Proceedings of the European Control Conference 2007
Kos, Greece, July 2-5, 2007

A Kalman decomposition to detect temporal linear system structure

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Abstract—**Feedback controllers for non-linear systems are often based on a linearized dynamic model. Such a linearized model may be *temporarily* uncontrollable and/or unreconstructable. This paper introduces the so-called *differential* Kalman decomposition of time-varying linear systems. It is based on *differential* controllability and *differential* reconstructability in conjunction with a linear time-varying continuous-time system description that allows the system *structure* and *dimensions* to change at certain time-instants. We show how these together enable the *detection* of what will be called *temporal system structure*. This structure among other things reveals the temporal loss of controllability and/or reconstructability. Moreover this paper shows how time-varying state-dimensions enable a satisfactory realization theory for time-varying linear systems and how our Kalman decomposition is linked to the conventional ones.**

I. INTRODUCTION

A general approach to control non-linear systems is to compute an optimal control and state trajectory off-line using a non-linear systems model. To accommodate for disturbances the *linearised* dynamic model about these trajectories is used to design e.g. a linear quadratic perturbation feedback controller that operates on-line [1]. This approach depends critically on the controllability and reconstructability of the linearised dynamic model that is generally *time-varying*. If the systems model or the optimal control is not sufficiently smooth, e.g. if the control is *bang-bang*, *partly singular* or *digital*, the time-varying linearised dynamic model may be *temporarily* uncontrollable and/or unreconstructable [2]. This implies that over the associated time-intervals the feedback controller is partly ineffective and the system may become unstable. Therefore the *detection* of *temporal* uncontrollability and unreconstructability is highly important to control engineers.

Most of linear systems theory and control system design is concerned only with the properties controllability, reachability, reconstructability and observability. These properties can be detected from Kalman decompositions of the linear system. As demonstrated in this paper these decompositions do not detect the *temporal loss* of such properties that is associated with temporal changes in the system structure. In this paper continuous-time systems with variable structure and dimensions are introduced along with the so-called differential Kalman decomposition. Together these enable the detection of what is called temporal system structure. Among other things this structure reveals immediately the temporal loss of familiar system properties.

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II. ILLUSTRATIVE EXAMPLE

Example 1

Consider the following time-varying linear system,

$$\begin{aligned}\dot{x}(t) &= A(t)x(t)+B(t)u(t)\\ y(t) &= C(t)x(t),\ t\in[0,1]\end{aligned} \tag{1}$$

where

$$A(t)=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}, t\in[0,0.25], A(t)=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}, t\in(0.25,0.5]$$
$$A(t)=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}, t\in(0.5,0.75], A(t)=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}, t\in(0.75,1] \tag{2}$$
$$B(t)=\begin{bmatrix}1 & 0\end{bmatrix}^T, C(t)=\begin{bmatrix}0 & 1\end{bmatrix}, t\in[0,1]$$

If time in (1) would be restricted to $(0.25,0.5)$ the system would be unreachable as well as uncontrollable. Similarly if time in (1) would be restricted to $(0.5,0.75)$ the system would be unobservable and unreconstructable. If according to (1) $t\in[0,1]$ then we might call the system *temporarily* uncontrollable/unreachable over $(0.25,0.5)$ because the second state variable is not influenced by the input. Similarly we might call the system *temporarily* unreconstructable/unobservable over $(0.5,0.75)$ because the first state variable does not influence the output. Since moreover the second state variable is unstable over $(0.25,0.5)$ and since the first state variable is unstable over $(0.5,0.75)$ they both cannot be stabilized by a controller over these time intervals. If we apply a similarity

ISBN: 978-960-89028-5-5

transformation at every time $t \in [0,1]$ to the system description (1)-(2) then the facts stated above are unchanged but no longer obvious from the system description. In this paper we will demonstrate that these facts will not become obvious either after application of any of the four conventional Kalman decompositions [3]. However they do become obvious after application of our *differential* Kalman decomposition that retrieves a system description similar to (1)-(2).

III. Continuous-time systems with variable structure and dimensions

Quoting Kalman from [4]: "The only possibility of getting a reasonably well-rounded realization theory is to generalize the notion of a dynamical system in such a way that the *dimension* of the state-space is allowed to vary with time". Remarkably, except for [5], continuous-time systems with variable dimensions seem to have been ignored. A reason for this might be that a general description of time-varying dimensions and system structure requires the following system description that is uncommon,

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t), \;\; y(t) = C(t)x(t), \\ &x(t) \in R^{n_i}, \; u(t) \in R^{m_i}, \; y(t) \in R^{l_i}, \\ &t \in (t_i, t_{i+1}), \;\; i = 0,1,..,N-1, \\ &x(t_i^+) = A_i x(t_i^-), \;\; A_i \in R^{n_{i+1} \times n_i}, \; i = 1,2,..,N-1. \end{aligned} \quad (3)$$

In equation (3) x denotes the state, u the input that is assumed to be bounded and y the output. Over every separate open interval (t_i, t_{i+1}) equation (3) equals a conventional linear system description since $A(t)$, $B(t)$ and $C(t)$ have constant dimensions. Moreover the controllability and reconstructability matrices are assumed to have a constant rank. As we shall see later in this paper this implies that the structure of the system is constant. *Changes* of the system structure and dimensions may only occur at the time instants t_i, $i = 1,2,..,N-1$. At these time instants the additional system matrices A_i, $i = 1,2,..,N-1$ describe the state transitions from t_i^- to t_i^+ where the superscripts $-,+$ denote respectively the right and left time limit. In equation (3) $x(t_0^+)$ should be identified as the *initial state.* Similarly $x(t_N^-)$ should be identified as the *terminal state.* According to [6] we may call the system (3) a *piecewise constant rank system (PCR system)*. The class of piecewise constant rank systems is very broad and contains among others piecewise time-invariant and piecewise analytic systems [6]. A PCR system differs from the switched linear systems in [7], [8] because the time-instants t_i, $i = 1,2,..,N-1$ are *a-priori fixed* and the linear system over each time-interval (t_i, t_{i+1}) is *time-varying*. The time domain of a PCR system is denoted by T given by,

$$\mathrm{T} = \bigcup (t_i, t_{i+1}), \; i = 0,1,..,N-1 \quad (4)$$

Three types of PCR systems will be considered with different time domains. We consider PCR systems with $t_0 = -\infty$, $t_N = +\infty$, with $t_0 = 0$, $t_N = +\infty$ and with, $t_0 = 0$, $t_N < +\infty$. These are denoted by -+, 0+ and 0N PCR systems respectively. In all the definitions in this paper concerning -+, 0+ and 0N PCR systems time should always be considered *restricted* to the associated time domain of the PCR system.

In practice we often start from a continuous-time system description with constant dimensions defined over $[t_0, t_N]$ such as example 1. Example 1 is *almost everywhere* equivalent to (3) if we select $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, $t_4 = t_N = 1$, $A_1 = A_2 = A_3 = I$, where I denotes the identity matrix. In general we should select t_i, $i = 1,2,..,N-1$ as the times where the system description *changes* from one constant rank system to another and $A_i = I$, $i = 1,2,..,N-1$. To do this we need to be able to *detect* these changes.

Let $s \in (t_j, t_{j+1})$, $t \in (t_k, t_{k+1})$, $t > s$, $0 \le j \le k \le N$. Then *the state transition matrix* $\Phi(t,s)$ of a PCR system satisfies,

$$\begin{aligned} &\Phi(t,s) = \Phi'(t,s), \; k = j, \\ &\Phi(t,s) = \Phi'(t, t_{j+1}^+) A_{j+1} \Phi'(t_{j+1}^-, s), \; k = j+1, \\ &\Phi(t,s) = \Phi'(t, t_k^+) A_k ... \Phi'(t_{j+2}^-, t_{j+1}^+) A_{j+1} \Phi'(t_{j+1}^-, s), \\ &k > j+1. \end{aligned} \quad (5)$$

In equation (5) the state transition matrices Φ' on the right are conventional state transition matrices for continuous-time linear systems with constant dimensions. The Kalman decomposition computes and uses *similarity transformations* to decompose the system at any time. For a PCR system these are described by,

$$\begin{aligned} &x'(t) = T(t)x(t), \; T(t) \in R^{n_i \times n_i}, \; rank(T(t)) = n_i, \\ &A'(t) = T(t)A(t)T^{-1}(t) + \dot{T}(t)T^{-1}(t), \\ &B'(t) = T(t)B(t), \; C'(t) = C(t)T^{-1}(t), \; t \in (t_i, t_{i+1}), \\ &A_i' = T(t_i^+) A_i T^{-1}(t_i^-), \; i = 1,2,..,N-1. \end{aligned} \quad (6)$$

In equation (6) the prime denotes quantities associated with the system obtained *after* the similarity transformation $T(t)$, $t \in (t_i, t_{i+1})$, $i = 0,1,..,N-1$.

Definition 1

Two PCR systems are called *equivalent* if one system can be obtained from the other through a similarity transformation $T(t)$, $t \in \mathrm{T}$. Over each interval (t_i, t_{i+1}),

$i = 0,1,..,N-1$, $T(t)$ may be non-smooth at a finite number of isolated times. Every time non-smoothness occurs the interval (t_i, t_{i+1}) is split up

Definition 2

The *input-output map* $G_{t_0,t_N} : U_{t_0,t_N} \mapsto Y_{t_0,t_N}$ of a PCR system maps any input sequence $U_{t_0,t_N} = \{u(t), t_0 < t < t_N\}$ to the associated output sequence $Y_{t_0,t_N} = \{y(t), t_0 < t < t_N\}$ as determined by equation (3) and in addition the initial state $x(t_0^+)$ if the system is of the type 0+ or 0N. Two PCR systems are called *input-output equivalent* if they have identical input-output maps almost everywhere on (t_0, t_N).

Definition 3

A PCR system with dimensions $n(t)$ is called *minimal* if no PCR system with dimensions $n'(t)$ has the same input-output map almost everywhere on (t_0, t_N) and the property that $n'(t) \le n(t)$ almost everywhere on (t_0, t_N) and $n'(t) < n(t)$ over some open interval inside (t_0, t_N). If a PCR system is minimal it is called *a minimal realization* of its associated input-output map and of any other PCR system with the same input-output map almost everywhere on (t_0, t_N).

Definition 4

A PCR system is called *reachable at time* t if there exists an $s < t$ such that any state $x(s)$ can be transferred to any state $x(t)$ through an appropriate choice of the input $U_{s,t}$. A PCR system is called *differentially reachable (d-reachable) at time* t if s can be selected arbitrarily close to t. A PCR system (3) is called *controllable from time* s if there exists a $t > s$ such that any state $x(s)$ can be transferred to any state $x(t)$ through an appropriate choice of the input. A PCR system (3) is called *differentially controllable (d-controllable) from time* s if t can be selected arbitrarily close to s.

Definition 5

A PCR system (3) is called *observable at time* s if there exists a $t > s$ such that the state $x(s)$ can be determined from the output $Y_{s,t}$. A PCR system is called *d-observable at time* s if t can be selected arbitrarily close to s. A PCR system is called *reconstructable from time* t if there exists an $s < t$ such that $x(s)$ can be determined from the output $Y_{s,t}$. A PCR system is called *d-reconstructable from time* t if s can be selected arbitrarily close to t.

Definition 6

The *reachability/controllability grammian* $W_{s,t}$, $t > s$, of the PCR system is given by,

$$\frac{dW_{s,t}}{dt} = A(t)W_{s,t} + W_{s,t}A^T(t) + B(t)B^T(t),\ t,s \in \mathrm{T},\ t > s, \quad (7)$$

$$W_{s,t_i^+} = A_i W_{s,t_i^-} A_i^T,\ i = 1,2,..,N-1,\ s \in \mathrm{T},\ s < t_i^-,\ W_{s,s} = 0. \quad (8)$$

The transition rule (8) of $W_{s,t}$ from $t = t_i^-$ to $t = t_i^+$ equals the discrete-time rule without an input. This follows from the last line of equation (3). The *observability/ reconstructability grammian* $M_{s,t}$, $t > s$, of the PCR system is given by,

$$-\frac{dM_{s,t}}{ds} = A^T(s)M_{s,t} + M_{s,t}A(s) + C^T(s)C(s), t,s \in \mathrm{T}, t > s, \quad (9)$$

$$M_{t_i^-,t} = A_i^T M_{t_i^+,t} A_i,\ i = 1,2,..,N-1, t \in \mathrm{T}, t > t_i^+, M_{t,t} = 0. \quad (10)$$

Lemma 1

For a PCR system and $s,t \in \mathrm{T}, t > s$ the following equivalence holds: any state $x(s)$ can be transferred to any state $x(t)$ through an appropriate choice of the input $U_{s,t}$ $\Leftrightarrow$ $W_{s,t} > 0$. For a PCR system and $s,t \in \mathrm{T}, t > s$ the following equivalence holds: the state $x(s)$ can be recovered from the output $Y_{s,t}$ $\Leftrightarrow$ $M_{s,t} > 0$.

Proof

For continuous-time linear systems with constant dimensions the above lemma is well known. The discrete time transition rules at the times t_i, $i = 1,2,..,N-1$, do not change this because lemma 1 also holds in discrete-time, even if the dimensions of the discrete-time system are variable [7] □

According to definition 4 and lemma 1, $W_{s,t} > 0$ implies that the system is reachable at time t as well as controllable from time s. Therefore $W_{s,t}$ is called the reachability/controllability grammian in definition 6 and dually $M_{s,t}$ is called the observability/reconstructability grammian.

Lemma 2

Over each time-interval (t_i, t_{i+1}), $i = 0,1,..,N-1$, a PCR system is either d-reachable from/d-controllable at *any* time $t \in (t_i, t_{i+1})$ or at *no* time $t \in (t_i, t_{i+1})$. A dual result applies to d-observability/d-reconstructability.

Proof

Over each time-interval (t_i, t_{i+1}), $i = 0,1,..,N-1$, the situation is comparable to the one for conventional constant rank systems. Then the result follows from [6] □

IV. THE DIFFERENTIAL KALMAN DECOMPOSITION

Kalman decomposition are introduced and applied in this section to decompose PCR systems and detect *global* and *local* system structure.

Given the modifications and definitions presented in the previous section, procedures to compute the Kalman decomposition now also apply to PCR systems. These procedures use two grammians as an input [10], [11]. Then from [3] observe that *four different conventional* Kalman decompositions may be computed from either:

1) $W_{t_0,t}$, M_{t,t_N}, 2) $W_{t_0,t}$, $M_{t_0,t}$, 3) W_{t,t_N}, M_{t,t_N}, 4) W_{t,t_N}, $M_{t_0,t}$

Procedure 1) decomposes the system at time $t \in \mathrm{T}$ into states that are a) reachable at time t and unobservable at time t b) reachable at time t and observable at time t c) unreachable at time t and unobservable at time t d) unreachable at time t and observable at time t. Procedure 2) does the same as 1) with "observable at" replaced by "reconstructable from". Procedure 3) does the same as 1) with "reachable at" replaced by "controllable from". Procedure 4) does the same as 1) with "reachable at" replaced by "controllable from" and "observable at" by "reconstructable from". In each case the system structure is of the following form,

$$
\begin{aligned}
&x'(t) = \begin{bmatrix} x_a'^T(t) & x_b'^T(t) & x_c'^T(t) & x_d'^T(t) \end{bmatrix}^T, \\
&x_a'(t) \in R^{n_a},\; x_b'(t) \in R^{n_b},\; x_c'(t) \in R^{n_c},\; x_d'(t) \in R^{n_d}, \\
&A'(t) = \begin{bmatrix} A'_{aa}(t) & A'_{ab}(t) & A'_{ac}(t) & A'_{ad}(t) \\ 0 & A'_{bb}(t) & 0 & A'_{bd}(t) \\ 0 & 0 & A'_{cc}(t) & A'_{cd}(t) \\ 0 & 0 & 0 & A'_{dd}(t) \end{bmatrix}, \\
&B'(t) = \begin{bmatrix} B_a'^T(t) & B_b'^T(t) & 0 & 0 \end{bmatrix}^T, \\
&C'(t) = \begin{bmatrix} 0 & C_b'(t) & 0 & C_d'(t) \end{bmatrix},\; t \in (t_i, t_{i+1}), \\
&i = 0,1,..,N-1.
\end{aligned}
\tag{11}
$$

Theorem 1

1) Using our definition of PCR systems (3) the system decompositions (11) may be interpreted as decompositions into four *PCR sub-systems* defined over T having time-varying dimensions in general. Over each time interval (t_i, t_{i+1}), $i = 0,1,..,N-1$, the number of states n_i, n_a, n_b, n_c, n_d of the PCR system and sub-systems are constant.

2) The input-output map of a PCR system is solemnly determined by PCR sub-system b) generated by Kalman decomposition 1). This sub-system is a *minimal realization* of the PCR system if $x(t_0^+) = 0$. If $x(t_0^+) \neq 0$ the reachability grammian $W_{t_0^+,t}$ should be replaced by the so-called *weak* reachability grammian $W'_{t_0^+,t}$ that is also described by equations (7), (8) except for the initial condition $W'_{t_0^+,t_0^+} = x(t_0^+)x(t_0^+)^T$.

Proof

1) Conventional constant rank systems have constant dimensions, grammians with boundary conditions equal to zero and the property that the four Kalman decompositions produce sub-systems having equal and constant dimensions [6]. For PCR systems the boundary conditions of the grammians over each time interval (t_i, t_{i+1}), $i = 1,2,..,N-2$, are non-zero in general. They affect the dimensions of the sub-systems obtained from the four Kalman decompositions, which therefore can no longer be guaranteed equal. But they do *not* change the fact that these sub-systems have constant dimensions.

2) This follows from definitions 2 and 3 and application of the results in [6] over each separate interval (t_i, t_{i+1}), $i = 0,1,..,N-1$. For $x(t_0^+) \neq 0$ the result follows from [12] □

Now, to introduce the differential Kalman decomposition and its importance, reconsider example 1. Observe that, according to definitions 4 and 5, the system is reachable at *any* time t, controllable from *any* time t, observable at *any* time t and reconstructable from *any* time t. After applying a similarity transformation therefore, the four Kalman decompositions will generally *not* reproduce the system structure of example 1 for $t \in (0.25, 0.75)$. The following lemma that follows immediately from definitions 4, 5 and lemma 2 applies to the time-intervals $(0.25, 0.5)$ and $(0.5, 0.75)$ of example 1 respectively.

Lemma 3

A PCR system being d-unreachable over (t_i, t_{i+1}) is *equivalent* with the PCR system being d-uncontrollable over (t_i, t_{i+1}) for some $0 \le i \le N-1$. A dual result applies to d-unobservability and d-unreconstructability.

So what has been termed *temporal* unreachability/uncontrollability in section II is now formalized as being d-uncontrollable/d-unreachable over (t_i, t_{i+1}) for some $0 \le i \le N-1$. A dual result applies with respect to d-unobservability/d-unreconstructability. So now the question is: can we device a Kalman decomposition based on d-reachability/d-controllability and d-observability/d-reconstructability? This type of Kalman decomposition has actually already been presented in [6]. In this paper we call it the *differential Kalman decomposition*. In [6] this decomposition is defined only for conventional constant rank systems. According to lemma 2 these systems are either d-controllable *everywhere* on their time domain or *nowhere* [6]. Therefore they *cannot* be *temporarily* uncontrollable/unreachable. This reveals that the description and detection of temporal uncontrollability *requires PCR systems*!

Definition 7

The *d-reachability/d-controllability grammian* W_t of a PCR system at every time $t \in \mathrm{T}$ is given by,

$$\begin{aligned} &W_t = C_j(t)C_j^T(t), C_j(t) = \left[P_0(t) \quad P_1(t) \quad .. \quad P_j(t)\right], \\ &P_0(t) = B(t), P_{i+1}(t) = -A(t)P_i(t) + \dot{P}_i(t), \\ &i = 0,1,..,j-1, \end{aligned} \tag{12}$$

with j the smallest value for which $rank\left(C_{j+1}(t)\right) = rank\left(C_j(t)\right)$.Dually the *d-observability/d-reconstructability grammian* M_t of the PCR system at time t is given by,

$$\begin{aligned} &M_t = O_k^T(t)O_k(t), O_k = \left[S_0^T(t) \quad S_1^T(t) \quad .. \quad S_k^T(t)\right]^T, \\ &O_0(t) = C(t), O_{i+1}(t) = O_i(t)A(t) + \dot{O}_i(t), \\ &i = 0,1,..,k-1, \end{aligned} \tag{13}$$

with k the smallest value such that $rank\left(O_{k+1}(t)\right) = rank\left(O_k(t)\right)$.

Definition 8

The *differential* Kalman decomposition at very time t uses as an input the *d-reachability/d-controllability grammian* W_t and the *d-observability/d-reconstructability grammian* M_t of the PCR system.

Theorem 2

1) The differential Kalman decomposition at every time $t \in \mathrm{T}$ decomposes a PCR system according to (11) into states that are a) d-reachable/d-controllable and d-unobservable/d-unreconstructable at time t b) d-reachable /d-controllable and d-observable/d-reconstructable at time t c) d-unreachable/d-uncontrollable and d-unobservable/ d-unreconstructable at time t d) d-unreachable/d-uncontroll. and d-observable/d-reconstructable at time t.

2) Over each time interval (t_i, t_{i+1}), $i = 0,1,..,N-1$, the number of states n_a, n_b, n_c, n_d generated by the differential Kalman decomposition is constant. So a) b) c) and d) may be regarded as PCR *sub-systems* of the PCR system. Like the original PCR system they have constant dimensions over (t_i, t_{i+1}), $i = 0,1,..,N-1$.

Proof

Identical to the proof of lemma 2 □

From theorem 2 the differential Kalman decomposition *does* reproduce the system structure of example 1 after a similarity transformation has been applied. So the differential Kalman decomposition *is* able to detect d-unreachability/d-uncontrollability over a time-interval as well as d-unobservability/d-unreconstructability.

Theorem 3

The d-reachability/d-controllability grammian W_t of a PCR system at every time $t \in (t_i, t_{i+1})$ may be *interchanged* with the controllability/reachability grammian $W_{t_i^+,t}$ with "initial condition" $W_{t_i^+,t_i^+} = 0$ or "terminal condition" $W_{t_{i+1}^-,t_{i+1}^-} = 0$ to obtain Kalman decompositions with identical dimensions n_a, n_b, n_c, n_d. A dual result holds for the d-observability/d-reconstructability grammian M_t.

Proof

From the proof of lemma 2 conventional constant rank systems have the property that they are either d-reachable/d-controllable at *any* time or at *no* time [6]. But this implies that conventional constant rank systems are either "reachable at" as well as "controllable from" *any* time or *no* time. So for conventional constant rank systems interchanging the grammians does not affect the dimensions n_a, n_b, n_c, n_d. Over each time-interval (t_i, t_{i+1}), $i = 0,1,..,N-1$, a PCR system is comparable to a conventional constant rank system except for the possibly non-zero initial and terminal values of the grammians (see the proof of theorem 1). Therefore these initial and terminal values should be taken equal to zero □

Corollary 1

For a PCR system *five different* Kalman decompositions may be computed using the differential equations (7), (9). These five Kalman decompositions *only differ with respect to their boundary conditions at* t_i^+ *and* t_{i+1}^-. The four conventional Kalman decompositions [3] have the following generally *non-zero* boundary conditions respectively 1) $W_{t_i^+,t_0^+}$, $M_{t_{i+1}^-,t_N^-}$ 2) $W_{t_i^+,t_0^+}$, $M_{t_i^+,t_0^+}$, 3) $W_{t_{i+1}^-,t_N^-}$, $M_{t_{i+1}^-,t_N^-}$, 4) $W_{t_{i+1}^-,t_N^-}$, $M_{t_i^+,t_0^+}$. These boundary conditions satisfy the transition rules (8), (10). They transfer *global* reachability, controllability, observability and reconstructability properties from one open time-interval (t_i, t_{i+1}) to the next. Over (t_i, t_{i+1}) the dimensions n_a, n_b, n_c, n_d are constant but may be *different* for each of the four conventional Kalman decompositions.

The *differential* Kalman computes *local* d-reachability/d-controllability and d-observability/d-reconstructability properties. *Knowing* t_i, $i=1,2..,N-1$, it may be computed in the same manner from *zero* boundary conditions. Then the four decompositions become effectively one since they *do* produce identical values n_a, n_b, n_c, n_d over every interval (t_i, t_{i+1}), $i=0,1,..,N-1$.

Suppose we start from a system description over $[t_0, t_N]$ such as example 1 *not knowing* t_i, $i=1,2,..,N-1$ while these are *required* to arrive at the PCR system description (3). Then the *differential* Kalman decomposition computed at every time $t\in(t_0,t_N)$ from the d-reachability/d-controllability and d-observability/d-reconstructability grammians is the *only* Kalman decomposition capable of *detecting* t_i, $i=1,2,..,N-1$ as those isolated times where the *structure* changes, i.e. one or several of $n_a(t), n_b(t), n_c(t), n_d(t)$. The grammians require the d-controllability and d-reconstructability matrices $C_j(t)$, $O_k(t)$ in equations (12), (13). They in turn require knowledge of a sufficient number of derivatives of $A(t)$, $B(t)$ and $C(t)$. Now t_i, $i=1,2,..,N-1$, are also precisely those isolated times where formally the differential Kalman decomposition does not exist because some of these derivatives do not exist.

Example 2: Minimal realizations

Consider example 1 converted to fit the PCR system description (3), i.e. with $t_1=0.25$, $t_2=0.5$, $t_3=0.75$, $A_1=A_2=A_3=I$. One easily sees and computes $rank\left(W_{t_0^+,t}\right)=2$, $rank\left(M_{t,t_N^-}\right)=2$, $t\in(0,1)$. Therefore the system is minimal despite the temporal uncontrollability / unreachability and unreconstructability/unobservability. The associated states may *not* be dropped because e.g. over $(0.25,0.5)$ the second d-uncontrollable/d-unreachable state is non-zero in general because at $t=0.25^+$ it is non-zero in general. This information is *transferred* by $W_{0^+,0.25^+}$ that satisfies $rank\left(W_{0^+,0.25^+}\right)=2$.

Consider example 1 but with the system descriptions on the intervals $[0,0.25]$, $(0.25,0.5]$ swapped as well as those on the intervals $(0.5,0.75]$, $(0.75,1]$. One then easily computes $rank\left(W_{t_0^+,t}\right)=1$, $t\in(0,0.25)$, $rank\left(W_{t_0^+,t}\right)=2$, $t\in(0.25,1)$, $rank\left(M_{t,t_N^-}\right)=2$, $t\in(0,0.75)$, $rank\left(M_{t,t_N^-}\right)=1$, $t\in(0.75,1)$ and the associated minimal realization,

$$A(t)=1, B(t)=1, C(t)=0,\ t\in(0,0.25),$$

$$A(t)=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}, B(t)=\begin{bmatrix}1\\0\end{bmatrix}, C(t)=\begin{bmatrix}0\\1\end{bmatrix}^T, t\in(0.25,0.5),$$

$$A(t)=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}, B(t)=\begin{bmatrix}1\\0\end{bmatrix}, C(t)=\begin{bmatrix}0\\1\end{bmatrix}^T, t\in(0.5,0.75),$$

$$A(t)=1, B(t)=0, C(t)=1, t\in(0.75,1),$$

$$A_1=\begin{bmatrix}1 & 0\end{bmatrix}^T,\ A_2=I,\ A_3=\begin{bmatrix}1 & 0\end{bmatrix}. \quad (14)$$

Note that the minimal realization (14) has *time-varying state-dimensions* since over $(0,0.25)$ the second state is unreachable and over $(0.75,1)$ the first state is unobservable.

V. Conclusions

The introduction of PCR systems together with the differential Kalman decomposition enables the *description* and *detection* of *local, temporal system structure* of *time-varying* linear systems. This structure is associated with d-reachability/d-controllability and d-observability/d-reconstructability. It reveals the *temporal loss* of the associated *global* system properties reachability, controllability, observability and reconstructability. This is highly relevant to control engineers. Moreover this paper reveals that the differences between reachability versus controllability and observability versus reconstructability are entirely due to *changes* of the local system structure. Finally the time-varying PCR system dimensions enable the well rounded realization theory suggested by Kalman [4].

References

[1] M. Athans, 1971, "The role and use of the Linear- Quadratic-Gaussian problem in control system design", *IEEE Trans. Aut. Contr.*, 16, 529-552.

[2] L.G. Van Willigenburg, 1990, "First-order controllability and the time optimal control problem for rigid articulated arm robots with friction", *Int. J. Contr.*, 51, 6, pp. 1159-1171

[3] L. Weiss, R.E. Kalman, 1965, Contributions to linear systems theory, *Int. J. Eng. Sc.*, 3, pp. 141-171.

[4] R.E. Kalman, P.L. Falb, M. Arbib, 1969, *Topics in Mathematical Systems Theory*, Mc Graw Hill, New York.

[5] H. Sandberg, A. Rantzer, 2004, "Balanced truncation of linear time-varying systems", IEEE Trans. Aut. Contr., 49, 2, pp. 217-229.

[6] L.M. Silverman, H.E. Meadows, 1969, "Equivalent realizations of linear systems", *SIAM J. Appl. Math.*, 17, 2, pp.393-408.

[7] G. Xie, D.Zheng, L. Wang, 2002, "Controllability of switched linear systems", *IEEE Trans. Aut. Contr.*, 47, 8, pp. 1401-1405.

[8] G. Stikkel, J. Bokor, S. Szabo, 2004, Necessary and sufficient cond. of switching linear hybrid systems, *Automatica*, 40, 6, 1093-1098.

[9] I. Gohberg, M.A. Kaashoek, L. Lerer, 1992, "Minimality and realization of discrete time-varying systems", *Operator theory: Advances and applications*, 56, 261-296. Birkhauser.

[10] R.E. Kalman, 1963, "Mathematical description of linear systems", *SIAM J. Contr*, A, 1, 2, pp.152-192.

[11] D. Boley, 1984, "Computing the Kalman decomposition: An optimal method", *IEEE Trans. Aut. Contr.*, AC-29, 1, pp. 51-53.

[12] L.G. Van Willigenburg, W.L. De Koning, 2002, "Minimal and non-minimal optimal fixed-order compensators for time-varying discrete-time systems", *Automatica*, 38, pp.157-165.