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Numerical Algorithms and Issues Concerning the Discrete-Time Optimal Projection Equations

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The discrete-time optimal projection equations, which constitute necessary conditions for optimal reduced-order LQG compensation, are strengthened. *For the class of minimal stabilizing compensators the strengthened discrete-time optimal projection equations are proved to be equivalent to first-order necessary optimality conditions for optimal reduced-order LQG compensation. The conventional discrete-time optimal projection equations are proved to be* weaker. *As a result solutions of the conventional discrete-time optimal projection equations may* not *correspond to optimal reduced-order compensators. Through numerical examples it is demonstrated that, in fact, many solutions exist that do not correspond to optimal reduced-order compensators. To compute optimal reduced-order compensators two new algorithms are proposed. One is a homotopy algorithm and one is based on* iteration *of the strengthened discrete-time optimal projection equations. The latter algorithm is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration and therefore is highly efficient. Using different initializations of the iterative algorithm it is demonstrated that the reduced-order optimal LQG compensation problem, in general, may possess multiple extrema. Through two computer experiments it is demonstrated that the homotopy algorithm often, but not always, finds the global minimum.*

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1. Introduction

Controller reduction is a vital practical issue. Two approaches to controller reduction may be distinguished, direct versus indirect design [1]. Indirect design is characterized by the fact that the design is performed in two steps, instead of one. The two steps concern either model-reduction followed by full-order controller design or full-order controller design followed by controller reduction. A major disadvantage of these indirect approaches is that stability of the closed loop system, in general, cannot be guaranteed, and optimality, in general, is lost. Direct design on the other hand incorporates both stability of the closed loop system and optimality. Therefore if, given the design criteria, a direct design method is feasible, it should always be preferred. This paper deals with the direct design of optimal reduced-order LQG controllers for time-invariant discrete-time systems.

Necessary conditions for optimal reduced-order LQG compensation have been presented in both the continuous-time case [2] and the discrete-time case [3], as a set of four coupled matrix equations. They are known as the optimal projection equations, since an oblique projection is a fundamental part of these equations. Although this was not mentioned explicitly, a small flaw was removed from the discrete-time optimal projection equations in [4], see also [5]. In all these

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papers the necessary conditions were *derived* from first-order necessary optimality conditions assuming the compensator to be minimal and stabilizing. The papers suggested, but did not prove, the *equivalence* of the optimal projection equations with first-order necessary optimality conditions for the class of minimal stabilizing compensators. In this paper it is demonstrated that, in the discrete-time case, this equivalence does not hold. From this analysis so-called strengthened discrete-time optimal projection equations (SDOPE) are obtained, which differ from the conventional discrete-time optimal projection equations (CDOPE) as presented in [4] and [5]. As opposed to the CDOPE, the SDOPE are proved to be equivalent to first-order necessary optimality conditions together with the condition that the compensator is minimal. Although the differences between the SDOPE, the CDOPE and the result presented in [3] are rather small and subtle, they are crucial when it comes to calculating numerical solutions, as demonstrated in this paper. Note that the strengthening of the discrete-time optimal projection equations also applies to the discrete-time optimal reduced-order modelling and filtering problem [3].

The optimal projection equations, for the first time, provided a good insight in the reduced-order LQG problem. They revealed relations with standard (full-order) LQG theory. Furthermore they provided an attractive alternative to compute optimal reduced-order LQG compensators [3,4,6,7]. Until then optimal reduced-order LQG compensators could only be computed through constrained non-linear parameter optimization which exhibits many drawbacks related to non-linear optimization. In the continuous-time case attempts have been made to find necessary and sufficient conditions for optimal reduced-order LQG compensation [6,7]. These results were carried over to the discrete-time case in [4]. The results of this paper indicate that these attempts, so far, have failed.

Based on the SDOPE two new algorithms are proposed to compute discrete-time optimal reduced-order LQG compensators. One is a homotopy algorithm. Although homotopy algorithms have been proposed before [4,7], the homotopy is different in our case. The other algorithm iterates the SDOPE and is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration. In two computer experiments, using different initializations of the iterative algorithm, it is demonstrated that the reduced-order optimal LQG compensation problem, in general, may possess *multiple extrema*. The computer experiments also show that the homotopy algorithm, which finds only one solution, often, but not always, finds the global minimum. The iterative algorithm is shown to be highly efficient compared to both the homotopy algorithm and constrained non-linear parameter optimization. An example for which the optimal full-order compensator is not minimal is included. For this example the algorithms generate a minimal realization of the full-order compensator. This shows that the algorithms *automatically* reduce the order of the compensator if a minimal optimal compensator with the prescribed order cannot be found.

Finally a numerical example is presented which shows that the CDOPE have *many* solutions which do not correspond to optimal reduced-order compensators. Furthermore an analytical argument is used to show that the *desired* solutions, in general, are *never* obtained through *iteration* of the CDOPE. Also it is clarified why initially this phenomenon was overlooked.

2. The Optimal Reduced-Order LQG Compensation Problem

Consider the system,

$$x_{i+1} = \Phi x_i + \Gamma u_i + \nu_i, \tag{1.1}$$

$$y_i = C x_i + w_i, \quad i = 0, 1, 2, \dots, \tag{1.2}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ is the control, $y_i \in R^l$ is the observation, $\nu_i \in R^n$ the system noise, $w_i \in R^l$ the observation noise and Φ, Γ, C are real matrices of appropriate dimensions. The processes $\{\nu_i\}$, $\{w_i\}$ are uncorrelated zero-mean white noise sequences with covariance $V \geq 0$ and $W > 0$ respectively. The initial condition x_0 is a stochastic variable with mean $\bar{x}_0$ and covariance P_0 and is uncorrelated with $\{\nu_i\}$ and $\{w_i\}$. System (1) is denoted by (Φ, Γ, C). As controller we choose the following dynamic compensator:

$$\hat{x}_{i+1} = F\hat{x}_i + K y_i, \tag{2.1}$$

$$u_i = -L\hat{x}_i, \quad i = 0, 1, 2, \dots, \tag{2.2}$$

where $\hat{x}_i \in R^{n_c}$ is the compensator state, and F, K, L are real matrices of appropriate dimension. The initial condition $\hat{x}_0$ is deterministic. It is assumed that $n \geq n_c$. Compensator (2) is denoted by (F, K, L).

Definition 1. (Φ, Γ, C) is called n_c-compensatable if there exists a compensator (F, K, L) of dimension n_c such that the closed loop system is stable. □

A number of properties concerning reduced-order compensatability are stated in the following theorem.

Theorem 1 [4]

(a) Φ stable $\Rightarrow$ (Φ, Γ, C) n_c-compensatable, $\forall n_c$,

(b) (Φ, Γ, C) n_c-compensatable $\Leftrightarrow (\Phi^T, C^T, \Gamma^T)$ n_c-compensatable,
(c) (Φ, Γ, C) n-compensatable $\Leftrightarrow (\Phi, \Gamma)$ and (Φ^T, C^T) both stabilizable,
(d) $n_1 > n_2 \Rightarrow$ (n_2-compensatabilty $\Rightarrow n_1$-compensatability). □

A prerequisite for the compensator is that it stabilizes the closed loop system. If it does the compensator is called a stabilizing compensator. The derivation of the (strengthened) optimal projection equations presumes the invertibility of the second-moment matrix of the compensator and the dual compensator [2,9,13]. This invertibility is equivalent to the minimality of the compensator [13]. Since the input–output behavior of the compensator is independent of its realization and since minimal realizations require less storage and computation, they are also preferred from a practical point of view. Therefore the optimal reduced-order LQG compensation problem is to find the minimal stabilizing compensator (F^*, K^*, L^*), of given dimension n_c, that minimizes the criterion

$$\begin{aligned} &\sigma_\infty(F, K, L) \\ &= \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i)\right\}, \\ &Q \geq 0, \quad R \geq 0, \end{aligned} \tag{3}$$

and to find the minimum $\sigma_\infty^* = \sigma_\infty(F^*, K^*, L^*)$. In exceptional cases the optimal compensator with dimension $n_c \leq n$ is not minimal [12,13] and formally falls outside the scope of the problem formulation. This situation is discussed after Theorem 3 in Section 3 and is illustrated by, and resolved, in Section 6, Example 3, see also [13].

3. The Strengthened Discrete-Time Optimal Projection Equations

As in [2] first-order necessary optimality conditions can be obtained by applying the matrix minimum principle [8] to the closed loop system and performance. The main theorem in this section is obtained by rearrangement of these first-order necessary optimality conditions. To state the main theorem in this section the following lemma is needed.

Lemma 1 [9]. Suppose $\hat{P}, \hat{S} \in R^{n\times n}$ are non-negative and $\mathrm{rank}(\hat{P}\hat{S}) = n_c$. Then there exist $G, H \in R^{n_c\times n}$ and invertible $M \in R^{n_c\times n_c}$ such that

$$\hat{P}\hat{S} = G^T M H, \tag{4.1}$$

$$HG^T = I_{n_c}. \tag{4.2}$$

Define

$$\tau = G^T H \tag{4.3}$$

and let # denote the group inverse which is unique [10]. Then,

$$(\hat{P}\hat{S})^{\#} = G^T M^{-1} H, \tag{4.4}$$

$$\tau^2 = \tau = \hat{P}\hat{S}(\hat{P}\hat{S})^{\#}, \tag{4.5}$$

$$\begin{aligned} \mathrm{rank}(G) = \mathrm{rank}(M) &= \mathrm{rank}(H) \\ &= \mathrm{rank}(\tau) = n_c: \end{aligned} \tag{4.6}$$

so τ is an oblique projection (idempotent matrix) uniquely determined by $\hat{P}$ and $\hat{S}$. G, M, H are unique up to a change of basis in R^{n_c}. The triple (G, M, H) is called a projective factorization of $\hat{P}\hat{S}$. □

$\hat{P}\hat{S}$ in Lemma 1 is diagonalizable and has n_c non-zero eigenvalues which are positive [9]. Hence G, M, H and τ can be computed from an eigenvalue decomposition of $\hat{P}\hat{S}$ as follows:

$$\hat{P}\hat{S} = U_{\hat{P}\hat{S}} \Lambda_{\hat{P}\hat{S}} U_{\hat{P}\hat{S}}^{-1}, \tag{5.1}$$

$$G = \begin{bmatrix} A^T & 0 \end{bmatrix} U_{\hat{P}\hat{S}}^T, \tag{5.2}$$

$$M = A^{-1} \Theta_{\hat{P}\hat{S}} A, \tag{5.3}$$

$$H = \begin{bmatrix} A^{-1} & 0 \end{bmatrix} U_{\hat{P}\hat{S}}^{-1}, \tag{5.4}$$

$$\tau = U_{\hat{P}\hat{S}} \begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix} U_{\hat{P}\hat{S}}^{-1}, \tag{5.5}$$

where the columns of $U_{\hat{P}\hat{S}}$ are eigenvectors of $\hat{P}\hat{S}$ and the elements of the diagonal matrix

$$\Lambda_{\hat{P}\hat{S}} = \begin{bmatrix} \Theta_{\hat{P}\hat{S}} & 0 \\ 0 & 0 \end{bmatrix}$$

are the eigenvalues of $\hat{P}\hat{S}$. The n_c non-zero diagonal elements of $\Lambda_{\hat{P}\hat{S}}$ are the diagonal elements of $\Theta_{\hat{P}\hat{S}}$. $A \in R^{n_c\times n_c}$ in (5.2)–(5.4) is an arbitrary non-singular matrix. This reflects the uniqueness of G, M, H up to a change of basis in R^{n_c}.

For convenience the following notations are introduced:

$$W_P = W + CPC^T, \tag{6.1}$$

$$R_S = R + \Gamma^{\mathrm{T}} S \Gamma, \tag{6.2}$$

$$K_P = \Phi P C^{\mathrm{T}} W_P^{-1}, \tag{6.3}$$

$$L_S = R_S^{-1} \Gamma^{\mathrm{T}} S \Phi, \tag{6.4}$$

$$\Sigma_P^1 = \Phi P C^{\mathrm{T}} W_P^{-1} (\Phi P C^{\mathrm{T}})^{\mathrm{T}} = K_P W_P K_P^{\mathrm{T}}, \tag{6.5}$$

$$\Sigma_S^2 = (\Gamma^{\mathrm{T}} S \Phi)^{\mathrm{T}} R_S^{-1} \Gamma^{\mathrm{T}} S \Phi = L_S^{\mathrm{T}} R_S L_S, \tag{6.6}$$

$$\Phi_P^1 = \Phi - \Phi P C^{\mathrm{T}} W_P^1 C = \Phi - K_P C, \tag{6.7}$$

$$\Phi_S^2 = \Phi - \Gamma R_S^{-1} \Gamma^{\mathrm{T}} S \Phi = \Phi - \Gamma L_S, \tag{6.8}$$

$$\Psi^1_{S.\hat{P}.P} = \Phi_S^2 \hat{P} \Phi_S^{2^{\mathrm{T}}} + \Sigma_P^1, \tag{6.9}$$

$$\Psi^2_{P.\hat{S}.S} = \Phi_P^{1^{\mathrm{T}}} \hat{S} \Phi_P^1 + \Sigma_S^2, \tag{6.10}$$

$$\tau_\perp = I_n - \tau. \tag{6.11}$$

Theorem 2. A stabilizing compensator (F, K, L) satisfies the first-order necessary optimality conditions for optimal reduced-order LQG compensation and is minimal *if and only if* there exist non-negative symmetric $n \times n$ matrices $P, S, \hat{P}, \hat{S}$ such that for some projective factorization (G, M, H) of $\hat{P}\hat{S}$,

$$F = H[\Phi - K_P C - \Gamma L_S] G^{\mathrm{T}}, \tag{7.1}$$

$$K = H K_P, \tag{7.2}$$

$$L = L_S G^{\mathrm{T}}. \tag{7.3}$$

and such that $P, S, \hat{P}, \hat{S}, \tau$ satisfy

$$P = \Phi P \Phi^{\mathrm{T}} - \Sigma_P^1 + V + \tau_\perp \Psi^1_{S.\hat{P}.P} \tau_\perp^{\mathrm{T}}, \tag{8.1}$$

$$S = \Phi^{\mathrm{T}} S \Phi - \Sigma_S^2 + Q + \tau_\perp^{\mathrm{T}} \Psi^2_{P.\hat{S}.S} \tau_\perp, \tag{8.2}$$

$$\hat{P} = \tfrac{1}{2}\left[\tau \Psi^1_{S.\hat{P}.P} + \Psi^1_{S.\hat{P}.P} \tau^{\mathrm{T}}\right], \tag{8.3}$$

$$\hat{S} = \tfrac{1}{2}\left[\tau^{\mathrm{T}} \Psi^2_{P.\hat{S}.S} + \Psi^2_{P.\hat{S}.S} \tau\right], \tag{8.4}$$

$$\operatorname{rank}(\hat{P}) = \operatorname{rank}(\hat{S}) = \operatorname{rank}(\hat{P}\hat{S}) = n_c, \tag{8.5}$$

$$\tau = \hat{P}\hat{S}(\hat{P}\hat{S})^{\#}. \tag{8.6}$$

For the costs we have

$$\sigma_\infty = \sigma_{Q.R} = \sigma_{V.W}, \tag{9.1}$$

$$\sigma_{Q.R} = \operatorname{tr}\left[QP + (Q + L_S^{\mathrm{T}} R L_S)\hat{P}\right], \tag{9.2}$$

$$\sigma_{V.W} = \operatorname{tr}\left[VS + (V + K_P W K_P^{\mathrm{T}})\hat{S}\right]. \tag{9.3}$$

□

Proof. The proof of Theorem 2 is given in Appendix 1. □

Equations (8.1)–(8.6) are the SDOPE. The SDOPE differ from the CDOPE [4,5] which in turn differ from the discrete-time optimal projection equations originally presented in [3]. The differences concern the expressions for $\hat{P}$ and $\hat{S}$. In [3],

$$\hat{P} = \Phi_S^2 \tau \hat{P} \tau^{\mathrm{T}} \Phi_S^{2^{\mathrm{T}}} + \Sigma_P^1, \tag{10.1}$$

$$\hat{S} = \Phi_P^{1^{\mathrm{T}}} \tau^{\mathrm{T}} \hat{S} \tau \Phi_P^1 + \Sigma_S^2. \tag{10.2}$$

Although this was not mentioned explicitly a small flaw in the above equations was removed in [4], see also [5]. This resulted in

$$\begin{aligned} \hat{P} &= \tau\left(\Phi_S^2 \hat{P} \Phi_S^{2^{\mathrm{T}}} + \Sigma_P^1\right)\tau^{\mathrm{T}} \\ &= \tau \Psi^1_{S.\hat{P}.P} \tau^{\mathrm{T}}, \end{aligned} \tag{11.1}$$

$$\begin{aligned} \hat{S} &= \tau^{\mathrm{T}}\left(\Phi_P^{1^{\mathrm{T}}} \hat{S} \Phi_P^1 + \Sigma_S^2\right)\tau \\ &= \tau^{\mathrm{T}} \Psi^2_{P.\hat{S}.S} \tau. \end{aligned} \tag{11.2}$$

The difference between Eqs (8.3) and (8.4), which are part of the SDOPE, and Eqs (11.1) and (11.2), which are part of the CDOPE, relates to the following equalities which must hold if the first-order necessary optimality conditions are to be satisfied (see Section A4 of Appendix 1):

$$\hat{P} = \tau \Psi^1_{S.\hat{P}.P} = \Psi^1_{S.\hat{P}.P} \tau^{\mathrm{T}} = \tau \Psi^1_{S.\hat{P}.P} \tau^{\mathrm{T}}, \tag{12.1}$$

$$\hat{S} = \tau^{\mathrm{T}} \Psi^2_{P.\hat{S}.S} = \Psi^2_{P.\hat{S}.S} \tau = \tau^{\mathrm{T}} \Psi^2_{P.\hat{S}.S} \tau. \tag{12.2}$$

Now Eqs (8.3) and (8.4) ensure that these equalities hold. Equations (11.1) and (11.2) however do *not* guarantee the second and third equality in (12.1) and (12.2) to hold (see Section A4 of Appendix 1). Although the differences mentioned above are rather small and subtle, they are crucial when it comes to calculating numerical solutions, as demonstrated in Sections 5 and 6.

Notice that in the full-order case the optimal projection matrix τ becomes the identity matrix while its factors G and H can be chosen to be the identity matrix. In that case (8.1) and (8.2) reduce to the standard observation and control Riccati equations and from (7.1)–(7.3) the optimal full-order compensator is obtained. Equations (8.3)–(8.6) then express the proviso that the compensator be minimal. The coupling of the equations due to the projection illustrates the non-optimality of sequential controller reduction or model reduction schemes, because in the reduced-order case there is no longer separation between observation and control operations.

To state the following theorems, it is convenient to introduce the notion of detectability and observability of a triple, instead of a pair of matrices.

Definition 2. (Φ, Γ, C) is called detectable if (Φ, C) and $(\Phi^{\mathrm{T}}, \Gamma^{\mathrm{T}})$ are both detectable. (Φ, Γ, C) is called observable if (Φ, C) and $(\Phi^{\mathrm{T}}, \Gamma^{\mathrm{T}})$ are both observable. □

Theorem 3. Assume $(\Phi, V^{1/2}, Q^{1/2})$ is detectable. Then all non-negative solutions $(P, S, \hat{P}, \hat{S})$ of (8) correspond to all minimal stabilizing compensators that satisfy the first-order necessary optimality conditions. □

Proof. This can be seen from [11, Theorem 3], where the more general case of systems with white parameters is considered. The order of the compensator plays no role. □

It can be shown that for the stability of the closed loop system, or for the compensator to be stabilizing, the detectability condition in Theorem 3 may be weakened to: $(\Phi, Q^{1/2})$ is detectable or $(\Phi^{T}, V^{1/2})$ is detectable.

From [12,13] the optimal compensator with prescribed dimensions may not be minimal, even if the system (Φ, Γ, C) is minimal and $(\Phi, V^{1/2}, Q^{1/2})$ is observable. These compensators do not fall within the scope of Theorems 2 and 3. However, if the optimal compensator is not minimal the prescribed compensator dimensions can be reduced, without loss of performance. Example 3 in Section 6 illustrates that the algorithms *automatically* reduce the order of the compensator if an optimal compensator with the prescribed dimension cannot be found.

So far first-order necessary optimality conditions have been considered. In the continuous-time case attempts have been made to find necessary and sufficient conditions for optimal reduced-order LQG compensation [6,7]. These results were carried over to the discrete-time case [4]. Along with the introduction of two numerical algorithms, in the next section these results, which rely on homotopy degree theory, will be reconsidered.

4. Numerical Algorithms

Let S_n denote the space of n-dimensional real symmetric matrices. Define the following non-linear transformation based on the SDOPE:

$$
\begin{aligned}
&\Re X : S_n \times S_n \times S_n \times S_n \to S_n \times S_n \times S_n \times S_n, \\
&\Re X = \Big(\Phi X_1 \Phi^{\mathrm{T}} - \Sigma^1_{X_1} + V + \tau_\perp \Psi^1_{X_2,X_3,X_1} \tau_\perp^{\mathrm{T}}, \\
&\qquad \Phi^{\mathrm{T}} X_2 \Phi - \Sigma^2_{X_2} + Q + \tau_\perp^{\mathrm{T}} \Psi^2_{X_1,X_4,X_2} \tau_\perp, \\
&\qquad \tfrac{1}{2}\left[\tau \Psi^1_{X_2,X_3,X_1} + \Psi^1_{X_2,X_3,X_1} \tau^{\mathrm{T}}\right], \\
&\qquad \tfrac{1}{2}\left[\tau^{\mathrm{T}} \Psi^2_{X_1,X_4,X_2} + \Psi^2_{X_1,X_4,X_2} \tau\right]\Big),
\end{aligned} \tag{13.1}
$$

where

$$X = (X_1, X_2, X_3, X_4), \quad X_1, X_2, X_3, X_4 \in S_n, \tag{13.2}$$

$$\tau = U_{X_3X_4} \begin{bmatrix} I_{n_c'} & 0 \\ 0 & 0 \end{bmatrix} U^{-1}_{X_3X_4}, \tag{13.3}$$

$$n_c' = \min(n_c, \operatorname{rank}(X_3X_4)). \tag{13.4}$$

In Eq. (13.3) $U_{X_3X_4}$ is obtained from the eigenvalue decomposition

$$X_3X_4 = U_{X_3X_4} \Lambda_{X_3X_4} U^{-1}_{X_3X_4}, \tag{13.5}$$

where the eigenvalues, i.e. the diagonal elements of the diagonal matrix $\Lambda_{X_3X_4}$, appear in ascending order of their real part. If during the iteration, for some reason, $\operatorname{rank}(X_3X_4) < n_c$, Eq. (13.4) ensures that τ in (8.6) is still properly computed. This situation occurs in Example 3 of Section 6. There the implications of

Eq. (13.4) will be discussed further. Observe that $(P, S, \hat{P}, \hat{S}) = \Re(P, S, \hat{P}, \hat{S})$ is equivalent to (8) if $\operatorname{rank}(\hat{P}\hat{S}) = n_c$. Now consider $(X_{1i}, X_{2i}, X_{3i}, X_{4i}) = \Re^i(X_{10}, X_{20}, X_{30}, X_{40})$, $i = 0, 1, 2, \ldots$ If $n_c = n$ then X_{1i} and X_{2i} are iterations of the well-known *uncoupled* observation and control Riccati equations. It is well known that $\{X_{1i}\}$ and $\{X_{2i}\}$ are monotonic if $X_{10} = X_{20} = 0$ in the sense that $X_{1i} \le X_{1_j}$ and $X_{2i} \le X_{2_j}$ if $i < j$. This property may be used to prove convergence of $\{X_{1i}\}$ and $\{X_{2i}\}$ and provides an easy way to compute a solution of the algebraic observation and control Riccati equations. If $n_c < n$ however, $\{X_{1i}\}$ and $\{X_{2i}\}$ are *not* monotonic due to the coupling between the corresponding equations. Fortunately it is still possible to obtain convergence using the method of homotopic continuation. This method embeds an original problem in a parameterized family of problems, where the parameter value varies continuously from 0 to 1. The idea is that for the parameter value 0 an easy problem with a known solution, in our case the full-order LQG problem, is obtained while for the parameter value 1 the original problem, in our case the reduced-order LQG problem, is obtained. We may follow the solution path as the easy problem is deformed into the original problem. For more information on homotopies and their computation, we refer to [14]. In order to use the above mentioned method define the nonlinear transformation

$$
\begin{aligned}
&\Re_\alpha X : S_n \times S_n \times S_n \times S_n \to S_n \times S_n \times S_n \times S_n, \\
&\quad \alpha \in [0, 1], \\
&\Re_\alpha X = \Big(\Phi X_1 \Phi^{\mathrm{T}} - \Sigma^1_{X_1} + V + \tau_{\alpha\perp} \Psi^1_{X_2, X_3, X_1} \tau^{\mathrm{T}}_{\alpha\perp}, \\
&\qquad \Phi^{\mathrm{T}} X_2 \Phi - \Sigma^2_{X_2} + Q + \tau^{\mathrm{T}}_{\alpha\perp} \Psi^2_{X_1, X_4, X_2} \tau_{\alpha\perp}, \\
&\qquad \tfrac{1}{2}\left[\tau_\alpha \Psi^1_{X_2, X_3, X_1} + \Psi^1_{X_2, X_3, X_1} \tau^{\mathrm{T}}_\alpha\right], \\
&\qquad \tfrac{1}{2}\left[\tau^{\mathrm{T}}_\alpha \Psi^2_{X_1, X_4, X_2} + \Psi^2_{X_1, X_4, X_2} \tau_\alpha\right] \Big),
\end{aligned}
\tag{14.1}
$$

where

$$
\tau_\alpha = U_{X_3 X_4} \begin{bmatrix} I_{n_c'} & 0 \\ 0 & (1-\alpha) I_{n-n_c'} \end{bmatrix} U^{-1}_{X_3 X_4}, \quad \alpha \in [0, 1],
\tag{14.2}
$$

with n_c' given by (13.4). In Eq. (14.2) $U_{X_3 X_4}$ is obtained from the eigenvalue decomposition (13.5) with the associated ordering of the eigenvalues. Call $X = (X_1, X_2, X_3, X_4)$ non-negative if $X_1, X_2, X_3, X_4 \ge 0$. Denote the parameterized equation $Y^\alpha = \Re_\alpha Y^\alpha$ by $H(Y^\alpha, \alpha) = 0$, where Y^α denotes the non-negative solution of $X = \Re_\alpha X$. The function $H(Y^\alpha, \alpha)$ is called a homotopy. For $\alpha = 1$ we have the original coupled SDOPE associated with the reduced-order problem, and for $\alpha = 0$ the uncoupled control and observation Riccati equations associated with the full-order problem. Based on the homotopy $H(Y^\alpha, \alpha)$ the following discrete homotopy algorithm is proposed.

Algorithm 1

Initialization:

$X_1^0 = 0,\ X_2^0 = 0,\ X_3^0 = I_n,\ X_4^0 = I_n,$
$\alpha = 0,\ \Delta\alpha = 1/N,\ N \ge 1$ and integer.

Compute $Y^\alpha = \lim_{i\to\infty} \Re^i_\alpha(X^0)$ through iteration.

Loop:

$\alpha := \alpha + \Delta\alpha$

Determine, through iteration, whether

$Y^\alpha = \lim\limits_{i\to\infty} \Re^i_\alpha(Y^{\alpha-\Delta\alpha})$ exists.

Stop when $\alpha = 1$. □

Consider again the homotopy $H(Y^\alpha, \alpha)$. If the number of solutions of the equation $H(Y^\alpha, \alpha) = 0$ from $\alpha = 0$ to $\alpha = 1$ remains constant, then, given the uniqueness of the optimal full-order compensator, Theorem 3 would give us necessary and sufficient conditions for the existence of a unique optimal reduced-order compensator. Similarly the algorithm that computes the unique solution of the two Riccati equations of full-order LQG control through iteration, could be carried over to an algorithm that computes, what would be the unique non-negative solution of the SDOPE, through iteration [4]. In the continuous-time case results have been published concerning conditions under which the number of solutions along the solution path remains constant [6,7]. These conditions were carried over to the discrete-time case [4]. On the other hand convex analysis of systems controlled by static output-feedback [15] seems to indicate that, in general, multiple nonnegative solutions satisfying the SDOPE may exist. Through numerical examples the latter is confirmed in Section 7. Despite this result we could still pursue the idea of iterating the SDOPE.

Algorithm 2

Initialization:

$$X_0^1 = 0,\ X_2^1 = 0,\ X_3^1 = \Lambda_1,\ X_4^1 = \Lambda_2$$

with $\Lambda_1, \Lambda_2 \ge 0$, symmetric, random and with rank n_c.

Computation:

Determine, through iteration, whether $Y^1 = \lim_{i\to\infty} \Re^i_1(X^1)$ exists. □

Theorem 4. If $(\Phi, V^{1/2}, Q^{1/2})$ is detectable then Algorithms 1 and 2, if they converge to $Y^1 \ge 0$, generate minimal stabilizing compensators, given by (7), with a minimal dimension equal to $n_c' = \operatorname{rank}(Y_3^1 Y_4^1) \le n_c$ and costs σ_∞, given by (9), where $(P, S, \hat{P}, \hat{S}) = (Y_1^1, Y_2^1, Y_3^1, Y_4^1)$. These compensators are local or global *minima* of the optimal reduced-order LQG compensation problem with prescribed compensator order n_c'. □

Proof. If the algorithms converge $\operatorname{rank}(Y_3^1 Y_4^1) \le n_c$. Then from (14) and Theorem 3, if $Y^1 \ge 0$, Y^1 corresponds to a minimal stabilizing compensator with dimension $n_c' = \operatorname{rank}(Y_3^1 Y_4^1) \le n_c$ which satisfies the first-order necessary optimality conditions when the prescribed compensator order equals n_c'. Because both Algorithm 1 and 2 are *generalizations* of the algorithm that solves the two Riccati equations of full-order LQG control through iteration, they converge to local (global) minima, *not* to local (global) maxima, which also satisfy the first-order necessary optimality conditions. □

The discussion in Section 6, related to example 3, and the large number of examples in Section 7 show that, in general, $n_c' = n_c$ in Theorem 4. Furthermore all the examples considered in Section 6 and also the large number of examples considered in Section 7 all share the property that, if Algorithm 1 or 2 converges, it converges to a non-negative solution. This suggests that, if the algorithms converge, $Y^1 \ge 0$. From Definition 1, Theorems 1 and 4 the following numerical test, representing *sufficient* conditions for n_c-compensatability, is obtained.

Compensatability Test. Check if Φ is stable. If so (Φ, Γ, C) is n_c-compensatable $\forall n_c$. If not, choose $R = I$, $W = I$, $Q = I$, $V = I$. Then (Φ, Γ, C) is n_c-compensatable if Algorithm 1 or 2, for some Λ_1, Λ_2, converges to $Y^1 \ge 0$. If not, nothing can be concluded with respect to the n_c-compensatability of (Φ, Γ, C). □

5. The Strengthened Versus the Conventional Discrete-Time Optimal Projection Equations

Consider the following non-linear transformation which is comparable to (13) but, instead of the SDOPE, is based on the CDOPE:

$$\Re_c X : S_n \times S_n \times S_n \times S_n \to S_n \times S_n \times S_n \times S_n,$$

$$\Re_c X = \Big(\Phi X_1 \Phi^T - \Sigma^1_{X_1} + V + \tau_\perp \Psi^1_{X_2, X_3, X_1} \tau_\perp^T, \\ \Phi^T X_2 \Phi - \Sigma^2_{X_2} + Q + \tau_\perp^T \Psi^2_{X_1, X_4, X_2} \tau_\perp, \\ \tau \Psi^1_{X_2, X_3, X_1} \tau^T, \tau^T \Psi^2_{X_1, X_4, X_2} \tau\Big). \quad (15)$$

where X and τ are given by (13.2)–(13.4). Since (X_1, X_2, X_3, X_4) corresponds to $(P, S, \hat{P}, \hat{S})$ we will refer to the latter if it suits us. Also the iteration index i is used whenever it suits us.

Lemma 2. Consider $(P_i, S_i, \hat{P}_i, \hat{S}_i) = \Re_c^i(0, 0, \Lambda_1, \Lambda_2)$ $i = 0, 1, 2, \ldots$ with $\Lambda_1, \Lambda_2 \ge 0$, symmetric, random and with rank n_c. Assume $\operatorname{rank}(\hat{P}_i \hat{S}_i) = n_c$, $i = 0, 1, 2, \ldots$ Then $\tau_i = \tau_0$, $i \ge 0$, i.e. τ_i is invariant under $\Re_c$. □

Proof. Since $\operatorname{rank}(\hat{P}_i \hat{S}_i) = n_c$, $i \ge 0$ from (13.3), (13.4) $\operatorname{rank}(\tau_i) = n_c$, $i \ge 0$. Then from (15),

$$\hat{P}_{i+1} \hat{S}_{i+1} = G_i^T H_i \Psi_i^1 H_i^T G_i \Psi_i^2 G_i^T H_i, \quad i \ge 0. \quad (16)$$

From (4) we have

$$\hat{P}_{i+1} \hat{S}_{i+1} = G_{i+1}^T M_{i+1} H_{i+1} \quad (17.1)$$

with

$$G_{i+1} = G_i \in R^{n_c \times n}, \quad (17.2)$$

$$H_{i+1} = H_i \in R^{n_c \times n}, \quad (17.3)$$

$$M_{i+1} = H_i \Psi_i^1 H_i^T G_i \Psi_i^2 G_i^T \in R^{n_c \times n_c}, \quad (17.4)$$

while M_{i+1} is positive definite because of the assumption $\operatorname{rank}(\hat{P}_{i+1} \hat{S}_{i+1}) = n_c$. Since M_{i+1} is positive definite, from Lemma 1, (17.1) is a projective factorization of $\hat{P}_{i+1} \hat{S}_{i+1}$. From Lemma 1, (15) and (8.6), it follows that

$$\tau_{i+1} = G_{i+1}^T H_{i+1} = G_i^T H_i = \tau_i, \quad i \ge 0. \quad (18)$$

Since τ_i is uniquely determined by $\hat{P}_i \hat{S}_i$ through Eq. (8.6) other projective factorizations do not alter this result. □

Lemma 2 implies that, in general, iterations of the CDOPE leave τ unchanged, so τ does not converge to an optimal value. In Section 6 it is demonstrated through numerical examples, that despite this property, $\Re_c^i(0, 0, \Lambda_1, \Lambda_2)$ often converges. Only if the initial value of τ, determined by Λ_1, Λ_2, is optimal, these solutions of the CDOPE correspond to optimal reduced-order compensators, otherwise they do not.

6. Numerical Issues and Examples

The function eig (Matlab reference guide [16]) together with the function esort (Control system toolbox for use with Matlab [17]) was used to compute the eigenvalue decomposition (13.5). In (13.4) $\operatorname{rank}(X_3X_4)$ is computed as the number of eigenvalues of X_3X_4 with a magnitude larger than 10^{-6} times the largest. Since (13) is initialized with symmetric matrices, X_3 and X_4 are always symmetric and so the eigenvalues of X_3X_4 are real numbers. The eigenvectors associated to the zero eigenvalues may be complex so Matlab may produce complex matrices $U_{X_3X_4}$, $\Lambda_{X_3X_4}$. The computation of τ and its factors G and H only requires real parts of $U_{X_3X_4}$, $\Lambda_{X_3X_4}$. G and H are computed according to (5.2) and (5.4) with $\hat{P}\hat{S} = X_3X_4$ and $A = I_{n_c}$.

The iterations of Algorithms 1 and 2 are numerically stable in general. In critical situations, e.g. if the closed loop system is at the edge of stability, the numerical stability is greatly enhanced if we make the following modifications: (1) After each iteration compute,

$$X_{1_i} := \tfrac{1}{2}\left(X_{1_i} + X_{1_i}^{\mathrm{T}}\right),$$
$$X_{2_i} := \tfrac{1}{2}\left(X_{2_i} + X_{2_i}^{\mathrm{T}}\right), \quad i = 0, 1, 2, \ldots \tag{19}$$

to further enhance the symmetry of X_{1i}, X_{2i}. (2) Immediately after (13) and (17) modify $X_i = (X_{1_i}, X_{2_i}, X_{3_i}, X_{4_i})$ according to

$$X_{j_i} := (1-a)X_{j_i} + aX_{j_{i-1}},$$
$$j = 1, 2, 3, 4, \quad i = 1, 2, \ldots, \quad 0 \le a < 1, \tag{20}$$

which implements a *numerical damping* that greatly enhances the convergence properties. In the following examples these computations were applied with $a = 0.25$.

Example 1 (Taken from [4])

$$\Phi = \begin{bmatrix} 0.1051 & 0.1841 & 0.2543 & 0.2004 & 0.2529 \\ 0.0226 & 0.2493 & 0.3222 & 0.3297 & 0.0441 \\ 0.3259 & 0.3989 & 0.0037 & 0.2827 & 0.3139 \\ 0.3261 & 0.0166 & 0.1840 & 0.4466 & 0.1997 \\ 0.4487 & 0.0237 & 0.0321 & 0.4062 & 0.3366 \end{bmatrix},$$

$$\Gamma^{\mathrm{T}} = [0.9103 \quad 0.7622 \quad 0.2625 \quad 0.0475 \quad 0.7361],$$
$$C = [0.3282 \quad 0.6326 \quad 0.7564 \quad 0.9910 \quad 0.3653],$$
$$V = \operatorname{diag}[0.6316 \quad 0.8847 \quad 0.2727 \quad 0.4364 \quad 0.7665],$$
$$Q = \operatorname{diag}[0.2470 \quad 0.9826 \quad 0.7227 \quad 0.7534 \quad 0.6515],$$
$$W = 0.4777, \quad R = 0.0727.$$

The spectral radius of Φ equals 1.1443 so the system is unstable. The numerical result presented in [4], based on the CDOPE, was computed as $\Re_c^i(0, 0, I_n, I_n)$ with the exception that $\tau_0 = I_n$ was used, instead of the value τ_0 which follows from $\hat{P}_0 = \hat{S}_0 = I_n$ and (13.3) and (13.4). The latter implies that only after the first iteration does the algorithm leave τ unchanged (see Lemma 2). The value of i, for which convergence is assumed, is based on the relative difference between consecutive values of $\operatorname{tr}(P+S)$ [11]. If this relative difference falls below a certain tolerance, in our case 10^{-6}, convergence is assumed. In [4] only the outcome of $\sigma_{V,W}$ and F, K, L for $n_c = 1$ was considered. Table 1 lists the outcome of $\Re_c^i(0, 0, I_n, I_n)$ with $\tau_0 = I_n$, for all possible compensator orders. Comparison of $\sigma_{Q,R}$ and $\sigma_{V,W}$ and substitution of F, K, L in (A6.5) (see Appendix 1) reveals that, although a solution of the CDOPE is obtained, the necessary optimality conditions are not satisfied. However, in case we restrict our attention to $\sigma_{V,W}$, the result for $n_c = 1$, when compared to the full-order case, seems correct. From Table 1 observe that in *each* case $\Re_c^i(0, 0, I_n, I_n)$ with $\tau_0 = I_n$, converges! This convergence strongly suggested correctness of the results. Also the recursions $\Re_c^i(0, 0, \Lambda_1, \Lambda_2)$, which satisfy Lemma 2, for different Λ_1, Λ_2 converge to solutions of the CDOPE which do *not* correspond to optimal reduced-order compensators. Therefore *many* solutions, each with a different value of τ, satisfy the CDOPE while they do *not* correspond to optimal reduced-order compensators. Table 2 lists the correct solutions obtained from the recursions $\Re^i(0, 0, \Lambda_1, \Lambda_2)$, based on the SDOPE. In this case the first-order necessary optimality conditions are satisfied in each case, which is illustrated by the fact that $\sigma_{Q,R}$ and $\sigma_{V,W}$ are equal while Eq. (A6.5) is satisfied, both within terms of numerical accuracy. The numerical accuracy increases if the tolerance (10^{-6} here) decreases.

Example 1 did not show very clearly the effect of increasing minimum costs as the reduced-order of the compensator decreases. The next example does and also illustrates that the compensator order may be *less* than the number of inputs and outputs of the system. Finally from the point of view of numerical stability the example is a very difficult one because the system has a single uncontrollable mode with an associated spectral radius of 0.95 so the system is only *just* full-order compensatable. Furthermore the system is highly unstable since the spectral radius of Φ equals 2.5.

Example 2

$$\Phi = \begin{bmatrix} 0.3884 & 1.6578 & 0.0613 & 0.0137 & 0 \\ 0.0834 & 0.6802 & 0.0948 & 0.6800 & 0 \\ 1.2041 & 0.9213 & 0.9395 & 0.1186 & 0 \\ 1.2048 & 1.4738 & 1.1904 & 0.7405 & 0 \\ 0 & 0 & 0 & 0 & 0.95 \end{bmatrix},$$

Table 1. Solutions of the CDOPE for Example 1.

n_c	Prescribed compensator order
n'_c	Minimal dimension of compensator computed from Algorithm 2
$\sigma_{Q,R}$	Costs computed from (8.2)
$\sigma_{V,W}$	Costs computed from (8.3)
$N(\Delta P')$	Largest singular value of $\Phi' P' \Phi'^{\mathrm{T}} + V' - P'$, see Eq. (A6.5) in Appendix 1
i	Number of iterations necessary to reach convergence
CT	Computation time in second on a pentium 90 MHz PC using MATLAB 4.2c2

n_c	5	4	3	2	1
n'_c	5	4	3	2	1
$\sigma_{Q,R}$	3.51575	25.9136	3.89037	3.53926	3.51948
$\sigma_{V,W}$	3.51576	3.56192	3.51857	3.57526	3.55414
$N(\Delta P')$	7.395e −5	52.59	0.8855	0.2397	0.2469
i	8	12	12	13	13
CT	0.22	0.33	0.33	0.33	0.33

Table 2. Solutions of the SDOPE for Example 1.

n_c	5	4	3	2	1
n'_c	5	4	3	2	1
$\sigma_{Q,R}$	3.51575	3.51538	3.51577	3.51542	3.53370
$\sigma_{V,W}$	3.51576	3.51572	3.51573	3.51622	3.53379
$N(\Delta P')$	7.40e −5	1.23e −3	9.69e −5	9.66e −4	1.34e −4
i	8	8	13	13	17
CT	0.22	0.22	0.33	0.33	0.44

Table 3. Solutions of the SDOPE for Example 2.

n_c	5	4	3	2	1
n'_c	5	4	3	2	1
$\sigma_{Q,R}$	195.53	199.56	220.03	359.09	2516.2
$\sigma_{V,W}$	195.53	199.56	220.03	359.09	2516.6
$N(\Delta P')$	9.45e − 5	1.54e − 4	4.87e − 3	2.17e − 6	4.43e − 2
i	96	104	130	138	518
CT	2.42	2.69	3.24	3.51	13.13

$$\Gamma^{\mathrm{T}} = \begin{bmatrix} 0.5890 & 0.9304 & 0.8462 & 0.5269 & 0 \\ 0.0920 & 0.6539 & 0.4160 & 0.7012 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.9130 & 0.2625 & 0.7361 & 0.6326 & 0.9910 \\ 0.7622 & 0.0475 & 0.3282 & 0.7564 & 0.3653 \end{bmatrix},$$

$$V = \mathrm{diag}[0.2470 \quad 0.9826 \quad 0.7227 \quad 0.7534 \quad 0.6515],$$
$$Q = \mathrm{diag}[0.8847 \quad 0.2727 \quad 0.4364 \quad 0.7665 \quad 0.4777],$$
$$W = \mathrm{diag}(0.0727 \quad 0.6316),$$
$$R = \mathrm{diag}(0.2378 \quad 0.2749).$$

Table 3 lists the outcome for all compensator orders.

The optimal reduced-order compensation problem may be conceived as a constrained non-linear parameter optimization problem, where the optimization parameters are the elements of F, K and L. In this case, as the order of the compensator increases, the number of optimization parameters increases dramatically which soon renders the optimization practically impossible. In contrast to this, from Tables 2 and 3 observe that the computation of optimal reduced-order compensators, based on iteration of the SDOPE, takes less effort when the order of the compensator increases.

To illustrate the superiority of the numerical algorithm based on iteration of the SDOPE the outcome and computation times obtained for some of the problems above using constrained non-linear parameter optimization are presented. To successfully solve some of these problems, first an optimization is performed which tries to minimize the spectral radius of the closed loop system as a function of F, K and L. This optimization was performed using the function FMINS of the MATLAB optimization toolbox. The outcome of this optimization is used as the initial value for the constrained non-linear parameter optimization. This optimization was performed using the function CONSTR of the MATLAB optimization toolbox. We only implemented the constraint that the spectral radius of the closed loop system be smaller than 1. Table 4 lists the outcome. As can be seen, especially for higher compensator orders, the algorithm based on iteration of the SDOPE behaves highly superior. The parameter optimization method becomes more efficient when the compensator is represented using a canonical form, since this reduces the number of elements of F, K and L that have to be optimized. But also in this case, as the compensator dimensions grow, the optimal projection algorithms soon become more efficient.

From [13] a special situation occurs if the optimal full-order LQG compensator, computed from the two Riccati equations of full-order LQG control, is not minimal. This may happen even if the system

Table 4. Optimal reduced-order compensators from constrained non-linear parameter optimization.

	Example 1				Example 2		
n_c	1	2	3	4	1	2	3
σ^*_∞	3.53373	3.51593	3.51580	3.51578	2516.4	359.09	220.03
CT	9.89	50.48	214.98	545.02	16.54	109.96	545.09

(Φ, Γ, C) is minimal and $(\Phi, V^{1/2}, Q^{1/2})$ is observable [12]. Example 3 has exactly these properties. The optimal full-order compensator for Example 3 has a minimal realization with dimension 1. This dimension is called the minimal dimension of the optimal full-order compensator and will be denoted by n_m. Example 3 was generated using constrained non-linear parameter optimization implemented using the function CONSTR of the MATLAB optimization toolbox. All the parameters which make up the three-dimensional compensation problem, except for Φ, were fixed. The elements of Φ were varied to minimize the *second* of the *ordered* singular values of the controllability grammian of the corresponding optimal full-order compensator. If, at the end of this minimization, this singular value equals zero, as it did, an optimal full-order compensator with minimal dimension one is obtained. To ensure that the system (Φ, Γ, C) is minimal the constraint forced the *smallest* singular value of both the controllability and observability grammian of (Φ, Γ, C) to be greater than 0.05. Note that a slight change of Φ destroys the above mentioned properties. Therefore Φ is specified with high accuracy.

Example 3

$$\Phi = \begin{bmatrix} 0.05438650982239 & 0.15028168349930 & -0.26938908545548 \\ -0.16392263097279 & 0.35576999327113 & 0.47463527719833 \\ 0.39517637368839 & 0.29013007687628 & -0.48922225431850 \end{bmatrix},$$

$$\Gamma^T = [1 \quad 1 \quad 1],$$
$$C = [1 \quad 1 \quad 1],$$
$$V = I_3,$$
$$Q = I_3,$$
$$W = 1, \; R = 1.$$

□

Table 5 lists the outcome of Algorithm 2. As desired, regardless of the prescribed compensator order, a minimal realization with dimension one of the optimal full-order LQG compensator is found, which is globally optimal. Algorithm 1 produced the same answers.

Table 5. Solutions of the SDOPE for Example 3.

n_c	3	2	1
n_c'	1	1	1
$\sigma_{Q,R}$	4.2242	4.2242	4.2242
$\sigma_{V,W}$	4.2242	4.2242	4.2242

This *suggests* (in this case) that the SDOPE have no solution when $n_m < n_c \le n$ because the rank condition (6.5) cannot be met. In this case Eq. (13.4) ensures that, despite the loss of rank, the projection τ is still properly computed. Although Algorithm 2 produces the desired answer one should preferably choose $n_c \le n_m$, i.e. less than or equal to the minimal dimension of the optimal full-order compensator, which is globally optimal. In Example 3 this amounts to the choice $n_c = 1$.

With respect to (13.4) one may wonder whether, during the iteration, $\text{rank}(X_3 X_4)$ is able to increase, in other words, is able to 'recover'. Using Algorithm 2 with initial values Λ_1, Λ_2 with a rank *less* than n_c it is easily verified, e.g. using Example 2, that this is so. This is due to Eqs (8.3) and (8.4). Note that Ψ^1, Ψ^2 are symmetric by definition. Then, as long as $\tau\Psi^1$ is not symmetric and $\text{rank}(\Psi^1) \ge \text{rank}(\tau)$, in general, $\text{rank}(\tau\Psi^1 + \Psi^1\tau^T) > \text{rank}(\tau)$ in (8.3). Similar arguments apply to (8.4). Because of this property, in general, solutions generated by Algorithms 1 and 2 have the property $\text{rank}(Y_3^1 Y_4^1) = n_c$, as desired, unless such solutions do not exist. This fact is illustrated by Examples 1–3 and the huge number of random examples considered in two computer experiments described in Section 7. From Lemma 2 observe that iterations of the CDOPE cannot recover $\text{rank}(X_3 X_4)$. This can also be seen from (15) and the fact that $\text{rank}(\tau\Psi^1\tau^T) \le \text{rank}(\tau)$.

7. Local and Global Optimal Reduced-Order Compensators: Two Computer Experiments

Let n_u denote the dimension of the unstable subspace of Φ, i.e. the number of eigenvalues of Φ with a magnitude greater or equal to 1. Consider the following two types of restrictions regarding the choice of n, m, l and n_c:

$$\max(\min(n, m, l), n_u) \leq n_c, \tag{21}$$

$$m \leq n, \ l \leq n, \ \max(m, l, n_u) \leq n_c \leq n. \tag{22}$$

From a system theoretic and control system design perspective the number of inputs m and the number of outputs l should not exceed the system and compensator dimension, i.e. n and n_c. Also, to stabilize the system, it is natural to choose the compensator order greater or equal to the dimension of the unstable subspace of Φ. Then the conditions (21) turn into the conditions (22) which are therefore more restrictive. It has been argued in the continuous-time case [6,7] that, if (21) is satisfied and the system is stabilizable, $Q > 0$, $V > 0$ and Γ, C full-rank, the optimal projection equations have *at most* one non-negative solution. Similar arguments apply to the discrete-time case when (22) is satisfied and $(\Phi, V^{1/2}, Q^{1/2})$ is detectable [4]. Example 2 in [18] is a discrete-time reduced-order LQG example where the system has dimension two and which satisfies all the conditions mentioned above for both the continuous-time and discrete-time case. Still, if the controller has prescribed dimension one, two distinct minima are found. Several numerical examples presented in this section also contradict the discrete-time result, i.e. although (22) is satisfied and $(\Phi, V^{1/2}, Q^{1/2})$ is detectable the SDOPE have more than one non-negative solution in several cases. With respect to the CDOPE note that the SDOPE are stronger so each solution of the SDOPE is also a solution of the CDOPE. With respect to our numerical findings note that convex analysis of static output-feedback problems seems to indicate that, in general, in both the continuous and discrete-time case the problem *may* have multiple extrema [15].

Computer Experiment 1

$n = 2, 3, \ldots, 30$. $m = 1$. $l = 1$, $n_c = 1$, spectral radius $\Phi = 0.95$ (magnitude largest eigenvalue of $\Phi = 0.95$), i.e. $n_u = 0$ and therefore $(\Phi, V^{1/2}, Q^{1/2})$ is detectable and the system is n_c-compensatable $\forall n_c$.

For each value of n, 100 randomly generated reduced-order compensation problems with the properties mentioned above were generated and solved, using Algorithm 2. Details concerning the generation of these random problems can be found in Appendix 2. Each example was *recomputed* 10 times (with different random values for Λ_1, Λ_2), to try and find multiple extrema, using a numerical damping of 0.25.

The problems above that exhibited none or multiple extrema were also solved using Algorithm 1 with a numerical damping of 0.75. □

Computer Experiment 2

$n = 49, 50$, $m = 5$, $l = 5$, $n_c = n_u$, spectral radius $\Phi = 1.25$ (magnitude largest eigenvalue of $\Phi = 1.25$) i.e. the system is unstable.

For each value of n, 20 randomly generated reduced-order compensation problems with the properties mentioned above were generated and solved, using Algorithm 2. Details concerning the generation of these random problems can be found in Appendix 2. Each example was *recomputed* 10 times (with different random values for Λ_1, Λ_2), to try and find multiple extrema, using a numerical damping of 0.25.

The problems above that exhibited none or multiple extrema were also solved using Algorithm 1 with a numerical damping of 0.75. □

Note that both computer experiments deal exclusively with problems that satisfy (22) and therefore also (21). Note with respect to NS in Table 6 that the system is always n_c-compensatable in Computer Experiment 1, but not necessarily in Computer Experiment 2. Finally for Computer Experiment 2 it was verified that $(\Phi, V^{1/2}, Q^{1/2})$ is observable and thus detectable in all of the 40 examples.

For one of the compensation problems Fig. 1, which shows a grid representing the costs in the parameter space, clearly illustrates the non-uniqueness, in terms of the performance, of optimal reduced-order compensators. Note that in the case $n_c = 1$ the compensator is uniquely determined by F and $K * L$ and is minimal as long as both of these are unequal to zero. After solving the discrete Lyapunov Eq. (A3) the costs of each compensator were computed using Eq. (A4) (see Appendix 1). For the same example Fig. 2 illustrates

Table 6. Results of Computer Experiments 1 and 2.

NS Number of problems for which the algorithm diverged each of the 10 times
MS Number of problems for which multiple solutions *including infinity*, i.e. when the algorithm diverges, were found
LM Number of problems with multiple solutions for which the homotopy algorithm did not converge to the 'global minimum'

	Experiment 1 (3000 problems)			Experiment 2 (40 problems)		
NS	75	2.50%	(75/3000)	7	17.5%	(7/40)
MS	258	8.82%	(258/3000)	14	35%	(14/40)
LM	64	24.81%	(64/258)	8	57.14%	(8/14)

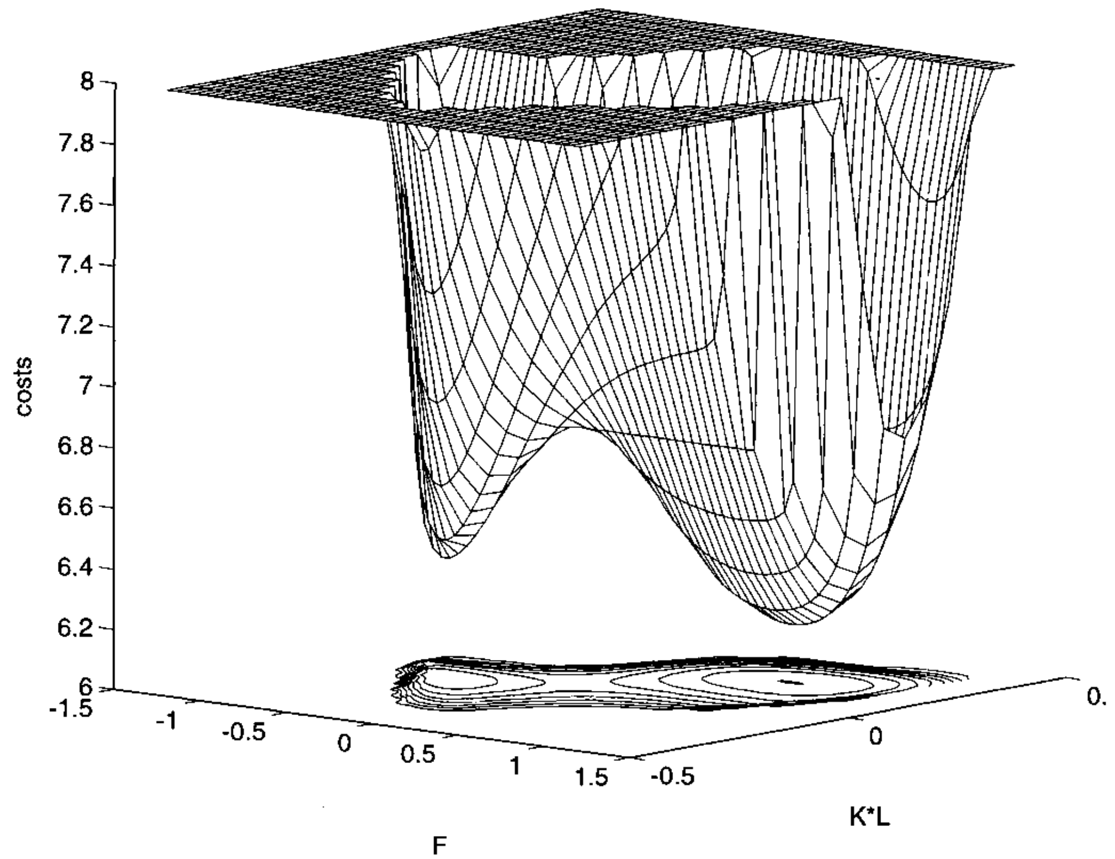

Fig. 1. Local optimality, nx = 10, nu = ny = 1, srp = 0.95, seed = 89 (see Appendix 2).

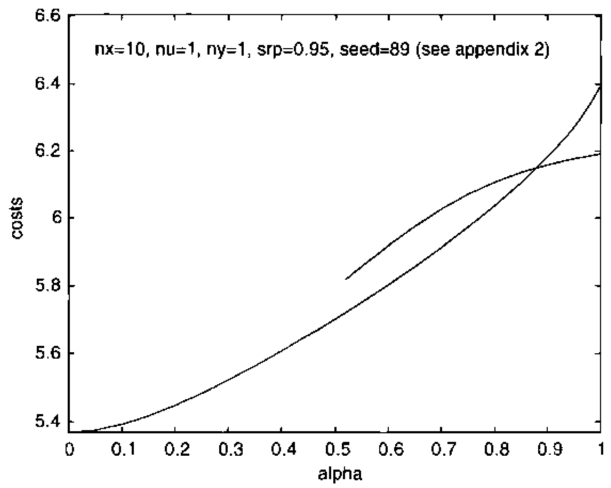

Fig. 2. Homotopy paths.

that the homotopy algorithm does not reach the 'global minimum'. The homotopy path related to the 'global minimum' *vanishes* around $\alpha = 0.51$. It was computed using Algorithm 1 in the backward direction, i.e. starting with the 'global minimum' at $\alpha = 1$ and then decreasing α with small discrete steps towards $\alpha = 0$. From the two computer experiments the following may be concluded:

(1) All solutions Y^1, found in both the computer experiments (and also all those in Section 6) using either Algorithm 1 or 2, within terms of numerical accuracy, have the property $Y^1 \geq 0$. This suggests that $Y^1 \geq 0$ is a (generic) property of the algorithms. Sometimes *during* the iteration the matrices become negative.

(2) Except for Example 3 in Section 6, all compensation problems have the property $n_m = n$, i.e. the optimal full-order compensator is minimal, and all solutions have the property $n_c' = n_c$. This suggests that the former is a generic property of random reduced-order compensation problems and shows that the latter is a generic property of the algorithms.

(3) Even if (22) is satisfied and $(\Phi, V^{1,2}, Q^{1/2})$ is detectable the SDOPE may have *multiple* non-negative solutions.

(4) A way to compute multiple non-negative solutions, if they exist, is to repeat Algorithm 2 several times, with different random initial values Λ_1, Λ_2.

(5) Algorithm 1, often but not always, finds what seems to be the global minimum.
(6) For random examples where the system is n_c-compensatable, the probability of finding local or no minima, using Algorithms 1 and 2, increases, when the order of the system increases, when the cost increase compared with the optimal full-order compensator increases, and when the spectral radius of Φ increases.
(7) Although not explicitly shown, for examples of computer Experiment 1, were Algorithm 2 did not converge, increasing the numerical damping of 0.25, usually resulted in convergence. As an example consider nx = 5, nu = 1, ny = 1, srp = 0.95, seed = 32 (see Appendix 2). The convergence of this example is very troublesome and is achieved only after choosing $a = 0.975$, a very large value for the numerical damping. The associated minimum costs of this not necessarily unique solution were computed to be 10.9452.

8. Conclusions

A strengthened version of the discrete-time optimal projection equations has been presented (SDOPE). For the class of stabilizing compensators this version of the optimal projection equations was proved to be *equivalent* to first-order necessary optimality conditions together with the condition that the compensator be minimal. The CDOPE were shown to be weaker and having *many* solutions which do not correspond to optimal reduced-order compensators.

Based on the SDOPE two new algorithms were proposed to compute optimal reduced-order compensator. One is a homotopy algorithm. The other algorithm iterates the SDOPE and is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration, and therefore is highly efficient. Through numerical examples and two computer experiments it was demonstrated that the SDOPE, in general, may have multiple solutions and that the homotopy algorithm often, but not always, finds the global minimum. The iterative algorithm was shown to be superior compared to constrained non-linear parameter optimization and the homotopy algorithm.

The huge number of numerical examples presented in this paper suggest the following important property of the algorithms. If the algorithms converge they converge to desired non-negative solutions. Furthermore it has been shown why these solutions have the desired property $n_c' = n_c$, except when $n_c > n_m$, i.e. if the prescribed compensator dimensions exceed those of a minimal realization of the optimal full-order compensator. With respect to the latter property, without loss of performance, it is preferable to prescribe $n_c \leq n_m$. Clearly the properties of the algorithms require further investigation and proof. This also applies to the convergence.

With respect to the possible non-uniqueness of the optimal reduced-order compensator the following practical approach is suggested. Apply Algorithm 2 several times with different random initial values. Pick the best solution. Of course one can never be sure that better solutions do not exist. However, compared to the performance of the optimal full-order compensator, which represents the global minimum obtainable with *any* compensator, the loss of performance may serve as a criterion for acceptance of a (locally) optimal reduced-order compensator.

The results of this paper pave the way to comparable results for optimal reduced-order compensation of linear discrete-time systems with white stochastic parameters [9,18,20]. Also the results open up the possibility to develop the SDOPE for time-varying finite-horizon LQG problems where the system has either deterministic or white stochastic parameters [11,13,19]. Moreover numerical algorithms to compute these compensators may be found, based on the results of this paper [19].

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Appendix 1: Proof of Theorem 2

To prove the equivalence mentioned in Theorem 2 we first show how the first-order necessary optimality conditions and the minimality of the compensator imply (7) and (8) in Theorem 2. Then we show how the conditions in Theorem 2 in return imply the first-order necessary optimality conditions and the minimality of the compensator. For clarity the proof of Theorem 2 is separated into five parts. In Section A1 the first-order necessary optimality conditions are obtained using the matrix minimum principle. From these first-order necessary optimality conditions, in Section A2, explicit expressions for the compensator gains are derived. In Section A3, the SDOPE are derived. In Section A4, we show how the conditions of Theorem 2 in return imply the first-order necessary optimality conditions and the minimality of the compensator. Finally in Section A5 the explicit expressions for the compensator costs are derived.

A1. First-Order Necessary Optimality Conditions

Introducing

$$x_i' = \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, \quad v_i' = \begin{bmatrix} v_i \\ Kw_i \end{bmatrix},$$
$$\Phi' = \begin{bmatrix} \Phi & -\Gamma L \\ KC & F \end{bmatrix}, \quad V' = \begin{bmatrix} V & 0 \\ 0 & KWK^{\mathrm{T}} \end{bmatrix} \tag{A1.1}$$

the closed loop system is described by

$$x_{i+1}' = \Phi' x_i' + v_i', \quad i = 0, 1, 2, \ldots, \tag{A1.2}$$

where $\{v_i'\}$ is a zero-mean white noise sequence with covariance V' and uncorrelated with the initial condition x_0'. Let $P_i' \in R^{(n+n_c)\times(n+n_c)}$ denote the closed loop second moment $E\{x_i' x_i'^{\mathrm{T}}\}$ then, from (A1),

$$P_{i+1}' = \Phi' P_i' \Phi'^{\mathrm{T}} + V'. \tag{A2}$$

If the closed loop system is stable, $P' = \lim_{i\to\infty} P_i'$ exists, $P' \geq 0$ and P' is the unique solution of

$$P' = \Phi' P' \Phi'^{\mathrm{T}} + V'. \tag{A3}$$

Furthermore the criterion (3) is finite and independent of initial conditions and can be expressed as

$$\sigma_\infty(F, K, L) = \mathrm{tr}(Q'P'), \tag{A4.1}$$

where $Q' \in R^{(n+n_c)\times(n+n_c)}$ given by

$$Q' = \begin{bmatrix} Q & 0 \\ 0 & L^{\mathrm{T}}RL \end{bmatrix}. \tag{A4.2}$$

Because the eigenvalues of Φ' continuously depend on (F, K, L) and since the set of stabilizing compensators is open we may apply the matrix minimum principle [8] to find first-order necessary conditions for the solution of the optimal reduced-order dynamic compensation problem. To that end define the

Hamiltonian

$$H(F,K,L,P',S') = \mathrm{tr}\left[Q'P' + \left(\Phi'P'\Phi'^{\mathrm{T}} + V' - P'\right)S'\right], \tag{A5}$$

where the symmetric matrix $S' \in R^{(n+n_c)\times(n+n_c)}$ is a Lagrange multiplier. Then the first-order necessary conditions for optimal reduced-order dynamic compensation are

$$\frac{\partial H}{\partial F} = \frac{\partial}{\partial F}\mathrm{tr}\left(\Phi'P'\Phi'^{\mathrm{T}}S'\right) = 0, \tag{A6.1}$$

$$\frac{\partial H}{\partial K} = \frac{\partial}{\partial K}\mathrm{tr}\left[V'S' + \Phi'P'\Phi'^{\mathrm{T}}S'\right] = 0, \tag{A6.2}$$

$$\frac{\partial H}{\partial L} = \frac{\partial}{\partial L}\mathrm{tr}\left[Q'P' + \Phi'P'\Phi'^{\mathrm{T}}S'\right] = 0, \tag{A6.3}$$

$$\frac{\partial H}{\partial P'} = \Phi'^{\mathrm{T}}S'\Phi' + Q' - S' = 0, \tag{A6.4}$$

$$\frac{\partial H}{\partial S'} = \Phi'P'\Phi'^{\mathrm{T}} + V' - P' = 0, \tag{A6.5}$$

where $S' \geq 0$, $P' \geq 0$.

A2. Explicit Expressions for the Compensator Gains

Partition P', S' according to the partitioning of Φ' in (A1),

$$P' = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}, \quad S' = \begin{bmatrix} S_1 & S_{12} \\ S_{12}^{\mathrm{T}} & S_2 \end{bmatrix}. \tag{A7.1}$$

If the compensator is minimal, $P_2 > 0$, $S_2 > 0$ [9,13]. Define the $n \times n$ non-negative definite matrices

$$P = P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}}, \tag{A7.2}$$

$$\hat{P} = P_{12}P_2^{-1}P_{12}^{\mathrm{T}}, \tag{A7.3}$$

$$S = S_1 - S_{12}S_2^{-1}S_{12}^{\mathrm{T}}, \tag{A7.4}$$

$$\hat{S} = S_{12}S_2^{-1}S_{12}^{\mathrm{T}}, \tag{A7.5}$$

$$G = P_2^{-1}P_{12}^{\mathrm{T}}, \tag{A7.6}$$

$$M = -P_{12}^{\mathrm{T}}S_{12}, \tag{A7.7}$$

$$H = -S_2^{-1}S_{12}^{\mathrm{T}}, \tag{A7.8}$$

$$\tau = G^{\mathrm{T}}H. \tag{A7.9}$$

It will be shown in Section A3, after Eq. (A23), that these definitions correspond to P, S, $\hat{P}$, $\hat{S}$, G, H and τ in Theorem 2 and to $\hat{P}$, $\hat{S}$, G, M, H and τ in Lemma 1. From (A7),

$$\hat{P} = G^{\mathrm{T}}P_2G, \tag{A8.1}$$

$$P_1 = P + \hat{P}, \tag{A8.2}$$

$$\hat{S} = H^{\mathrm{T}}S_2H, \tag{A8.3}$$

$$S_1 = S + \hat{S}. \tag{A8.4}$$

Equation (A6.1) equals

$$S_{12}^{\mathrm{T}}\Phi P_{12} - S_{12}^{\mathrm{T}}\Gamma LP_2 + S_2FP_2 + S_2KCP_{12} = 0. \tag{A9}$$

From S_2^{-1} (A9) P_2^{-1}, (A7.6) and (A7.8) we obtain (7.1). Equation (8.2) equals

$$S_{12}^{\mathrm{T}}\Phi P_1C^{\mathrm{T}} - S_{12}^{\mathrm{T}}\Gamma LP_{12}^{\mathrm{T}}C^{\mathrm{T}} + S_2FP_{12}^{\mathrm{T}}C^{\mathrm{T}} + S_2KCP_1C^{\mathrm{T}} + S_2KW = 0. \tag{A10}$$

Using (7.1) Eq. (A10) equals

$$S_2K\left(CPC^{\mathrm{T}} + W\right) + S_{12}^{\mathrm{T}}\Phi PC^{\mathrm{T}} = 0. \tag{A11}$$

Using (A7.8) from S_2^{-1}(A11) we obtain (7.2).

Equation (8.3) equals

$$-\Gamma^{\mathrm{T}}S_1\Phi P_{12} - \Gamma^{\mathrm{T}}S_{12}KCP_{12} - \Gamma^{\mathrm{T}}S_{12}FP_2 + \Gamma^{\mathrm{T}}S_1\Gamma LP_2 + RLP_2 = 0. \tag{A12}$$

Using (7.1) Eq. (A12) equals

$$\Gamma^{\mathrm{T}}S\Phi P_{12} - \left(\Gamma^{\mathrm{T}}S\Gamma + R\right)LP_2 = 0. \tag{A13}$$

Using (A7.6) from (A13) P_2^{-1} we obtain (7.3).

A3. The Strengthened Optimal Projection Equations

Expanding Eq. (A6.4) using (7.1) (7.3) yields the following three equations:

$$\Phi^{\mathrm{T}} S\Phi + (\Phi - K_P C)^{\mathrm{T}} \hat{S}(\Phi - K_P C) + Q = S_1, \tag{A14}$$

$$\Big[(\Phi - K_P C)^{\mathrm{T}} \hat{S}(\Phi - K_P C) + \Phi^{\mathrm{T}} S \Gamma L_S\Big] G^{\mathrm{T}} = -S_{12}, \tag{A15}$$

$$G\Big[(\Phi - K_P C)^{\mathrm{T}} \hat{S}(\Phi - K_P C) + L_S^{\mathrm{T}}(R + \Gamma^{\mathrm{T}} S\Gamma) L_S\Big] G^{\mathrm{T}} = S_2. \tag{A16}$$

Expanding Eq. (A6.5) using (7.1)–(7.3) yields the following three equations:

$$\Phi P\Phi^{\mathrm{T}} + (\Phi - \Gamma L_S)\hat{P}(\Phi - \Gamma L_S)^{\mathrm{T}} + V = P_1, \tag{A17}$$

$$\Big[(\Phi - \Gamma L_S)\hat{P}(\Phi - \Gamma L_S)^{\mathrm{T}} + \Phi P C^{\mathrm{T}} K_P^{\mathrm{T}}\Big] H^{\mathrm{T}} = P_{12}, \tag{A18}$$

$$H\Big[(\Phi - \Gamma L_S)\hat{P}(\Phi - \Gamma L_S)^{\mathrm{T}} + K_P(CPC^{\mathrm{T}} + W)K_P^{\mathrm{T}}\Big] H^{\mathrm{T}} = P_2. \tag{A19}$$

Then from G(A15) S_2^{-1} – (A15) S_2^{-1}, or alternatively H(A18)P_2^{-1} - $(A18)P_2^{-1}$ we obtain

$$HG^{\mathrm{T}} = GH^{\mathrm{T}} = I_{n_c}. \tag{A20}$$

From (A20) and (A7.9),

$$\tau^2 = \tau; \tag{A21}$$

so (A20) defines an oblique projection. From (A7) and (A21) we obtain the following properties:

$$\hat{P}\hat{S} = -P_{12} G H^{\mathrm{T}} S_{12}^{\mathrm{T}} = -P_{12} S_{12}^{\mathrm{T}}, \tag{A22.1}$$

$$\hat{P} = \tau\hat{P} = \hat{P}\tau^{\mathrm{T}} = \tau\hat{P}\tau^{\mathrm{T}}, \tag{A22.2}$$

$$P_{12} = \hat{P}H^{\mathrm{T}}, \tag{A22.3}$$

$$P_2 = H\hat{P}H^{\mathrm{T}}, \tag{A22.4}$$

$$\hat{S} = \hat{S}\tau = \tau^{\mathrm{T}}\hat{S} = \tau^{\mathrm{T}}\hat{S}\tau, \tag{A22.5}$$

$$S_{12} = -\hat{S}G^{\mathrm{T}}, \tag{A22.6}$$

$$S_2 = G\hat{S}G^{\mathrm{T}}. \tag{A22.7}$$

From (A7) and (A22) we obtain

$$\begin{aligned} \operatorname{rank}(G) &= \operatorname{rank}(M) = \operatorname{rank}(H) \\ &= \operatorname{rank}(\tau) = \operatorname{rank}(\hat{P}) = \operatorname{rank}(\hat{S}) \\ &= \operatorname{rank}(\hat{P}\hat{S}) = \operatorname{rank}(P_{12}) = \operatorname{rank}(P_2) \\ &= \operatorname{rank}(S_{12}) = \operatorname{rank}(S_2). \end{aligned} \tag{A23}$$

Now, as indicated in Section A2 after Eq. (A7), given (A7), (A20), (A21) and (A23) we may identify P, S, $\hat{P}$, $\hat{S}$, G, H and τ with those in Theorem 2 and $\hat{P}$, $\hat{S}$, G, M, H and τ with those in Lemma 1.

Now $\frac{1}{2}((\text{A16})H + ((\text{A16})H)^{\mathrm{T}})$ equals (8.4) and $(\text{A15}) + H^{\mathrm{T}}G(\text{A16})H - (\text{A16})H - ((\text{A16})H)^{\mathrm{T}}$ equals (8.2). Similarly $\frac{1}{2}((\text{A18})G + ((\text{A18})G)^{\mathrm{T}})$ equals (8.3) and $(\text{A17}) + G^{\mathrm{T}}H(\text{A18})G - (\text{A18})G - ((\text{A18})G)^{\mathrm{T}}$ equals (8.1).

Summarizing, the conditions (7) and (8) in Theorem 2 are *implied* by (A6) and the minimality of the compensator.

A4. Equivalence

To prove the *equivalence* of (A6) and (7), (8) we now reverse the derivation. After substitution of (7.1)–(7.3) in (A9), (A10) and (A12), or equivalently (A6.1)–(A6.3), these relations still hold, so (7.1)–(7.3) are equivalent to (A6.1)–(A6.3). As a result (A14)–(A19) are equivalent to (A6.4) and (A6.5). Observe that (8.5) and (8.6) imply (A22.2) and (A22.5). From (8.3), (8.4) and (A22.2), (A22.5) we have

$$\Psi^2_{P,\hat{S},S}\tau = \tau^{\mathrm{T}}\Psi^2_{P,\hat{S},S} = \tau^{\mathrm{T}}\Psi^2_{P,\hat{S},S}\tau, \tag{A24}$$

$$\Psi^1_{S,\hat{P},P}\tau = \tau\Psi^1_{S,\hat{P},P} = \tau\Psi^1_{S,\hat{P},P}\tau^{\mathrm{T}}. \tag{A25}$$

Using (A24), (A25) and (8.7) we have that (8.2) + (8.4) equals (A14), $(7.4)G^{\mathrm{T}}$ equals (A15) and $G(8.4)G^{T}$ equals (A16). Similarly (8.1) + (8.3) equals (A17), $(8.3)H^{\mathrm{T}}$ equals (A18) and $H(8.3)H^{\mathrm{T}}$ equals (A19). Summarizing, (7) and (8) imply (A6). Finally from (A7.3) and (A7.5) observe that the rank condition in

(8.5) implies $P_2 > 0$, $S_2 > 0$, i.e. the invertibility of the second-moment matrix of the compensator and its dual. This implies the minimality of the compensator [13].

Note that recovering Eqs (A14)–(A19) from the conventional optimal projection equations, i.e. (8.3) and (8.4) replaced by

$$\hat{P} = \tau \Psi^1_{S,\hat{P},P} \tau^{\mathrm{T}}, \tag{A26}$$

$$\hat{S} = \tau^{\mathrm{T}} \Psi^2_{P,\hat{S},S} \tau \tag{A27}$$

is not possible because (A26) and (A27) do *not* imply (A24) and (A25). Therefore the conventional optimal projection equations are only *implied* by the first-order necessary optimality conditions (A6), they are *not* equivalent to (A6). Numerical evidence of this fact was presented in Section 6. Equations (A24) and (A25) might suggest that (8.3) and (8.4) may be replaced by e.g.

$$\hat{P} = \tau \Psi^1_{S,\hat{P},P}, \tag{A28}$$

$$\hat{S} = \Psi^2_{P,\hat{S},S} \tau. \tag{A29}$$

Replacing (8.3) and (8.4) by (A28) and (A29) in the algorithms revealed that nonnegative solutions which satisfy (8.1), (8.2) and (A28), (A29) exist which do not satisfy (A22.2), (A22.5) and, as a result, (A24) and (A25), because $\hat{P}$ and/or $\hat{S}$ are not symmetric. From (A7.3) and (A7.5) note that $\hat{P}$ and $\hat{S}$ are symmetric by definition. This symmetry and the crucial equalities (A24) and (A25) are both implied by (8.3) and (8.4). If, as in Theorem 2, the symmetry of $\hat{P}$ and $\hat{S}$ is *presumed* then (A28) and (A29) may replace (8.3) and (8.4). Without this presumption from (A24) and (A25) observe that (8.3) and (8.4) may also be replaced by

$$\hat{P} = \Psi^1_{S,\hat{P},P} - \tau_\perp \Psi^1_{S,\hat{P},P} \tau_\perp^{\mathrm{T}}, \tag{A30}$$

$$\hat{S} = \Psi^2_{P,\hat{S},S} - \tau_\perp^{\mathrm{T}} \Psi^2_{P,\hat{S},S} \tau_\perp. \tag{A31}$$

However, replacing (8.3) and (8.4) by (A30) and (A31) in Algorithms 1 and 2 results in instability of the algorithms. Therefore (8.3) and (8.4) have been mentioned in Theorem 2.

A5. Explicit Expressions for the Minimum Costs

From Eqs (A6.4) and (A6.5) we obtain the following equalities:

$$\mathrm{tr}[S'P'] = \mathrm{tr}\left[Q'P' + \Phi'^{\mathrm{T}} S' \Phi' P'\right]. \tag{A32}$$

$$\mathrm{tr}[P'S'] = \mathrm{tr}\left[V'S' + \Phi' P' \Phi'^{\mathrm{T}} S'\right] \tag{A33}$$

From (A32), (A33) and (A4.1) we obtain

$$\sigma_\infty = \mathrm{tr}[Q'P'] = \mathrm{tr}[V'S'] \tag{A34}$$

which is *equivalent* to Eq (9).

Appendix 2: Generation of Random LQG Problems

The following MATLAB m file was developed to generate the random LQG problems in Computer Experiments 1 and 2 with MATLAB version 4.2c2.

```
% DRLQG.M: Discrete-time random LQG problem
  generation.
%
%        function [p,g,c,v,w,q,r] = drlqg(nx,nu,ny,srp,
         seed);
%
% Input:
%        nx: dimension x
%        nu: dimension u
%        ny: dimension y
%        srp: optional, desired spectral radius p
              (system matrix).
%        seed: optional, seed for random number
               generator
% Output:
%        p,g,c,v,w,q,r: LQG problem parameters
%
% L.G. Van Willigenburg, W.L. De Koning, 28-11-95.
%
  function [p,g,c,v,w,q,r] = drlqg(nx,nu,ny,srp,seed);
  if nargin = = 5; elseif nargin = = 4 seed = [];
   elseif nargin = = 3
  srp = []; seed = = [];
  else error(Wrong number of input arguments);
   end;
  if max(size(seed)) ~ = 0; rand(seed, seed); end;
  [p,g,c] = drmodel(nx,ny,nu);
  if max(size(srp)) ~ = 0 & srp > 0; p = srp*p/
   sperad(p); end;
  v = diag(rand(nx,1)); w = diag(rand(ny,1));
  q = diag(rand(nx,1)); r = diag(rand(nu,1));
%  Rounding of matrices to 4 decimal digits
  p = round(1e4*p)/1e4; g = round(1e4*g)/1e4;
   c = round(1e4*c)/1e4;
  v = round(1e4*v)/1e4; w = round(1e4*w)/1e4;
  q = round(1e4*q)/1e4; r = round(1e4*r)/1e4;
```

The m file uses the m file drmodel.m from the MATLAB Control Toolbox, which generates random discrete-time stable systems. Note that this function sometimes generates systems for which the input or output matrix is the zero matrix. These systems were excluded from the experiments. In Computer Experiment 1 seed $= 0, 1, 2, \ldots, 99$ was used. In Computer Experiment 2 seed $= 0, 1, 2, \ldots, 19$ was used.

To reproduce the examples the random number generator function rand must produce the same values for equal seeds. To verify this here is the outcome of p,g,c,v,w,q,r for nx = 1, nu = 1, ny = 1, srp = 1, seed = 1:

$p = -1$, $g = 0.7092$, $c = 0.1160$, $v = 0.4714$, $w = 0.1449$, $q = 0.7178$, $r = 0.6617$.