Numerical Algorithms and Issues Concerning the Discrete-Time Optimal Projection Equations

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The discrete-time optimal projection equations, which constitute necessary conditions for optimal reduced-order LQG compensation, are strengthened. For the class of minimal stabilizing compensators the strengthened discrete-time optimal projection equations are proved to be equivalent to first-order necessary optimality conditions for optimal reduced-order LQG compensation. The conventional discrete-time optimal projection equations are proved to be weaker. As a result solutions of the conventional discrete-time optimal projection equations may not correspond to optimal reduced-order compensators. Through numerical examples it is demonstrated that, in fact, many solutions exist that do not correspond to optimal reduced-order compensators. To compute optimal reduced-order compensators two new algorithms are proposed. One is a homotopy algorithm and one is based on iteration of the strengthened discrete-time optimal projection equations. The latter algorithm is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration and therefore is highly efficient. Using different initializations of the iterative algorithm it is demonstrated that the reduced-order optimal LQG compensation problem, in general, may possess multiple extrema. Through two computer experiments it is demonstrated that the homotopy algorithm often, but not always, finds the global minimum.

Keywords: Optimal reduced-order dynamic compensation; Fixed-order LQG control; Discrete-time systems; Numerical algorithms

1. Introduction

Controller reduction is a vital practical issue. Two approaches to controller reduction may be distinguished, direct versus indirect design [1]. Indirect design is characterized by the fact that the design is performed in two steps, instead of one. The two steps concern either model-reduction followed by full-order controller design or full-order controller design followed by controller reduction. A major disadvantage of these indirect approaches is that stability of the closed loop system, in general, cannot be guaranteed, and optimality, in general, is lost. Direct design on the other hand incorporates both stability of the closed loop system and optimality. Therefore if, given the design criteria, a direct design method is feasible, it should always be preferred. This paper deals with the direct design of optimal reduced-order LQG controllers for time-invariant discrete-time systems.

Necessary conditions for optimal reduced-order LQG compensation have been presented in both the continuous-time case [2] and the discrete-time case [3], as a set of four coupled matrix equations. They are known as the optimal projection equations, since an oblique projection is a fundamental part of these equations. Although this was not mentioned explicitly, a small flaw was removed from the discrete-time optimal projection equations in [4], see also [5]. In all these

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Received 5 January 1999; Accepted 16 July 1999.
Recommended by M. Schock and M. Gevers.
papers the necessary conditions were derived from first-order necessary optimality conditions assuming the compensator to be minimal and stabilizing. The papers suggested, but did not prove, the equivalence of the optimal projection equations with first-order necessary optimality conditions for the class of minimal stabilizing compensators. In this paper it is demonstrated that, in the discrete-time case, this equivalence does not hold. From this analysis so-called strengthened discrete-time optimal projection equations (SDOPE) are obtained, which differ from the conventional discrete-time optimal projection equations (CDOPE) as presented in [4] and [5]. As opposed to the CDOPE, the SDOPE are proved to be equivalent to first-order necessary optimality conditions together with the condition that the compensator is minimal. Although the differences between the SDOPE, the CDOPE and the result presented in [3] are rather small and subtle, they are crucial when it comes to calculating numerical solutions, as demonstrated in this paper. Note that the strengthening of the discrete-time optimal projection equations also applies to the discrete-time optimal reduced-order modelling and filtering problem [3].

The optimal projection equations, for the first time, provided a good insight in the reduced-order LQG problem. They revealed relations with standard (full-order) LQG theory. Furthermore they provided an attractive alternative to compute optimal reduced-order LQG compensators [3,4,6,7]. Until then optimal reduced-order LQG compensators could only be computed through constrained non-linear parameter optimization which exhibits many drawbacks related to non-linear optimization. In the continuous-time case attempts have been made to find necessary and sufficient conditions for optimal reduced-order LQG compensation [6,7]. These results were carried over to the discrete-time case in [4]. The results of this paper indicate that these attempts, so far, have failed.

Based on the SDOPE two new algorithms are proposed to compute discrete-time optimal reduced-order LQG compensators. One is a homotopy algorithm. Although homotopy algorithms have been proposed before [4,7], the homotopy is different in our case. The other algorithm iterates the SDOPE and is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration. In two computer experiments, using different initializations of the iterative algorithm, it is demonstrated that the reduced-order optimal LQG compensation problem, in general, may possess multiple extrema. The computer experiments also show that the homotopy algorithm, which finds only one solution, often, but not always, finds the global minimum.

The iterative algorithm is shown to be highly efficient compared to both the homotopy algorithm and constrained non-linear parameter optimization. An example for which the optimal full-order compensator is not minimal is included. For this example the algorithms generate a minimal realization of the full-order compensator. This shows that the algorithms automatically reduce the order of the compensator if a minimal optimal compensator with the prescribed order cannot be found.

Finally a numerical example is presented which shows that the CDOPE have many solutions which do not correspond to optimal reduced-order compensators. Furthermore an analytical argument is used to show that the desired solutions, in general, are never obtained through iteration of the CDOPE. Also it is clarified why initially this phenomenon was overlooked.

2. The Optimal Reduced-Order LQG Compensation Problem

Consider the system,

\[ x_{i+1} = \Phi x_i + \Gamma u_i + \nu_i, \quad (1.1) \]

\[ y_i = C x_i + w_i, \quad i = 0, 1, 2, \ldots, \quad (1.2) \]

where \( x_i \in \mathbb{R}^n \) is the state, \( u_i \in \mathbb{R}^m \) is the control, \( y_i \in \mathbb{R}^l \) is the observation, \( \nu_i \in \mathbb{R}^r \) is the system noise, \( w_i \in \mathbb{R}^q \) the observation noise and \( \Phi, \Gamma, C \) are real matrices of appropriate dimensions. The processes \{\nu_i\}, \{w_i\} are uncorrelated zero-mean white noise sequences with covariance \( V \geq 0 \) and \( W > 0 \) respectively. The initial condition \( x_0 \) is a stochastic variable with mean \( \bar{x}_0 \) and covariance \( P_0 \) and is uncorrelated with \{\nu_i\} and \{w_i\}. System (1) is denoted by \( (\Phi, \Gamma, C) \). As controller we choose the following dynamic compensator:

\[ \dot{x}_i = F \hat{x}_i + K y_i, \quad (2.1) \]

\[ u_i = -L \hat{x}_i, \quad i = 0, 1, 2, \ldots, \quad (2.2) \]

where \( \hat{x}_i \in \mathbb{R}^n \) is the compensator state, and \( F, K, L \) are real matrices of appropriate dimension. The initial condition \( \bar{x}_0 \) is deterministic. It is assumed that \( u \geq u_c \). Compensator (2) is denoted by \( (F, K, L) \).

Definition 1. \((\Phi, \Gamma', C)\) is called \( n_c \)-compensatable if there exists a compensator \((F, K, L)\) of dimension \( n_c \) such that the closed loop system is stable.

A number of properties concerning reduced-order compensatability are stated in the following theorem.

Theorem 1 [4]

(a) \( \Phi \) stable \( \Rightarrow \) \((\Phi, \Gamma, C)\) \( n_c \)-compensatable, \( \forall n_c \).
Define and let \( \# \) denote the group inverse which is unique [10]. Then,
\[
\begin{align*}
\text{rank}(C) &= \text{rank}(M) = \text{rank}(H) = \text{rank}(T) = \text{rank}(\tau) = n_c.
\end{align*}
\]
so \( \tau \) is an oblique projection (idempotent matrix) uniquely determined by \( P \) and \( S \). \( G, M, H \) are unique up to a change of basis in \( \mathbb{R}^{n_c} \). The triple \((G, M, H)\) is called a projective factorization of \( \hat{P}S \).

\( \hat{P}S \) in Lemma 1 is diagonalizable and has \( n_c \) non-zero eigenvalues which are positive [9]. Hence \( G, M, H \) and \( \tau \) can be computed from an eigenvalue decomposition of \( \hat{P}S \) as follows:
\[
\begin{align*}
\hat{P}S &= U_{\hat{P}S} \Lambda_{\hat{P}S} U_{\hat{P}S}^{-1}, \quad (5.1) \\
G &= [A^T \ 0] U_{\hat{P}S}^{-1}, \quad (5.2) \\
M &= A^{-1} \Theta_{\hat{P}S} A, \quad (5.3) \\
H &= [A^{-1} \ 0] U_{\hat{P}S}^{-1}, \quad (5.4) \\
\tau &= U_{\hat{P}S} \begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix} U_{\hat{P}S}^{-1}, \quad (5.5)
\end{align*}
\]
where the columns of \( U_{\hat{P}S} \) are eigenvectors of \( \hat{P}S \) and the elements of the diagonal matrix
\[
\Lambda_{\hat{P}S} = \begin{bmatrix} \Theta_{\hat{P}S} & 0 \\ 0 & 0 \end{bmatrix}
\]
are the eigenvalues of \( \hat{P}S \). The \( n_c \) non-zero diagonal elements of \( \Lambda_{\hat{P}S} \) are the diagonal elements of \( \Theta_{\hat{P}S} \). \( A \in \mathbb{R}^{n_c \times n_c} \) in (5.2)–(5.4) is an arbitrary non-singular matrix. This reflects the uniqueness of \( G, M, H \) up to a change of basis in \( \mathbb{R}^{n_c} \).

For convenience the following notations are introduced:
\[
W_P = W + CPC^T, \quad (6.1)
\]
\[ R_S = R + \Gamma^T S \Gamma, \quad \text{(6.2)} \]
\[ K_P = \Phi P C^T W_p^{-1}, \quad \text{(6.3)} \]
\[ L_S = R_S^{-1} \Gamma^T S \Phi, \quad \text{(6.4)} \]
\[ \Sigma_P^1 = \Phi P C^T W_P^{-1} (\Phi P C^T)^T = K_P W_P K_P^T, \quad \text{(6.5)} \]
\[ \Sigma_S^2 = (\Gamma^T S \Phi)^T R_S^{-1} \Gamma^T S \Phi = L_S^T R_S L_S, \quad \text{(6.6)} \]
\[ \Phi_P^1 = \Phi - \Phi P C^T W_p C = \Phi - K_P C, \quad \text{(6.7)} \]
\[ \Phi_S^2 = \Phi - \Gamma R_S^{-1} \Gamma^T S \Phi = \Phi - \Gamma L_S, \quad \text{(6.8)} \]
\[ \Psi_{P,P}^1 = \Phi_S^2 \hat{P} \Phi_S^2 + \Sigma_P^1, \quad \text{(6.9)} \]
\[ \Psi_{P,S}^2 = \Phi_P^1 \hat{S} \Phi_P^1 + \Sigma_S^2, \quad \text{(6.10)} \]
\[ \tau = I - \tau. \quad \text{(6.11)} \]

**Theorem 2.** A stabilizing compensator \((F, K, L)\) satisfies the first-order necessary optimality conditions for optimal reduced-order LQG compensation and is minimal if and only if there exist non-negative symmetric \(n \times n\) matrices \(P, S, \hat{P}, \hat{S}\) such that for some projective factorization \((G, M, H)\) of \(\hat{P} \hat{S}\),

\[ F = H^T (\Phi - K_P C - \Gamma L_S) G, \quad \text{(7.1)} \]
\[ K = HK_P, \quad \text{(7.2)} \]
\[ L = L_S G^T. \quad \text{(7.3)} \]

and such that \(P, S, \hat{P}, \hat{S}, \tau\) satisfy

\[ P = \Phi P \Phi^T - \Sigma_P^1 + V + \tau \Psi_{P,P}^1 \Psi_{P,P}^1^T, \quad \text{(8.1)} \]
\[ S = \Phi^T S \Phi - \Sigma_S^2 + Q + \tau \Psi_{P,S}^2 \Psi_{P,S}^2^T, \quad \text{(8.2)} \]
\[ \hat{P} = \frac{1}{2} \left[ \Psi_{P,P}^1 \Psi_{P,P}^1^T + \Psi_{P,P}^1^T \Psi_{P,P}^1 \right], \quad \text{(8.3)} \]
\[ \hat{S} = \frac{1}{2} \left[ \Psi_{P,S}^2 \Psi_{P,S}^2^T + \Psi_{P,S}^2^T \Psi_{P,S}^2 \right], \quad \text{(8.4)} \]

\[ \text{rank} (\hat{P}) = \text{rank} (\hat{S}) = \text{rank} (\hat{P} \hat{S}) = n_c, \quad \text{(8.5)} \]

\[ \tau = \hat{P} \hat{S} (\hat{P} \hat{S})^T. \quad \text{(8.6)} \]

For the costs we have

\[ \sigma^\infty = \sigma_Q, R = \sigma_V, W, \quad \text{(9.1)} \]
\[ \sigma_Q, R = \text{tr} [QP + (Q + L^T R L_S) \hat{P}], \quad \text{(9.2)} \]
\[ \sigma_V, W = \text{tr} [VS + (V + K_P W K_P^T) \hat{S}]. \quad \text{(9.3)} \]

Proof. The proof of Theorem 2 is given in Appendix 1.

Equations (8.1)–(8.6) are the SDOPE. The SDOPE differ from the CDOPE [4, 5] which in turn differ from the discrete-time optimal projection equations originally presented in [3]. The differences concern the expressions for \(\hat{P}\) and \(\hat{S}\). In [3],

\[ \hat{P} = \Phi_S^2 \hat{P} \tau \Phi_S^2 + \Sigma_P^1, \quad \text{(10.1)} \]
\[ \hat{S} = \Phi_P^1 \tau \hat{S} \Phi_P^1 + \Sigma_S^2. \quad \text{(10.2)} \]

Although this was not mentioned explicitly a small flaw in the above equations was removed in [4], see also [5]. This resulted in

\[ \hat{P} = \tau (\Phi_S^2 \hat{P} \Phi_S^2 + \Sigma_P^1) \tau^T \]
\[ = \tau \Psi_{P,P}^1 \Psi_{P,P}^1 \tau^T, \quad \text{(11.1)} \]
\[ \hat{S} = \tau (\Phi_P^1 \tau \hat{S} \Phi_P^1 + \Sigma_S^2) \tau \]
\[ = \tau \Psi_{P,S}^2 \Psi_{P,S}^2 \tau^T. \quad \text{(11.2)} \]

The difference between Eqs (8.3) and (8.4), which are part of the SDOPE, and Eqs (11.1) and (11.2), which are part of the CDOPE, relates to the following equalities which must hold if the first-order necessary optimality conditions are to be satisfied (see Section A4 of Appendix 1):

\[ \hat{P} = \tau \Psi_{P,P}^1 \tau \Phi_P^1 \tau, \quad \text{(12.1)} \]
\[ \hat{S} = \tau \Psi_{P,S}^2 \tau \Phi_P^1 \tau \Phi_P^1 \tau, \quad \text{(12.2)} \]
Now Eqs (8.3) and (8.4) ensure that these equalities hold. Equations (11.1) and (11.2) however do not guarantee the second and third equality in (12.1) and (12.2) to hold (see Section A4 of Appendix I). Although the differences mentioned above are rather small and subtle, they are crucial when it comes to calculating numerical solutions, as demonstrated in Sections 5 and 6.

Notice that in the full-order case the optimal projection matrix \( T \) becomes the identity matrix while its factors \( G \) and \( H \) can be chosen to be the identity matrix. In that case (8.1) and (8.2) reduce to the standard observation and control Riccati equations and from (7.1)-(7.3) the optimal full-order compensator is obtained. Equations (8.3)-(8.6) then express the proviso that the compensator be minimal. The coupling of the equations due to the projection illustrates the non-optimality of sequential controller reduction or model reduction schemes, because in the reduced-order case there is no longer separation between observation and control operations.

To state the following theorems, it is convenient to introduce the notion of detectability and observability of a triple, instead of a pair of matrices.

**Definition 2.** \((\Phi, \Gamma, C)\) is called detectable if \((\Phi, C)\) and \((\Phi^T, \Gamma^T)\) are both detectable. \((\Phi, \Gamma, C)\) is called observable if \((\Phi, C)\) and \((\Phi^T, \Gamma^T)\) are both observable.

**Theorem 3.** Assume \((\Phi, \nu^{1/2}, Q^{1/2})\) is detectable. Then all non-negative solutions \((P, S, \Phi, \Sigma)\) of (8) correspond to all minimal stabilizing compensators that satisfy the first-order necessary optimality conditions.

**Proof.** This can be seen from [11, Theorem 3], where the more general case of systems with white parameters is considered. The order of the compensator plays no role.

It can be shown that for the stability of the closed loop system, or for the compensator to be stabilizing, the detectability condition in Theorem 3 may be weakened to: \((\Phi, Q^{1/2})\) is detectable or \((\Phi^T, \nu^{1/2})\) is detectable.

From [12,13] the optimal compensator with prescribed dimensions may not be minimal, even if the system \((\Phi, \Gamma, C)\) is minimal and \((\Phi, \nu^{1/2}, Q^{1/2})\) is observable. These compensators do not fall within the scope of Theorems 2 and 3. However, if the optimal compensator is not minimal the prescribed compensator dimensions can be reduced, without loss of performance. Example 3 in Section 6 illustrates that the algorithms automatically reduce the order of the compensator if an optimal compensator with the prescribed dimension cannot be found.

So far first-order necessary optimality conditions have been considered. In the continuous-time case attempts have been made to find necessary and sufficient conditions for optimal reduced-order LQG compensation [6,7]. These results were carried over to the discrete-time case [4]. Along with the introduction of two numerical algorithms, in the next section these results, which rely on homotopy degree theory, will be reconsidered.

### 4. Numerical Algorithms

Let \( S_n \) denote the space of \( n \)-dimensional real symmetric matrices. Define the following non-linear transformation based on the SDOPE:

\[
\mathbb{R}X : S_n \times S_n \times S_n \times \cdots \times S_n \rightarrow S_n \times S_n \times S_n \times \cdots ,
\]

\[
\mathbb{R}X = \begin{pmatrix} \Phi X_1 \Phi^T - \Sigma_{11}^1 + V + \tau_1 \psi_{X_1,x_1,x_1} & \Phi^T X_2 \Phi - \Sigma_{12}^2 + Q + \tau_1 \psi_{X_2,x_2,x_2} \\ \frac{1}{2} \left[ \tau_1 \psi_{X_1,x_1,x_1} + \psi_{X_2,x_2,x_2} \right] \\ \frac{1}{2} \left[ \tau_1 \psi_{X_1,x_2,x_2} + \psi_{X_2,x_1,x_1} \right] \end{pmatrix},
\]

(13.1)

where

\[
X = (X_1, X_2, X_3, X_4), \quad X_1, X_2, X_3, X_4 \in S_n,
\]

(13.2)

\[
\tau = U_{x,X_4} \begin{bmatrix} L_x & 0 \\ 0 & 0 \end{bmatrix} U_{x,X_4}^{-1}.
\]

(13.3)

\[
n_4' = \min(n_x, \text{rank}(X_3 X_4)).
\]

(13.4)

In Eq. (13.3) \( U_{x,X_4} \) is obtained from the eigenvalue decomposition

\[
X_3 X_4 = U_{x,X_4} \Lambda_{X_3,X_4} U_{x,X_4}^{-1},
\]

(13.5)

where the eigenvalues, i.e. the diagonal elements of the diagonal matrix \( \Lambda_{X_3,X_4} \), appear in ascending order of their real part. If during the iteration, for some reason, \( \text{rank}(X_3 X_4) < n_x \), Eq. (13.4) ensures that \( \tau \) in (8.6) is still properly computed. This situation occurs in Example 3 of Section 6. There the implications of
Eq. (13.4) will be discussed further. Observe that 
\( (P, S, \hat{P}, \hat{S}) = \mathcal{R}(P, S, \hat{P}, \hat{S}) \) is equivalent to (8) if 
\( \text{rank}(\hat{P}\hat{S}) = n_c \). Now consider 
\( (X_{1i}, X_{2i}, X_{3i}, X_{4i}) = \mathcal{R}(X_{10}, X_{20}, X_{30}, X_{40}), i = 0, 1, 2, \ldots \) If 
\( n_c = n \) then \( X_{1i} \) and \( X_{2i} \) are iterations of the well-known uncoupled 
observation and control Riccati equations. It is well known that \( \{X_{1i}\} \) and \( \{X_{2i}\} \) are monotonic if 
\( X_{10} = X_{20} = 0 \) in the sense that \( X_{1i} \leq X_{1j} \) and \( X_{2i} \leq X_{2j} \) if 
\( i < j \). This property may be used to prove convergence 
of \( \{X_{1i}\} \) and \( \{X_{2i}\} \) and provides an easy way to com­
pute a solution of the algebraic observation and control 
Riccati equations. If \( n_c < n \) however, \( \{X_{1i}\} \) and 
\( \{X_{2i}\} \) are not monotonic due to the coupling between 
the corresponding equations. Fortunately it is still po­
sible to obtain convergence using the method of homo­
topic continuation. This method embeds an original 
problem in a parameterized family of problems, where 
the parameter value varies continuously from 0 to 1. 
The idea is that for the parameter value 0 an easy prob­
lem with a known solution, in our case the full-order 
LQG problem, is obtained while for the parameter 
value 1 the original problem, in our case the reduced­
order LQG problem, is obtained. We may follow the 
solution path as the easy problem is deformed into 
the original problem. For more information on homo­
topies and their computation, we refer to [14]. In order 
to use the above mentioned method define the non­
linear transformation 
\[
\mathcal{R}_\alpha X : S_n \times S_n \times S_n \times S_n \rightarrow S_n \times S_n \times S_n \times S_n,
\]
\( \alpha \in [0, 1] \), 
\[
\mathcal{R}_\alpha X = \begin{pmatrix}
\Phi X_1 \Phi^T - \Sigma_{X_1} + \Psi_{X_1} \Phi X_2 \Phi^T - \Sigma_{X_2} + \Psi_{X_2} \Phi X_3 \Phi^T - \Sigma_{X_3} + \Psi_{X_3} \Phi X_4 \Phi^T - \Sigma_{X_4} + \Psi_{X_4}
\end{pmatrix}
\]
where 
\( \tau_\alpha = U_{X_1X_4} \begin{bmatrix} I_{n_c} & 0 \\ 0 & (1-\alpha)I_{n-n_c} \end{bmatrix} U_{X_1X_4}^T, \alpha \in [0, 1] \), 
\[
(14.1)
\]
with \( n_c \) given by (13.4). In Eq. (14.2) \( U_{X_1X_4} \) is obtained 
from the eigenvalue decomposition (13.5) with 
the associated ordering of the eigenvalues. Call \( X = 
(X_1, X_2, X_3, X_4) \) non-negative if \( X_1, X_2, X_3, X_4 \geq 0 \). 
Denote the parameterized equation \( Y^\alpha = \mathcal{R}_\alpha X \) by 
\( H(Y^\alpha, \alpha) = 0 \), where \( Y^\alpha \) denotes the non-negative solution 
of \( X = \mathcal{R}_0 X \). The function \( H(Y^\alpha, \alpha) \) is called a 
homotopy. For \( \alpha = 1 \) we have the original coupled 
SDOPE associated with the reduced-order problem, 
and for \( \alpha = 0 \) the uncoupled control and observation 
Riccati equations associated with the full-order 
problem. Based on the homotopy \( H(Y^\alpha, \alpha) \) the following 
discrete homotopy algorithm is proposed.

**Algorithm 1**

Initialization: 
\( X_0^\alpha = 0, X_2^\alpha = 0, X_3^\alpha = 1, X_4^\alpha = 1 \), 
\( \alpha = 0, \Delta \alpha = 1/N, N \geq 1 \) and integer. 
Compute \( Y^\alpha = \lim_{\alpha \rightarrow \infty} \mathcal{R}_\alpha(X^0) \) through iteration. 
Loop: 
\( \alpha := \alpha + \Delta \alpha \)
Determine, through iteration, whether 
\( Y^\alpha = \lim_{\alpha \rightarrow \infty} \mathcal{R}_\alpha(Y^{\alpha-\Delta \alpha}) \) exists. 
Stop when \( \alpha = 1 \). 

Consider again the homotopy \( H(Y^\alpha, \alpha) \). If the number 
of solutions of the equation \( H(Y^\alpha, \alpha) = 0 \) from \( \alpha = 0 \) to 
\( \alpha = 1 \) remains constant, then, given the uniqueness of 
the optimal full-order compensator, Theorem 3 would 
give us necessary and sufficient conditions for the exis­
tence of a unique optimal reduced-order compensator. 
Similarly the algorithm that computes the unique solu­
tion of the two Riccati equations of full-order LQG 
control through iteration, could be carried over to an 
algorithm that computes, what would be the unique 
non-negative solution of the SDOPE, through iteration 
[4]. In the continuous-time case results have been 
published concerning conditions under which the 
number of solutions along the solution path remains 
constant [6,7]. These conditions were carried over to 
the discrete-time case [4]. On the other hand convex 
analysis of systems controlled by static output-feed­
back [15] seems to indicate that, in general, multiple 
nonnegative solutions satisfying the SDOPE may 
exist. Through numerical examples the latter is con­
firmed in Section 7. Despite this result we could still 
pursue the idea of iterating the SDOPE.

**Algorithm 2**

Initialization: 
\( X_0^\alpha = 0, X_2^\alpha = 0, X_3^\alpha = \Lambda_1, X_4^\alpha = \Lambda_2 \)
with \( \Lambda_1, \Lambda_2 \geq 0 \), symmetric, random and with rank \( n_c \). 
Computation: 
Determine, through iteration, whether \( Y^\alpha = \lim_{\alpha \rightarrow \infty} \mathcal{R}_\alpha(X^\alpha) \) exists. 

**Theorem 4.** If \( (\bar{\Phi}, \bar{\Psi}_1^{\alpha}, \bar{Q}_1^{\alpha}) \) is detectable then Algorithms 1 and 2, if they converge to \( Y^\alpha \geq 0 \), generate
minimal stabilizing compensators, given by (7), with a minimal dimension equal to \( n'_c = \text{rank}(Y'_1 Y'_2) \leq n_c \) and costs \( \sigma_n \), given by (9), where \((P, S, \hat{P}, \hat{S}) = (Y'_1, Y'_2, Y'_3, Y'_4)\). These compensators are local or global minima of the optimal reduced-order LQG compensation problem with prescribed compensator order \( n'_c \).

**Proof.** If the algorithms converge \( \text{rank}(Y'_1 Y'_2) \leq n_c \). Then from (14) and Theorem 3, if \( Y'_1 \geq 0 \), \( Y'_1 \) corresponds to a minimal stabilizing compensator with dimension \( n'_c = \text{rank}(Y'_1 Y'_2) \leq n_c \) which satisfies the first-order necessary optimality conditions when the prescribed compensator order equals \( n'_c \).

Because both Algorithm 1 and 2 are generalizations of the algorithm that solves the two Riccati equations of full-order LQG control through iteration, they converge to local (global) minima, not to local (global) maxima, which also satisfy the first-order necessary optimality conditions. \( \square \)

The discussion in Section 6, related to example 3, and the large number of examples in Section 7 show that, in general, \( n'_c = n_c \) in Theorem 4. Furthermore all the examples considered in Section 6 and also the large number of examples considered in Section 7 all share the property that, if Algorithm 1 or 2 converges, it converges to a non-negative solution. This suggests that, if the algorithms converge, \( Y'_1 \geq 0 \). From Definition 1, Theorems 1 and 4 the following numerical test, representing sufficient conditions for \( n_c \)-compensatability, is obtained.

Compensatability Test. Check if \( \Phi \) is stable. If so \((\Phi, \Gamma, C)\) is \( n_c \)-compensatable \( \forall \tau \). If not, choose \( R = I, \tau = I, Q = I, V = I \). Then \((\Phi, \Gamma, C)\) is \( n_c \)-compensatable if Algorithm 1 or 2, for some \( \Lambda_1, \Lambda_2 \) converges to \( Y'_1 \geq 0 \). If not, nothing can be concluded with respect to the \( n_c \)-compensatability of \((\Phi, \Gamma, C)\). \( \square \)

5. The Strengthened Versus the Conventional Discrete-Time Optimal Projection Equations

Consider the following non-linear transformation which is comparable to (13) but, instead of the SDOPE, is based on the CDOPE:

\[
\begin{align*}
\mathbb{R}_c X : S_n \times S_n \times S_n \times S_n & \to S_n \times S_n \times S_n \times S_n, \\
\mathbb{R}_c X = (\Phi X_1 \Phi^T - \Sigma_{x_1}^T + V + \tau_l \Psi_{\lambda_2, \lambda_1} \tau_{l \lambda_1}^T, \\
& \quad \Phi^T X_2 \Phi - \Sigma_{x_2}^T + Q + \tau_l \Psi_{\lambda_1, \lambda_2} \tau_{l \lambda_1}^T, \\
& \quad \tau_l \Psi_{\lambda_1, \lambda_1} \tau_{l \lambda_1}^T, \tau_l \Psi_{\lambda_1, \lambda_2} \tau_{l \lambda_1}^T).
\end{align*}
\]  \( (15) \)

where \( X \) and \( \tau \) are given by (13.2)-(13.4). Since \((X_1, X_2, X_3, X_4)\) corresponds to \((P, S, \hat{P}, \hat{S})\) we will refer to the latter if it suits us. Also the iteration index \( i \) is used whenever it suits us.

**Lemma 2.** Consider \((P_i, S_i, \hat{P}_i, \hat{S}_i) = \mathbb{R}_c(0, 0, \Lambda_1, \Lambda_2) i = 0, 1, 2, \ldots \) with \( \Lambda_1, \Lambda_2 \geq 0 \), symmetric, random and with rank \( n_c \). Assume rank \((\hat{P}_i, \hat{S}_i) = n_c, i = 0, 1, 2, \ldots \) Then \( \tau_i = \tau_0, i \geq 0 \), i.e. \( \tau_i \) is invariant under \( \mathbb{R}_c \). \( \square \)

**Proof.** Since \( \text{rank}(\hat{P}_i, \hat{S}_i) = n_c, i \geq 0 \) from (13.3), (13.4) \( \text{rank}(\tau_i) = n_c, i \geq 0 \). Then from (15),

\[
\hat{P}_{i+1} \hat{S}_{i+1} = G_i^T H_{\lambda \lambda} H_i^T G_i \Psi_{\lambda}^2 G_i^T \Psi_{\lambda}, \quad i \geq 0.
\]  \( (16) \)

From (4) we have

\[
\hat{P}_{i+1} \hat{S}_{i+1} = G_{i+1} M_{i+1} H_{i+1}
\]  \( (17.1) \)

with

\[
G_{i+1} = G_i \in \mathbb{R}^{n_c \times n_c}, \quad (17.2)
\]

\[
H_{i+1} = H_i \in \mathbb{R}^{n_c \times n_c}, \quad (17.3)
\]

\[
M_{i+1} = H_i \Psi_{\lambda}^2 H_i^T G_i \Psi_{\lambda}^2 G_i^T \in \mathbb{R}^{n_c \times n_c}, \quad (17.4)
\]

while \( M_{i+1} \) is positive definite because of the assumption \( \text{rank}(\hat{P}_i, \hat{S}_i) = n_c \). Since \( M_{i+1} \) is positive definite, from Lemma 1, (17.1) is a projective factorization of \( \hat{P}_{i+1} \hat{S}_{i+1} \). From Lemma 1, (15) and (8.6), it follows that

\[
\tau_{i+1} = G_{i+1}^T H_{i+1} = G_i^T H_i = \tau_i, \quad i \geq 0.
\]  \( (18) \)

Since \( \tau_i \) is uniquely determined by \( \hat{P}_i, \hat{S}_i \) through Eq. (8.6) other projective factorizations do not alter this result. \( \square \)

Lemma 2 implies that, in general, iterations of the CDOPE leave \( \tau \) unchanged, so \( \tau \) does not converge to an optimal value. In Section 6 it is demonstrated through numerical examples, that despite this property, \( \mathbb{R}_c(0, 0, \Lambda_1, \Lambda_2) \) often converges. Only if the initial value of \( \tau \), determined by \( \Lambda_1, \Lambda_2 \), is optimal, these solutions of the CDOPE correspond to optimal reduced-order compensators, otherwise they do not.

6. Numerical Issues and Examples

The function `eig` (Matlab reference guide [16]) together with the function `esort` (Control system
toolbox for use with Matlab [17]) was used to compute the
eigenvalue decomposition (13.5). In (13.4) rank($X_iX_j$) is computed as the number of eigenvalues
of $X_iX_j$ with a magnitude larger than $10^{-6}$ times the
largest. Since (13) is initialized with symmetric matrices, $X_3$ and $X_4$ are always symmetric and so the
eigenvalues of $X_3X_4$ are real numbers. The eigenvectors associated to the zero eigenvalues may be complex
so Matlab may produce complex matrices, $U_{X_iX_j}$, and $X_{X_iX_j}$. The computation of $\tau$ and its factors $G$ and $H$
only requires real parts of $U_{X_iX_j}$, $X_{X_iX_j}$, $G$ and $H$ are computed according to (5.2) and (5.4) with
$P_S = X_3X_4$ and $A = I_n$.

The iterations of Algorithms 1 and 2 are numerically
stable in general. In critical situations, e.g. if the
closed loop system is at the edge of stability, the
numerical stability is greatly enhanced if we make the
following modifications: (1) After each iteration compute,

$$X_i := \frac{1}{2}(X_{i+1} + X_i^T),$$
$$X_2 := \frac{1}{2}(X_2 + X_2^T), \quad i = 0, 1, 2, \ldots \quad (19)$$

to further enhance the symmetry of $X_{ii}$, $Y_{ii}$. (2) Immediately after (13) and (17) modify

$$X_i = (X_i, X_2, X_3, X_4)$$

according to

$$X_i := (1-a)X_i + aX_{i-1},$$

$$j = 1, 2, 3, 4, \quad i = 1, 2, \ldots, \quad 0 \leq a < 1, \quad (20)$$

which implements a numerical damping that greatly enhances the convergence properties. In the following examples these computations were applied with $a = 0.25$.

Example 1 (Taken from [4])

$$\Phi = \begin{bmatrix}
0.1051 & 0.1841 & 0.2543 & 0.2004 & 0.2529 \\
0.0226 & 0.2493 & 0.3222 & 0.3297 & 0.0441 \\
0.3259 & 0.3989 & 0.0037 & 0.2827 & 0.3139 \\
0.3261 & 0.0166 & 0.1840 & 0.4466 & 0.1997 \\
0.4487 & 0.0237 & 0.0321 & 0.4062 & 0.3366
\end{bmatrix}$$

$$\Gamma^T = \begin{bmatrix}
0.9103 & 0.7622 & 0.2625 & 0.0475 & 0.7361 \\
0.3282 & 0.6326 & 0.7564 & 0.9910 & 0.3653 \\
0.6316 & 0.8847 & 0.2727 & 0.4364 & 0.7665 \\
0.2470 & 0.9826 & 0.7227 & 0.7534 & 0.6515 \\
0.4777 & 0.0727
\end{bmatrix},$$

$$C = \begin{bmatrix}
0.3884 & 1.6578 & 0.0613 & 0.0137 & 0 \\
0.0834 & 0.6802 & 0.0948 & 0.6800 & 0 \\
1.2041 & 0.9213 & 0.9395 & 0.1186 & 0 \\
1.2048 & 1.4738 & 1.1904 & 0.7405 & 0 \\
0 & 0 & 0 & 0 & 0.95
\end{bmatrix}.$$
Table 1. Solutions of the CDOPE for Example I.

<table>
<thead>
<tr>
<th>$n_c$</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n'_c$</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{Q,R}$</td>
<td>3.51575</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
</tr>
<tr>
<td>$\sigma_{V,W}$</td>
<td>3.51575</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
</tr>
<tr>
<td>$N(\Delta P')$</td>
<td>7.395e-5</td>
<td>52.59</td>
<td>0.8855</td>
<td>0.2397</td>
<td>0.2469</td>
</tr>
<tr>
<td>$i$</td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>$CT$</td>
<td>0.22</td>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 2. Solutions of the SDOPE for Example I.

<table>
<thead>
<tr>
<th>$n_c$</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n'_c$</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{Q,R}$</td>
<td>3.51575</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
</tr>
<tr>
<td>$\sigma_{V,W}$</td>
<td>3.51575</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
<td>3.51576</td>
</tr>
<tr>
<td>$N(\Delta P')$</td>
<td>7.40e-5</td>
<td>1.23e-3</td>
<td>9.69e-5</td>
<td>9.66e-4</td>
<td>1.34e-4</td>
</tr>
<tr>
<td>$i$</td>
<td>8</td>
<td>8</td>
<td>13</td>
<td>13</td>
<td>17</td>
</tr>
<tr>
<td>$CT$</td>
<td>0.22</td>
<td>0.22</td>
<td>0.33</td>
<td>0.33</td>
<td>0.44</td>
</tr>
</tbody>
</table>

Table 3. Solutions of the SDOPE for Example 2.

<table>
<thead>
<tr>
<th>$n_c$</th>
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<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n'_c$</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{Q,R}$</td>
<td>195.53</td>
<td>199.56</td>
<td>220.03</td>
<td>359.09</td>
<td>2516.2</td>
</tr>
<tr>
<td>$\sigma_{V,W}$</td>
<td>195.53</td>
<td>199.56</td>
<td>220.03</td>
<td>359.09</td>
<td>2516.2</td>
</tr>
<tr>
<td>$N(\Delta P')$</td>
<td>9.45e-5</td>
<td>1.54e-4</td>
<td>4.87e-3</td>
<td>2.17e-6</td>
<td>4.43e-2</td>
</tr>
<tr>
<td>$i$</td>
<td>96</td>
<td>104</td>
<td>130</td>
<td>138</td>
<td>518</td>
</tr>
<tr>
<td>$CT$</td>
<td>2.42</td>
<td>2.69</td>
<td>3.24</td>
<td>3.51</td>
<td>13.13</td>
</tr>
</tbody>
</table>

The optimal reduced-order compensation problem may be conceived as a constrained non-linear parameter optimization problem, where the optimization parameters are the elements of $F, K$ and $L$. In this case, as the order of the compensator increases, the number of optimization parameters increases dramatically which soon renders the optimization practically impossible. In contrast to this, from Tables 2 and 3 observe that the computation of optimal reduced-order compensators, based on iteration of the SDOPE, takes less effort when the order of the compensator increases.

To illustrate the superiority of the numerical algorithm based on iteration of the SDOPE for Example I.

and computation times obtained for some of the problems above using constrained non-linear parameter optimization are presented. To successfully solve some of these problems, first an optimization is performed which tries to minimize the spectral radius of the closed loop system as a function of $F, K$ and $L$. This optimization was performed using the function FMINS of the MATLAB optimization toolbox. The outcome of this optimization is used as the initial value for the constrained non-linear parameter optimization. This optimization was performed using the function CONSTR of the MATLAB optimization toolbox. We only implemented the constraint that the spectral radius of the closed loop system be smaller than 1. Table 4 lists the outcome. As can be seen, especially for higher compensator orders, the algorithm based on iteration of the SDOPE behaves highly superior. The parameter optimization method becomes more efficient when the compensator is represented using a canonical form, since this reduces the number of elements of $F, K$ and $L$ that have to be optimized. But also in this case, as the compensator dimensions grow, the optimal projection algorithms soon become more efficient.

From [13] a special situation occurs if the optimal full-order LQG compensator, computed from the two Riccati equations of full-order LQG control, is not minimal. This may happen even if the system...
Table 4. Optimal reduced-order compensators from constrained non-linear parameter optimization.

<table>
<thead>
<tr>
<th>Example 1</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_c$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$\sigma_{c}$</td>
<td>3.53373</td>
<td>3.51593</td>
<td>3.51580</td>
<td>3.51578</td>
</tr>
<tr>
<td>$CT$</td>
<td>9.89</td>
<td>50.48</td>
<td>214.98</td>
<td>545.02</td>
</tr>
</tbody>
</table>

Example 2

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2516.4$</td>
<td>$359.09$</td>
<td>$220.03$</td>
<td></td>
</tr>
</tbody>
</table>

Example 3

Table 5. Solutions of the SDOPE for Example 3.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_c$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$n_c'$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{Q,k}$</td>
<td>4.2242</td>
<td>4.2242</td>
<td>4.2242</td>
</tr>
<tr>
<td>$\sigma_{T,W}$</td>
<td>4.2242</td>
<td>4.2242</td>
<td>4.2242</td>
</tr>
</tbody>
</table>

This suggests (in this case) that the SDOPE have no solution when $n_m < n_c \leq n$ because the rank condition (6.5) cannot be met. In this case Eq. (13.4) ensures that, despite the loss of rank, the projection $\tau$ is still properly computed. Although Algorithm 2 produces the desired answer one should preferably choose $n_c \leq n_m$, i.e. less than or equal to the minimal dimension of the optimal full-order compensator, which is globally optimal. In Example 3 this amounts to the choice $n_c = 1$.

With respect to (13.4) one may wonder whether, during the iteration, $\text{rank}(X_3X_4)$ is able to increase, in other words, is able to 'recover'. Using Algorithm 2 with initial values $\Lambda_1, \Lambda_2$ with a rank less than $n_c$ it is easily verified, e.g. using Example 2, that this is so. This is due to Eqs (8.3) and (8.4). Note that $\Psi^1, \Psi^2$ are symmetric by definition. Then, as long as $\Psi^1$ is not symmetric and $\text{rank}(\Psi^1) \geq \text{rank}(\tau)$, in general, $\text{rank}(\tau \Psi^1 + \Psi^1 \tau^T) > \text{rank}(\tau)$ in (8.3). Similar arguments apply to (8.4). Because of this property, in general, solutions generated by Algorithms 1 and 2 have the property $\text{rank}(Y_1 Y_2^T) = n_c$, as desired, unless such solutions do not exist. This fact is illustrated by Examples 1 - 3 and the huge number of random examples considered in two computer experiments described in Section 7. From Lemma 2 observe that iterations of the CDOPE cannot recover $\text{rank}(X_3X_4)$. This can also be seen from (15) and the fact that $\text{rank}(\tau \Psi^1 \tau^T) \leq \text{rank}(\tau)$.

7. Local and Global Optimal Reduced-Order Compensators: Two Computer Experiments

Let $n_u$ denote the dimension of the unstable subspace of $\Phi$, i.e. the number of eigenvalues of $\Phi$ with a magnitude greater or equal to 1. Consider the following two types
of restrictions regarding the choice of \( n, m, l \) and \( n_c \):

\[
\max(\min(n, m, l), n_u) \leq n_c, \quad (21)
\]

\[
m \leq n, \ l \leq n, \ \max(m, l, n_u) \leq n_c \leq n. \quad (22)
\]

From a system theoretic and control system design perspective the number of inputs \( m \) and the number of outputs \( l \) should not exceed the system and compensator dimension, i.e. \( n \) and \( n_c \). Also, to stabilize the system, it is natural to choose the compensator order greater or equal to the dimension of the unstable subspace of \( \Phi \). Then the conditions (21) turn into the conditions (22) which are therefore more restrictive.

It has been argued in the continuous-time case [6,7] that, if (21) is satisfied and the system is stabilizable, \( Q > 0, V > 0 \) and \( \Gamma, C \) full-rank, the optimal projection equations have at most one non-negative solution. Similar arguments apply to the discrete-time case when (22) is satisfied and \( (\Phi, V^{1/2}, Q^{1/2}) \) is detectable [4]. Example 2 in [18] is a discrete-time reduced-order LQG example where the system has dimension two and which satisfies all the conditions mentioned above for both the continuous-time and discrete-time case.

Still, if the controller has prescribed dimension one, two distinct minima are found. Several numerical examples presented in this section also contradict the discrete-time result, i.e. although (22) is satisfied and \( (\Phi, V^{1/2}, Q^{1/2}) \) is detectable the SDOPE have more than one non-negative solution in several cases. With respect to the CDOPE note that the SDOPE are stronger so each solution of the SDOPE is also a solution of the CDOPE. With respect to our numerical findings note that convex analysis of static output-feedback problems seems to indicate that, in general, in both the continuous and discrete-time case the problem may have multiple extrema [15].

Computer Experiment 1

\( n = 2, 3, \ldots, 30, m = 1, l = 1, n_c = 1, \) spectral radius \( \Phi = 0.95 \) (magnitude largest eigenvalue of \( \Phi = 0.95 \)), i.e. \( n_u = 0 \) and therefore \( (\Phi, V^{1/2}, Q^{1/2}) \) is detectable and the system is \( n_c \)-compensatable \( \forall n_c \).

For each value of \( n \), 100 randomly generated reduced-order compensation problems with the properties mentioned above were generated and solved, using Algorithm 2. Details concerning the generation of these random problems can be found in Appendix 2. Each example was recomputed 10 times (with different random values for \( A_1, A_2 \),) to try and find multiple extrema, using a numerical damping of 0.25.

The problems above that exhibited none or multiple extrema were also solved using Algorithm 1 with a numerical damping of 0.75.

Computer Experiment 2

\( n = 49, 50, \ m = 5, \ l = 5, \ n_c = n_u, \) spectral radius \( \Phi = 1.25 \) (magnitude largest eigenvalue of \( \Phi = 1.25 \)) i.e. the system is unstable.

For each value of \( n \), 20 randomly generated reduced-order compensation problems with the properties mentioned above were generated and solved, using Algorithm 2. Details concerning the generation of these random problems can be found in Appendix 2. Each example was recomputed 10 times (with different random values for \( A_1, A_2 \)) to try and find multiple extrema, using a numerical damping of 0.25.

The problems above that exhibited none or multiple extrema were also solved using Algorithm 1 with a numerical damping of 0.75.

Note that both computer experiments deal exclusively with problems that satisfy (22) and therefore also (21). Note with respect to NS in Table 6 that the system is always \( n_c \)-compensatable in Computer Experiment 1, but not necessarily in Computer Experiment 2. Finally for Computer Experiment 2 it was verified that \( (\Phi, V^{1/2}, Q^{1/2}) \) is observable and thus detectable in all of the 40 examples.

For one of the compensation problems Fig. 1, which shows a grid representing the costs in the parameter space, clearly illustrates the non-uniqueness, in terms of the performance, of optimal reduced-order compensators. Note that in the case \( n_c = 1 \) the compensator is uniquely determined by \( F \) and \( K * L \) and is minimal as long as both of these are unequal to zero. After solving the discrete Lyapunov Eq. (A3) the costs of each compensator were computed using Eq. (A4) (see Appendix 1). For the same example Fig. 2 illustrates

<table>
<thead>
<tr>
<th>Table 6: Results of Computer Experiments 1 and 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NS</strong></td>
</tr>
<tr>
<td><strong>MS</strong></td>
</tr>
<tr>
<td><strong>LM</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Experiment 1 (3000 problems)</th>
<th>Experiment 2 (40 problems)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NS</strong></td>
<td>75</td>
</tr>
<tr>
<td><strong>MS</strong></td>
<td>258</td>
</tr>
<tr>
<td><strong>LM</strong></td>
<td>64</td>
</tr>
</tbody>
</table>
Fig. 1. Local optimality, \(nx = 10, nu = ny = 1, srp = 0.95, seed = 89\) (see Appendix 2).

that the homotopy algorithm does not reach the 'global minimum'. The homotopy path related to the 'global minimum' vanishes around \(\alpha = 0.51\). It was computed using Algorithm 1 in the backward direction, i.e. starting with the 'global minimum' at \(\alpha = 1\) and then decreasing \(\alpha\) with small discrete steps towards \(\alpha = 0\). From the two computer experiments the following may be concluded:

1. All solutions \(Y^1\), found in both the computer experiments (and also all those in Section 6) using either Algorithm 1 or 2, within terms of numerical accuracy, have the property \(Y^1 \geq 0\). This suggests that \(Y^1 \geq 0\) is a (generic) property of the algorithms. Sometimes during the iteration the matrices become negative.

2. Except for Example 3 in Section 6, all compensation problems have the property \(n_m = n\), i.e. the optimal full-order compensator is minimal, and all solutions have the property \(n_e' = n_e\). This suggests that the former is a generic property of random reduced-order compensation problems and shows that the latter is a generic property of the algorithms.

3. Even if (22) is satisfied and \((\Phi, V^{1,2}, Q^{1,2})\) is detectable the SDOPE may have multiple non-negative solutions.

4. A way to compute multiple non-negative solutions, if they exist, is to repeat Algorithm 2 several times, with different random initial values \(\Lambda_1, \Lambda_2\).
Algorithm 1, often but not always, finds what seems to be the global minimum.

For random examples where the system is $n_c$-compensatable, the probability of finding local or no minima, using Algorithms 1 and 2, increases, when the order of the system increases, when the cost increase compared with the optimal full-order compensator increases, and when the spectral radius of $\Phi$ increases.

Although not explicitly shown, for examples of computer Experiment 1, were Algorithm 2 did not converge, increasing the numerical damping of 0.25, usually resulted in convergence. As an example consider $n_x = 5, n_u = 1, n_y = 1, s_{re} = 0.95$, seed = 32 (see Appendix 2). The convergence of this example is very troublesome and is achieved only after choosing $s = 0.975$, a very large value for the numerical damping. The associated minimum costs of this not necessarily unique solution were computed to be 10.9452.

8. Conclusions

A strengthened version of the discrete-time optimal projection equations has been presented (SDOPE). For the class of stabilizing compensators this version of the optimal projection equations was proved to be equivalent to first-order necessary optimality conditions together with the condition that the compensator be minimal. The CDOPE were shown to be weaker and having many solutions which do not correspond to optimal reduced-order compensators.

Based on the SDOPE two new algorithms were proposed to compute optimal reduced-order compensator. One is a homotopy algorithm. The other algorithm iterates the SDOPE and is a generalization of the algorithm that solves the two Riccati equations of full-order LQG control through iteration, and therefore is highly efficient. Through numerical examples and two computer experiments it was demonstrated that the SDOPE, in general, may have multiple solutions and that the homotopy algorithm often, but not always, finds the global minimum. The iterative algorithm was shown to be superior compared to constrained non-linear parameter optimization and the homotopy algorithm.

The huge number of numerical examples presented in this paper suggest the following important property of the algorithms. If the algorithms converge they converge to desired non-negative solutions. Furthermore it has been shown why these solutions have the desired property $n_c^* = n_c$, except when $n_c > n_m$, i.e. if the prescribed compensator dimensions exceed those of a minimal realization of the optimal full-order compensator. With respect to the latter property, without loss of performance, it is preferable to prescribe $n_c \leq n_m$. Clearly the properties of the algorithms require further investigation and proof. This also applies to the convergence.

With respect to the possible non-uniqueness of the optimal reduced-order compensator the following practical approach is suggested. Apply Algorithm 2 several times with different random initial values. Pick the best solution. Of course one can never be sure that better solutions do not exist. However, compared to the performance of the optimal full-order compensator, which represents the global minimum obtainable with any compensator, the loss of performance may serve as a criterion for acceptance of a (locally) optimal reduced-order compensator.

The results of this paper pave the way to comparable results for optimal reduced-order compensation of linear discrete-time systems with white stochastic parameters [9,18,20]. Also the results open up the possibility to develop the SDOPE for time-varying finite-horizon LQG problems where the system has either deterministic or white stochastic parameters [11,13,19]. Moreover numerical algorithms to compute these compensators may be found, based on the results of this paper [19].

References

Furthermore the criterion (3) is finite and independent of initial conditions and can be expressed as

\[ p_i = \lim_{i \to \infty} x_i \]

exists, \( p_i = x \) is the unique solution of

\[ \begin{equation} \tag{A2} \end{equation} \]

In Section A3, the SDOPE are derived. In Section A4, we show how the conditions of Theorem 2 in return imply the first-order necessary optimality conditions and the minimality of the compensator. Finally in Section A5 the explicit expressions for the compensator costs are derived.

### A1. First-Order Necessary Optimality Conditions

Introducing

\[ x_i' = \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, \quad y_i' = \begin{bmatrix} v_i \\ Kw_i \end{bmatrix}, \]

\[ \Phi' = \begin{bmatrix} \Phi & -\Gamma L \\ Kw & F \end{bmatrix}, \quad V' = \begin{bmatrix} V & 0 \\ 0 & KWK^T \end{bmatrix} \]

the closed loop system is described by

\[ x_{i+1}' = \Phi' x_i' + y_i', \quad i = 0, 1, 2, \ldots, \tag{A1.2} \]

where \( \{v_i\} \) is a zero-mean white noise sequence with covariance \( V' \) and uncorrelated with the initial condition \( x_0' \). Let \( P_i' \in R^{(n+s_i) \times (n+s_i)} \) denote the closed loop second moment \( E\{x_i'x_i'^T\} \) then, from (A1),

\[ P_i' + \Phi' P_i' \Phi'^T + V'. \tag{A2} \]

If the closed loop system is stable, \( P' = \lim_{i \to \infty} P_i' \) exists, \( P' \geq 0 \) and \( P' \) is the unique solution of

\[ P' = \Phi' P' \Phi'^T + V'. \tag{A3} \]

Furthermore the criterion (3) is finite and independent of initial conditions and can be expressed as

\[ \sigma_\infty(F, K, L) = \text{tr}(Q' P'). \tag{A4.1} \]

where \( Q' \in R^{(n+s_i) \times (n+s_i)} \) given by

\[ Q' = \begin{bmatrix} Q & 0 \\ 0 & L^T R L \end{bmatrix}. \tag{A4.2} \]

Because the eigenvalues of \( \Phi' \) continuously depend on \( F, K, L \) and since the set of stabilizing compensators is open we may apply the matrix minimum principle [8] to find first-order necessary conditions for the solution of the optimal reduced-order dynamic compensation problem. To that end define the
Hamiltonian

\[ H(F, K, L, P', S') = \text{tr} \left[ Q' P' + (\Phi' P' \Phi'^T + V' - P') S' \right] \]

(A5)

where the symmetric matrix \( S' \in \mathbb{R}^{(n+n_t) \times (n+n_t)} \) is a Lagrange multiplier. Then the first-order necessary conditions for optimal reduced-order dynamic compensation are

\[
\begin{align*}
\frac{\partial H}{\partial F} &= \frac{\partial}{\partial F} \text{tr} \left( \Phi' P' \Phi'^T S' \right) = 0, \quad (A6.1) \\
\frac{\partial H}{\partial K} &= \frac{\partial}{\partial K} \text{tr} \left[ V' S' + \Phi' P' \Phi'^T S' \right] = 0, \quad (A6.2) \\
\frac{\partial H}{\partial L} &= \frac{\partial}{\partial L} \text{tr} \left[ Q' P' + \Phi' P' \Phi'^T S' \right] = 0, \quad (A6.3) \\
\frac{\partial H}{\partial P'} &= \Phi'^T S' \Phi' + Q' - S' = 0, \quad (A6.4) \\
\frac{\partial H}{\partial S'} &= \Phi' P' \Phi'^T + V' - P' = 0, \quad (A6.5)
\end{align*}
\]

where \( S' \geq 0, P' \geq 0. \)

A2. Explicit Expressions for the Compensator Gains

Partition \( P', S' \) according to the partitioning of \( \Phi' \) in (A1).

\[
P' = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \quad S' = \begin{bmatrix} S_1 & S_{12} \\ S_{12}^T & S_2 \end{bmatrix}
\]

(A7.1)

Using (7.1) Eq. (A10) equals

\[
S_{12}^T \Phi P_{12} - S_{12}^T \Gamma LP_2 + S_2 FP_2 + S_2 KCP_{12} = 0.
\]

(A9)

From \( S_{12}^{-1}(A9) P_{12}^{-1} \), (A7.6) and (A7.8) we obtain (7.1). Equation (8.2) equals

\[
S_{12}^T \Phi P_{12} - S_{12}^T \Gamma LP_2 + S_2 FP_2 + S_2 KCP_{12} = 0.
\]

(A10)

Using (7.1) Eq. (A10) equals

\[
S_2 K (CPCT + W) + S_{12}^T \Phi PC^T = 0.
\]

(A11)

If the compensator is minimal, \( P_2 > 0, S_2 > 0 \) \[9,13]\]. Define the \( n \times n \) non-negative definite matrices

\[
P = P_1 - P_{12} P_{12}^{-1} P_{12}^T, \quad (A7.2)
\]

\[
\hat{P} = P_{12} P_{12}^{-1} P_{12}^T, \quad (A7.3)
\]

\[
S - S_1 - S_{12} S_{12}^{-1} S_{12}^T, \quad (A7.4)
\]

Using (A7.6) from (A13) \( P_{12}^{-1} \) we obtain (7.3).
A3. The Strengthened Optimal Projection Equations

Expanding Eq. (A6.4) using (7.1)-(7.3) yields the following three equations:

\[ \Phi^T S \Phi + (\Phi - K \rho C)^T \hat{S}(\Phi - K \rho C) + Q = S_1, \]  
(A14)

\[ \left[ (\Phi - K \rho C)^T \hat{S}(\Phi - K \rho C) + \Phi^T S \Gamma L_S \right] G^T = -S_{12}, \]  
(A15)

\[ G \left[ (\Phi - K \rho C)^T \hat{S}(\Phi - K \rho C) + L_S^T (R + \Gamma \Gamma L_S) \right] G^T = S_2. \]  
(A16)

Expanding Eq. (A6.5) using (7.1)-(7.3) yields the following three equations:

\[ \Phi \hat{P} \Phi^T + (\Phi - \Gamma L_S) \hat{P}(\Phi - \Gamma L_S)^T + V = P_1, \]  
(A17)

\[ \left[ (\Phi - \Gamma L_S) \hat{P}(\Phi - \Gamma L_S)^T + \Phi \rho C \Gamma \hat{K}_p \right] H^T = P_{12}, \]  
(A18)

\[ H \left[ (\Phi - \Gamma L_S) \hat{P}(\Phi - \Gamma L_S)^T + K \rho (\Gamma \Gamma L_S + \Gamma \Gamma) \hat{K}_p \right] H^T = P_2. \]  
(A19)

Then from \( G(A15) S_2^{-1} - (A15) S_2^{-1} \), or alternatively \( H(A18) P_2^{-1} - (A18) P_2^{-1} \) we obtain

\[ HG^T = GH^T = I_n. \]  
(A20)

From (A20) and (A7.9),

\[ \tau^2 = \tau; \]  
(A21)

so (A20) defines an oblique projection. From (A7) and (A21) we obtain the following properties:

\[ \hat{P} \hat{S} = -P_{12} G H^T S_{12}^T = -P_{12} S_{12}^T, \]  
(A22.1)

\[ \hat{P} = \tau \hat{P} = \hat{P} \tau = \tau \hat{P} \tau, \]  
(A22.2)

\[ P_{12} = \hat{P} H^T. \]  
(A22.3)

From (A7) and (A22) we obtain

\[ \text{rank}(G) = \text{rank}(M) = \text{rank}(H) \]
\[ = \text{rank}(\tau) = \text{rank}(\hat{P}) = \text{rank}(\hat{S}) \]
\[ = \text{rank}(\hat{P} \hat{S}) = \text{rank}(P_{12}) = \text{rank}(P_2) \]
\[ = \text{rank}(S_{12}) = \text{rank}(S_2). \]  
(A23)

Now, as indicated in Section A2 after Eq. (A7), given (A7), (A20), (A21) and (A23) we may identify \( P, S, \hat{P}, S, G, H \) and \( \tau \) with those in Theorem 2 and \( \hat{P}, \hat{S}, G, M, H \) and \( \tau \) with those in Lemma 1.

Now \( \frac{1}{2}((A16)H + ((A16)H)^T) \) equals (8.4) and \( (A15) + H^T G (A16) H - (A16) H - (A16) H^T \) equals (8.2). Similarly \( \frac{1}{2}((A18)G + ((A18)G)^T) \) equals (8.3) and \( (A17) + G^T (A18) G - (A18) G - (A18) G^T \) equals (8.1).

Summarizing, the conditions (7) and (8) in Theorem 2 are implied by (A6) and the minimality of the compensator.

A4. Equivalence

To prove the equivalence of (A6) and (7), (8) we now reverse the derivation. After substitution of (7.1)-(7.3) in (A9), (A10) and (A12), or equivalently (A6.1)-(A6.3), these relations still hold, so (7.1)-(7.3) are equivalent to (A6.1)-(A6.3). As a result (A14)-(A19) are equivalent to (A6.4) and (A6.5). Observe that (8.5) and (8.6) imply (A22.2) and (A22.5).

From (8.3), (8.4) and (A22.2), (A22.5) we have

\[ \Psi^2_{P,S,S} = \tau^2 \Psi^2_{P,S,S} = \tau^2 \Psi^2_{P,S,S} \]  
(A24)

\[ \Psi^1_{P,P,P} = \tau \Psi^1_{P,P,P} = \tau \Psi^1_{P,P,P} \]  
(A25)

Using (A24), (A25) and (8.7) we have that (8.2) + (8.4) equals (A14), (7.4)G^T equals (A15) and G(8.4)G^T equals (A16). Similarly (8.1) + (8.3) equals (A17), (8.3)H^T equals (A18) and H(8.3)H^T equals (A19).

Summarizing, (7) and (8) imply (A6). Finally from (A7.3) and (A7.5) observe that the rank condition in
(8.5) implies \( P_2 > 0, S_2 > 0 \), i.e. the invertibility of the second-moment matrix of the compensator and its dual. This implies the minimality of the compensator \([13]\).

Note that recovering Eqs (A14)-(A19) from the conventional optimal projection equations, i.e. (8.3) and (8.4) replaced by

\[
\hat{P} = \tau \Psi_{s,p,p}^{T},
\]

(A26)

\[
\hat{S} = \tau^T \Psi_{s,s,s}^{T}
\]

(A27)

is not possible because (A26) and (A27) do not imply (A24) and (A25). Therefore the conventional optimal projection equations are only implied by the first-order necessary optimality conditions (A6), they are not equivalent to (A6). Numerical evidence of this fact was presented in Section 6. Equations (A24) and (A25) might suggest that (8.3) and (8.4) may be replaced by e.g.

\[
\hat{P} = \tau \Psi_{s,p,p}^{T}.
\]

(A28)

\[
\hat{S} = \Psi_{s,s,s}^{T}.
\]

(A29)

Replacing (8.3) and (8.4) by (A28) and (A29) in the algorithms revealed that nonnegative solutions which satisfy (8.1), (8.2) and (A28), (A29) exist which do not satisfy (A22.2), (A22.5) and, as a result, (A24) and (A25), because \( \hat{P} \) and/or \( \hat{S} \) are not symmetric. From (A7.3) and (A7.5) note that \( \hat{P} \) and \( \hat{S} \) are symmetric by definition. This symmetry and the crucial equalities (A24) and (A25) are both implied by (8.3) and (8.4). If, as in Theorem 2, the symmetry of \( \hat{P} \) and \( \hat{S} \) is presumed then (A28) and (A29) may replace (8.3) and (8.4). Without this presumption from (A24) and (A25) observe that (8.3) and (8.4) may also be replaced by

\[
\hat{P} = \tau \Psi_{s,p,p}^{T} - \tau_1 \Psi_{s,p,p}^{T},
\]

(A30)

\[
\hat{S} = \Psi_{s,s,s}^{T} - \tau_1 \Psi_{s,s,s}^{T}.
\]

(A31)

However, replacing (8.3) and (8.4) by (A30) and (A31) in Algorithms 1 and 2 results in instability of the algorithms. Therefore (8.3) and (8.4) have been mentioned in Theorem 2.

A5. Explicit Expressions for the Minimum Costs

From Eqs (A6.4) and (A6.5) we obtain the following equalities:

\[
\text{tr}[P'S'] = \text{tr}[Q'P' + \Phi'\Phi'S']
\]

(A32)

\[
\text{tr}[P'S'] = \text{tr}[V'S' + \Phi'\Phi'S']
\]

(A33)

From (A32), (A33) and (A4.1) we obtain

\[
\sigma_\infty = \text{tr}[Q'P'] = \text{tr}[V'S']
\]

(A34)

which is equivalent to Eq (9).

Appendix 2: Generation of Random LQG Problems

The following MATLAB m file was developed to generate the random LQG problems in Computer Experiments 1 and 2 with MATLAB version 4.2c2.

% DRLQG.M: Discrete-time random LQG problem generation.
% function [p,g,c,v,w,q,r] = drlqg(nx,nu,ny,srp,seed);
% % Input:
% \% nx: dimension x
% \% nu: dimension u
% \% ny: dimension y
% \% srp: optional, desired spectral radius p
% \% seed: optional, seed for random number generator
% % Output:
% \% p,g,c,v,w,q,r: LQG problem parameters
% % L.G. Van Willigenburg, W.L. De Koning, 28-11-95.
% if nargin == 5; seed = []; end;
% if max(size(seed)) == 0; rand(seed, seed); end;
% [p,g,c] = drmodel(nx,ny,nu);
% if max(size(srp)) == 0 & srp > 0; p = srp*p / sperad(p); end;
% v = diag(rand(nx,1)); w = diag(rand(ny,1));
% q = diag(rand(nx,1)); r = diag(rand(nu,1));
% p = round(1e4*p)/1e4; g = round(1e4*g)/1e4;
% c = round(1e4*c)/1e4;
% v = round(1e4*v)/1e4; w = round(1e4*w)/1e4;
% q = round(1e4*q)/1e4; r = round(1e4*r)/1e4;

The m file uses the m file drmodel.m from the MATLAB Control Toolbox, which generates random discrete-time stable systems. Note that this function
sometimes generates systems for which the input or output matrix is the zero matrix. These systems were excluded from the experiments. In Computer Experiment 1 seed = 0, 1, 2, ..., 99 was used. In Computer Experiment 2 seed = 0, 1, 2, ..., 19 was used.

To reproduce the examples the random number generator function rand must produce the same values for equal seeds. To verify this here is the outcome of

\[ p, g, c, v, w, q, r \text{ for } n_x = 1, n_y = 1, s_r p = 1, \text{seed} = 1: \]

\[ p = -1, \quad g = 0.7092, \quad c = 0.1160, \quad v = 0.4714, \quad w = 0.1449, \quad q = 0.7178, \quad r = 0.6617. \]