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DIGITAL OPTIMAL CONTROL AND LQG COMPENSATION OF ASYNCHRONOUS AND APERIODICALLY SAMPLED NON-LINEAR SYSTEMS

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Abstract

Because of system and controller constraints, frequent, synchronous and periodic sampling is undesirable or impossible in many practical applications. In case of asynchronous and aperiodic sampling the frequent, synchronous and periodic updating of controls and observations is no longer required as opposed to conventional digital control. In case of asynchronous and aperiodic sampling an *arbitrary* number of control variables is updated and an *arbitrary* number of outputs is sampled at *arbitrary* time instants. This sampling scheme generalizes most of the deterministic sampling schemes considered in the control literature. The design of digital controllers for non-linear systems may be considered at two levels. At the highest level a deterministic non-linear digital optimal control problem is solved. The computation of numerical solutions to these problems, in case of asynchronous and aperiodic sampling is the topic of this paper. At the second level a digital LQG compensator is designed to compensate on-line for errors during the actual control. Recently we solved the digital LQG compensation problem in case of asynchronous and aperiodic sampling and showed how to numerically compute the solution. Together with the results presented here this allows for the design and computation of asynchronous and aperiodically sampled non-linear digital optimal control systems.

1 Introduction

In many chemical, economical and mechanical processes, it is undesirable or impossible to synchronously update measurements at the desired control rate. It may also be undesirable or impossible to synchronously update all control variables. Reasons for this are intensive analyses and costs associated to measuring, costs associated to updating the controls and the locally distributed nature of the system. Furthermore industrial control equipment often uses one D/A and A/D converter to convert several digitally coded control variables and analog measurements respectively. Using a multiplexer these control variables can only be updated sequentially and kept constant using zero order hold circuits. In the same manner the observations can only be updated sequentially there digital values being stored in the computer. Only if for each control variable and each analog output a separate D/A and A/D converter is used synchronous sampling can actually be realized. Especially in fast systems, such as mechanical ones, the time which elapses between the updating of different controls and observations cannot be neglected.

Given these practical constraints, in general, the updating of an arbitrary number of control variables as well as the sampling of an arbitrary number of outputs may occur at arbitrary time instants. We will refer to this as asynchronous and aperiodic sampling. This notion, we believe, generalizes most of the deterministic sampling schemes that have been considered in the control literature. It for instance generalizes conventional sampling, multi-rate sampling, non-synchronous sampling and multiple-order sampling, sampling schemes considered by Kalman and Bertram in their theory of sampling systems [1].

In case of synchronous sampling a characteristic feature of digital LQ and LQG problems is that they can be transformed into equivalent discrete-time optimal control problems [2-6]. In the *equivalent* discrete-time problem the continuous-time (inter-sample) system behavior is explicitly accounted for thus eliminating the requirement of small sampling intervals. Recently Van Willigenburg and De Koning [7] showed that transformation of digital LQG problems into equivalent discrete-time problems is also possible in case of asynchronous sampling. In this paper we demonstrate that this also holds for deterministic non-linear digital optimal control problems. Furthermore we show how the equivalent discrete-time problem formulation allows for the application of dynamic optimization algorithms originally designed to solve conventional continuous-time and discrete-time optimal control problems and how these algorithms should be adapted. Together with [7-8] this constitutes a framework for the design and computation of asynchronous and aperiodically sampled digital optimal control systems [9].

2 Digital optimal control problems and their discrete-time equivalents in case of asynchronous and aperiodic sampling

Consider the deterministic non-linear system,

$$\dot{x}=f(x,u,t),\ x\in R^n, u\in R^m, \quad (1.1)$$

$$x(t_0)=x_0. \quad (1.2)$$

Asynchronous and aperiodic sampling is described in [7] where control and observation instants are distinguished. Since we consider a deterministic digital optimal control problem we will only be concerned with the set of control instants,

$$T_u=\{t_c,\ c=0,1,2,..,C-1,\ t_c>t_{c-1}\}, \quad (1.3)$$

At each control instant t_c, $c=0,1,2,..C-1$ one, several or all m control variables are updated while the others remain unchanged. The a-priori known sets U_c, $c=0,1,2,..,C-1$ describe which control variables are updated at each control instant. They contain the m_c indices, $1\le m_c\le m$, of updated control variables at t_{u_c} i.e.,

$$card(U_c)=m_c,\ 1\le m_c\le m,$$
$$i\in U_c\Rightarrow u_{c_i} \text{ is updated at } t_c,$$
$$i=1,2,..,m,\ c=0,1,..C-1. \quad (1.4)$$

In accordance with (1.3) t_C is the final time involved in the digital optimal control problem which satisfies,

$$t_C>t_{C-1}. \quad (1.5)$$

After each control instant all control variables remain unchanged until the next control instant through the use of zero-order hold circuits,

$$u(t)=u(t_c),\ t\in[t_c,t_{c+1}),\ c=0,1,2,..C-1. \quad (1.6)$$

Given the system (1), the objective is to minimize the costfunctional,

$$J(x(t_0),u(t))=\Phi(x(t_C),t_C)+\int_{t_0}^{t_C} L(x(t),u(t),t)dt. \quad (2)$$

If, for the moment, we consider *all* control variables to be updated at the control instants t_c the digital optimal control problem (1), (2) can be transformed in the following equivalent discrete-time optimal control problem [10],

$$x_{c+1}=f_c(x_c,u_c),\ x_c=x(t_c)\in R^n,$$
$$u_c=u(t_c)\in R^m,\ c=0,1,2,..,C-1, \quad (3.1)$$

$$J(u_c,x_0)=\Phi(x_C)+\sum_{c=0}^{c=C-1} L_c(u_c,x_c), \quad (3.2)$$

where,

$$f_c(x_c,u_c)=x_c+\int_{t_c}^{t_{c+1}} f(x(t),u_c,t)dt, \quad (3.3)$$

$$L_c(u_c,x_c)=\int_{t_c}^{t_{c+1}} L(x(t),u_c,t)dt. \quad (3.4)$$

In equation (3.3,3.4) $x(t)$, $t\in[t_c,t_{c+1})$ is the solution of the system equation (1) with $x(t_c)=x_c$ and $u(t_c)=u_c$. Note that although in general analytic expressions for $f_c(x_c,u_c)$ and $L_c(x_c,u_c)$ are not available they can be computed numerically for any x_c and u_c through simultaneous numerical integration.

In the equivalent discrete-time problem (3) u_c appears as the control. From equation (1.4) observe that a problem formulation is required in which only the updated control variables appear as the control. Therefore we rearrange the control variables u_c into u_c' which separates into a first part u_c^u containing the updated control variables and a second part u_c^o containing the unchanged control variables,

$$u_c'=\begin{bmatrix} u_c^u \\ u_c^o \end{bmatrix}. \quad (4.1)$$

For each control instant t_c, $c=0,1,..,C-1$ this rearrangement is defined by two one to one mappings $U_{u_c}(.)$ and $U_{o_c}(.)$,

$$U_{u_c}(i)=j,\ i\in\{1,2,..,m_c\},$$
$$j\in U_c,\ c=0,1,2,..,C-1, \quad (4.2)$$

indicates that the updated control variable $u_{c_i}^u$ corresponds to u_{c_j} and,

$$U_{o_c}(i)=j,\ i\in\{1,2,..,m-m_c\},$$
$$j\in U\backslash U_c,\ c=0,1,2,..,C-1, \quad (4.3)$$

indicates that the unchanged control variable $u_{c_i}^o$ corresponds to u_{c_j}. The set U_c defined by (1.4) contains the m_c indices of updated control variables at t_c and the set $U\backslash U_c$ contains the indices of the unchanged control variables at t_c with,

$$U=\{1,2,..,m\}. \quad (4.4)$$

From (4) we proceed to obtain the equivalent discrete-time problem formulation which contains the actual control u_c^u, given by (4). Through augmentation of the state x_c, $c=0,1,..,C-1$ with the unchanged control variables u_c^o we are able to describe their influence properly. The augmented equivalent discrete-time system becomes,

$$x_{c+1}^a=f_c^a(x_c^a,u_c^u),\ c=0,1,2,..,C-1, \quad (5.1)$$

$$x_c^a=\begin{bmatrix} x_c \\ u_c^o \end{bmatrix},\ x_c^a\in R^{(n+m-m_c)}, \quad (5.2)$$

and the equivalent discrete time costfunction becomes,

$$L_c(u_c^u,x_c^a)=\int_{t_c}^{t_{c+1}} L(x^a(t),u_c^u,t)dt, \quad (5.3)$$

$$x^a(t)=\begin{bmatrix} x(t)\\ u^o_c\end{bmatrix},\quad x^a\in R^{(n+m-m_c)},\quad t\in[t_c,t_{c+1}). \tag{5.4}$$

Note that the dimension of the equivalent discrete-time system *(5.1,5.2)* changes with time. From equation *(5.1,5.2)* observe that,

$$x^a_{c+1}=\begin{bmatrix} x_{c+1}\\ u^o_{c+1}\end{bmatrix}=f^a_c(x^a_c,u^u_c)=\begin{bmatrix} f^a_{c_1}(x_c,u^o_c,u^u_c)\\ f^a_{c_2}(u^o_c,u^u_c)\end{bmatrix}. \tag{6.1}$$

The functions $f^a_{c_1}$, $c=0,1,2,..,C-1$ in *(6.1)* describe the transitions of the original state x_c and are therefore equivalent to the functions f_c in equation *(3.3)* for corresponding values of u_c and u^u_c,u^o_c related by equation *(4)*. In equation *(6.1)* the functions $f^a_{c_2}$ describe the updating of the unchanged control variables u^o_c, $c=0,1,2,..,C-1$ at each sampling instant. From equation *(4)* we obtain,

$$f^a_{c_2}=\begin{bmatrix} I^1_c & I^2_c\end{bmatrix}\begin{bmatrix} u^u_c\\ u^o_c\end{bmatrix},$$

$$I^1_c\in R^{(m-m_{c+1})\times m_c},\quad I^2_c\in R^{(m-m_{c+1})\times(m-m_c)}, \tag{6.2}$$

$$I^1_{c_{ij}}=1 \text{ if } U_{o_{c+1}}(i)=U_{u_c}(j) \text{ else } I^1_{c_{ij}}=0, \tag{6.3}$$

$$I^2_{c_{ij}}=1 \text{ if } U_{o_{c+1}}(i)=U_{o_c}(j) \text{ else } I^2_{c_{ij}}=0. \tag{6.4}$$

3 Numerical solution of the equivalent discrete-time problems in case of asynchronous and aperiodic sampling

Necessary conditions for the solution of a general time-varying discrete-time optimal control problem are derived in [11]. The derivation is easily seen to hold also for systems with time-varying dimensions. Therefore the necessary conditions for the solution of the equivalent discrete-time problem *(5)* are,

$$x^a_{c+1}=f^a_c(x^a_c,u^u_c),$$
$$x^a_c\in R^{(n+m-m_c)},\quad c=0,1,2,..,C-1, \tag{7.1}$$

$$\lambda^a_c=(\partial f^a_c/\partial x^a_c)^T\lambda^a_{c+1}+\partial L_c/\partial x^a_c,$$
$$\lambda^a_c\in R^{(n+m-m_c)},\quad c=0,1,2,..,C-1, \tag{7.2}$$

$$u^u_c=\min_{v_c}\left(L_c(x^a_c,v_c)+\lambda^{a^T}_{c+1}f^a_c(x^a_c,v_c)\right),$$
$$v_c,u^u_c\in R^{m_c},\quad c=0,1,2,..,C-1. \tag{7.3}$$

$$x^a_0\in R^{(n+m-m_0)},\quad \lambda^a_C=\partial\Phi/\partial x^a_C,$$
$$x^a_C=x_C\in R^n,\ \lambda^a_C=\lambda_C\in R^n. \tag{7.4}$$

From equation *(5.2)* and since complete knowledge of the initial state x^a_0 is required the values of the unchanged control variables at the initial control instant t_0 must be known. In case control constraints are involved v_c in *(7.3)* is restricted to belong to the set Ω_c that represents the control constraints during each interval $[t_c,t_{c+1})$, $c=0,1,2,..,C-1$. In *(7.1-7.4)*, which constitutes a two point boundary value problem, $\lambda^a_c\in R^{(n+m-m_c)}$, $c=0,1,2,..,C$ are Lagrange multipliers associated to the constraints *(2.1)*. With respect to *(7.4)* note that the controls beyond the final time t_C play no part in the problem so state augmentation of the final state is unnecessary.

For our numerical examples we have chosen a conjugate gradient algorithm because this type of algorithm does not pose additional constraints on the optimal control problem and the initial guess, while it is reported to converge reasonably fast [12-14]. Furthermore during each iteration it does not require the solution of *(7.3)*. Since no analytic expressions for the functions L_c and f^a_c are available such a solution has to be computed numerically. This computation may fail and is computationally expensive since it generally requires many function evaluations of both L_c and f^a_c at each control instant which constitute numerical integrations. Instead the conjugate gradient algorithm uses the derivative with respect to the control of the right hand argument of equation *(7.3)* to compute an improved control. This derivative is given by,

$$(\partial f^a_c/\partial u^u_c)^T\lambda^a_{c+1}+\partial L_c/\partial u^u_c,\quad c=0,1,2,...,C-1. \tag{8}$$

Because analytic expressions for f^a_c and L_c are not available the derivatives $\partial f^a_c/\partial u^u_c$ and $\partial L_c/\partial u^u_c$ are computed through numerical integration and numerical differentiation. The same holds for the right hand side of equation *(7.2)*.

Close examination of the algorithm and equations *(3)-(8)* reveals that the algorithm may also be stated in terms of the functions f_c and L_c in equation *(3)*, the original state x_c, the updated and non-updated control variables u^u_c, u^o_c which are determined by equation *(4)*, I^1_c, I^2_c in equation *(6.2-6.4)* which are also determined by equation *(4)* and finally λ^a_c. To see this from equation *(6)* observe that

$$\partial f^a_c/\partial x^a_c=\begin{bmatrix}\partial f_c/\partial x_c & \partial f_c/\partial u^o_c\\ 0 & I^2_c\end{bmatrix}, \tag{9.1}$$

$$\partial f^a_c/\partial u^u_c=\begin{bmatrix}\partial f_c/\partial u^u_c\\ I^1_c\end{bmatrix}, \tag{9.2}$$

$$\partial L_c/\partial x^a_c=\begin{bmatrix}\partial L_c/\partial x_c\\ \partial L_c/\partial u^o_c\end{bmatrix}. \tag{9.3}$$

Since no analytic expressions for f_c and L_c are available also in equation *(9)* the derivatives have to be computed through numerical integration and numerical differentiation. Our implementation of the algorithm was based on equation *(9)*.

4 Numerical examples

We consider two numerical examples. First we consider the LQ version of the asynchronous and aperiodically sampled digital LQG tracking example presented in [7] where a numerical solution was computed. For the problem data we refer to [7].

Table 1 compares the optimal controls and criterion values computed from both numerical algorithms. Also results obtained from a static optimization performed using the IMSL routine BCPOL are included. Static optimization is based on a mathematical programming formulation of the digital optimal control problem [10]. Both the static and dynamic optimization where initiated with all controls equal to zero.

As a final example the maneuvering of a fire truck is chosen since the fire truck constitutes a highly non-linear three-input system. For detailed information we refer to Tilbury and Chelouah [15] and Bushnell *et. al.* [16]. The dynamics of this system may be represented by the following model,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\Phi}_0 \\ \dot{\theta}_0 \\ \dot{\Phi}_1 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_0) \\ 0 \\ \dfrac{\tan(\Phi_0)}{l_0} \\ 0 \\ \dfrac{-\sin(\Phi_1-\theta_0+\theta_1)}{l_1\cos(\Phi_1)} \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \nu + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} z$$

$$l_0=1,\ l_1=3. \tag{10.1}$$

The three control variables are v, the linear velocity of the truck ν, the angular velocity of the front wheels of the truck and z, the angular velocity of the rear wheels of the trailer. The three controls therefore correspond to driving and steering. The objective is to maneuver the truck in 30 seconds from the initial state

$$x(0)=x(t_0)=x_0=[0,4,0,\pi/2,0,\pi/2]^T, \tag{10.2}$$

near the prescribed final state,

$$x(30)=x(t_C)=x_C=[0,0,0,0,0,0]^T, \tag{10.3}$$

while compromising on the magnitude of the trucks velocity and using a penalty function on the state variables Φ_0 and Φ_1 which represent the angles of the front and rear wheels respectively. These angles are assumed to be limited to $\pm\pi/4$. These objectives are described by the costfunction *(2.1)* if we take,

$$L(x(t),u(t),t)=1+\frac{(\Phi_0-\pi/4)(\Phi_0+\pi/4)}{\sqrt{(\Phi_0-\pi/4)^2(\Phi_0+\pi/4)^2+\varepsilon}}$$

$$+1+\frac{(\Phi_1-\pi/4)(\Phi_1+\pi/4)}{\sqrt{(\Phi_1-\pi/4)^2(\Phi_1+\pi/4)^2+\varepsilon}}+0.5v^2,$$

$$\varepsilon=0.003, \tag{10.4}$$

$$\Phi(x(t_C))=x_C^T G x_C,\quad G=\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix}. \tag{10.5}$$

In equation *(10.5)* two diagonal elements of G are zero indicating that the angles of the wheels at the final time are considered unimportant. The control updating is characterized by a matrix,

$$M=\begin{bmatrix} 0.0 & 4.8 & 13.5 & 18.4 & 24.0 \\ 2 & 2 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 & 3 \\ 3 & 1 & 3 & 3 & 2 \end{bmatrix}. \tag{10.6}$$

M is defined as follows. $M_{1c}=t_c\in T_u$, $M_{2c}=m_c$, $c=0,1,2,..C-1$. The remaining elements in each column of M equal $U_{u_c}(i)$, $i=1,..,m_c$, $U_{o_c}(i)$, $i=1,..,m-m_c$, $c=0,1,2,..C-1$ respectively. This for instance implies that at $t_1=4.8$ control variable 2 and 3 are updated (see also [7]). For the unchanged control variable at the initial time we have,

$$u_3(t_0)=0, \tag{10.7}$$

Table 2 compares results obtained from static and dynamic optimization. The total number of updated control variables, as can be seen from equation *(10.6)* is equal to ten, which for digital optimal control problems is small. Yet the advantage of dynamic optimization is apparent.

5 Conclusions

In case of asynchronous sampling the discrete-time equivalents of digital optimal control problems presented in this paper allow for the application of a wide range of *dynamic* optimization algorithms when adapted appropriately. As an example we treated the conjugate gradient algorithm but our results are easily seen to carry over to many other *dynamic* optimization algorithms [12]. The advantage over *static* optimization has been demonstrated by two numerical examples. The approach in general lacks analytic expressions for the equivalent discrete-time system and costfunction. These functions have to be evaluated through numerical integration. As a result several first derivatives of these functions, which play a major role in the algorithm, have to be computed through numerical differentiation. Clearly the combination of numerical integration and differentiation constitutes a source of numerical errors. The influence of these and other numerical errors on the suboptimality of numerical solutions might be an area of future research.

Compared to there continuous-time counter parts digital optimal control problems loose performance and do not allow for a free final time. The loss of performance depends on the sampling scheme. Therefore the following procedure is suggested. First solve the corresponding continuous-time problem. Based on the sampling scheme of the digital optimal control problem choose as the fixed final time the earliest control instant greater than the final time computed from the continuous-time problem. Using the results of this paper compute the solution of the digital optimal control problem where the initial guess is based on the solution of the continuous-time problem. Use the computed loss of performance with respect to the continuous-time problem to decide on the selection of the sampling scheme and the final time.

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Tables

Table 1.

	$u(t_0)$	$u(t_1)$	$u(t_2)$	$u(t_3)$	J	CT 1%
LQ:	1.8172e-1	2.9801e-1	2.9801e-1	1.0992e+0	3.3658	-
	0.0000e+0	1.5690e+0	1.0139e+0	1.0139e+0		
CG:	1.8298e-1	2.9900e-1	2.9900e-1	1.0964e+0	3.3656	2.19
	0.0000e+0	1.5671e+0	1.0121e+0	1.0121e+0		
SO:	1.8357e-1	3.0092e-1	3.0092e-1	1.0948e+0	3.3656	11.75
	0.0000e+0	1.5703e+0	1.0118e+0	1.0118e+0		

Minimum cost and optimal control computed from the LQ problem formulation (LQ), the conjugate gradient algorithm (CG) and static optimization (SO) and the computation time in seconds necessary to reach the minimum within 1% (CT).

Table 2.

	$u(t_0)$	$u(t_1)$	$u(t_2$	$u(t_3)$	$u(t_4)$
CG:	-1.0961e-1	-1.0961e-1	-1.0961e-1	-2.6720e-1	-2.7227e-1
	-1.3283e-1	1.9042E-2	2.1816e-1	4.1995e-3	4.1995e-3
	0.0000e+0	-3.8135e-2	-3.8135e-2	-7.8131e-3	1.5284e-3
SO:	-1.0874e-1	-1.0873e-1	-1.0873e-1	-2.4109e-1	-2.9537e-1
	-1.3374e+-1	2.1575e-2	2.1084e-1	6.6844e-3	6.6844e-3
	0.0000e+0	-3.8182e-2	-3.8182e-2	-6.3772e-3	-2.65028e-3

	J	CT 50%	CT 10%	CT 1%
CG:	1.4230	71.40	303.18	368.71
SO:	1.4208	205.86	422.65	454.84

Minimum cost and optimal control computed from the conjugate gradient algorithm (CG) and static optimization (SO) and the computation time in seconds necessary to reach the minimum within 50%, 10% and 1% (CT).