,t), \quad f, x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad g, y \in \mathbb{R}^p \tag{5.69}
\]

**Exercise 1**

A very simple example of a time-varying system is a rocket that burns a fixed amount of fuel per time unit due to which its mass is decreasing linearly with time. Presuming the rocket moves straight it may be seen as a mass with a force acting on it, see Figure 13. Note that the mass is not constant now but decreasing linearly with time,

\[
m = c_1 - c_2 t \tag{5.70}
\]

where \( c_1, c_2 \) are two constants. Modify the original state-space model (3.33), (3.34), (3.64), (3.65) to describe this time dependence of the mass and show that the resulting state-space model is time-varying.

### 5.9 Summary

1. To determine several system properties first write the system in state-space form. In other words determine \( f(x,u), g(x,u) \).
2. To determine whether a system is linear first check whether \( f(0,0) = 0, g(0,0) = 0 \). If not the system is not linear. Next compute
   \[
   A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}, \quad C = \frac{\partial g}{\partial x}, \quad D = \frac{\partial g}{\partial u}.
   \]
   Check whether all these matrices do not depend on \( x,u \). As soon as one of the matrix elements does depend on \( x,u \) the system is not linear. If the system is linear then it is represented by
\[
\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du. \text{ If the linear system is time-varying at least one of the matrices } A, B, C, D \text{ depends explicitly on time.}
\]

(3) To check whether a steady state \( x^{ss}, u^{ss} \) is stable compute the \( A \) matrix of the linearised system around this steady state: \( A = \frac{\partial f}{\partial x} \bigg|_{x=x^{ss}, u=u^{ss}} \in \mathbb{R}^{n_x \times n_x} \).

Compute the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{n_x} \) of this matrix and check whether the real part of each eigenvalue is negative. If so both the linearised system and the associated steady state are stable, otherwise they are not. If the steady state is stable without the use of any control corrections small arbitrary state deviations (perturbations) tend to zero ultimately. If a linear system is stable without any control any state ultimately converges to the zero state.

(4) A steady state is called stabilisable if with the use of control corrections small arbitrary state deviations (perturbations) can be controlled to zero ultimately.

(5) A system is called linear in the state \( x \) if the matrices \( A = \frac{\partial f}{\partial x}, \ C = \frac{\partial g}{\partial x} \) do not depend on \( x \). Similarly a system is called linear in the control \( u \) if the matrices \( B = \frac{\partial f}{\partial u}, \ D = \frac{\partial g}{\partial u} \) do not depend on \( u \). A system is called linear in the parameters \( p \) if the matrices \( E = \frac{\partial f}{\partial p}, \ F = \frac{\partial g}{\partial p} \) do not depend on \( p \).

(6) A system is called time-varying if the time \( t \) appears explicitly in the function description of \( f \) or \( g \).

(7) The linearised system about an optimal state and control trajectory \( x^*(t), u^*(t) \) characterised by \( A(t) = \frac{\partial f}{\partial x} \bigg|_{x(t)=x^*(t),u=u^*(t)} \), \( B(t) = \frac{\partial f}{\partial u} \bigg|_{x(t)=x^*(t),u=u^*(t)} \)
\[
C(t) = \frac{\partial g}{\partial x} \bigg|_{x(t)=x^*(t),u=u^*(t)} \quad \text{ and } \quad D(t) = \frac{\partial g}{\partial u} \bigg|_{x(t)=x^*(t),u=u^*(t)}
\]
is a time-varying linear system that is used for the design of a feedback controller in an optimal control system, see Figure 16.