Survey of adaptive dual control methods

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Abstract: A survey of adaptive dual control methods, elaborated from the early 1960s until the present, is given. The development of dual control methods is considered in chronological order, taking into account its close interconnection with general progress in adaptive control theory and applications. Detailed classifications of stochastic adaptive control methods and dual control methods are presented. The properties of a neutral control system and the nature of the dual effect in adaptive control systems are described. The historical stages of the development of the theory and applications of dual control are reviewed.

1 Introduction

Most adaptive controllers are based on the separation of parameter estimation and controller design. In such cases, the certainty equivalence (CE) approach is applied when the uncertainty of estimation is not taken into consideration for the controller design, and the parameter estimates are used in the control law as if they were the real values of the unknown parameters. This heuristic approach is very simple to implement, and has been used in many adaptive control schemes from the beginning of the development of adaptive control theory (in the mid 1950s) until recent times: Feldbaum [1–4], in his early works, considered the problem of optimal adaptive control and indicated that systems based on the CE approach are not always optimal, but can indeed be far from optimality. He postulated two main properties of the control signal of optimal adaptive systems. The control signal should ensure that (i) the system output *cautiously* tracks the desired reference value, and (ii) it *excites* the plant sufficiently to accelerate the parameter estimation process, so that the quality of adaptive controllers designed on the parameter estimation process can be improved considerably in future time intervals. Adaptive control systems showing these two properties were named *adaptive dual control systems*, and these properties were named *dual properties* (or *dual features*).

The formal solution of the optimal adaptive dual control problem in the formulation considered by Feldbaum [5] can be obtained using the method of dynamic programming. However, the equations cannot be solved either analytically or numerically because of the growing dimension of the underlying space, even for simple examples (exact solutions of very simple dual control problems can be found in Sternby [6], where a system with only a few possible states was considered). These difficulties in finding the optimal solution lead to the appearance of various simplified approaches, which can be divided into two large groups: those based on various approximations of the optimal adaptive dual control problem, and approaches

based on the reformulation of the problem to obtain a simple solution so that the system maintains its dual properties. These approaches have also been named implicit and explicit adaptive dual control methods. The main idea of these adaptive dual control methods lies in the attempt to design adaptive systems that would not be optimal, but would at least have the main dual features of optimal adaptive control systems. The adaptive control approaches, which are based on approximations of the stochastic dynamic programming equations, are usually complex and require large computational efforts in real-time mode, or they are based on rough approximations so that the system loses the dual features and the control quality remains insufficient [7–10]. The problem reformulation methods are more flexible. Before the elaboration of the bicriterial design method for adaptive dual control systems (see, for example, [11], the reformulated adaptive dual control problems were considered on the basis of a special cost function, which consists of two added parts: control losses and an uncertainty measure (the measure of the precision of parameter estimation) [12, 13]. These methods allow one to design simple dual controllers, and the computational complexity of the control algorithms can then be compared to those of the generally used CE controllers.

Two surveys of adaptive dual control methods are available [14, 15]. Taking into account the growing interest in adaptive dual control, a more detailed survey of dual control methods, including very recent results, will be given here. The development of adaptive dual control methods will be considered in chronological order, together with the general progress of the theory and application of adaptive control. The definitions of a dual control system and a neutral control system, as well as a new definition of an adaptive control system, will be given. The interconnections of adaptive dual control with the general development of adaptive control methods will be emphasised. For this purpose, a classification of not only adaptive dual control systems, but also of adaptive control systems will be given. The applications of adaptive dual control will be summarised.

2 Unsolvability of the optimal adaptive control problem

The formulation of the unsolvable optimal adaptive control problem was originally suggested by Feldbaum [1–5]. This problem will be described below in a more general form,

including a model with time-varying parameters and the state-space representation.

2.1 Statement of optimal adaptive control problem

Consider the system described by the following discrete-time equations of states, parameters and observation

$$x(k+1) = \bar{f}_k(x(k), p(k), u(k), \xi(k)),$$
$$k = 0, 1, \ldots, N-1 \qquad (1)$$

$$p(k+1) = v_k(p(k), \varepsilon(k)) \qquad (2)$$

$$y(k) = h_k(x(k), \eta(k)) \qquad (3)$$

where $x(k) \in \mathfrak{R}^{n_x}$ is the state vector, $p(k) \in \mathfrak{R}^{n_p}$ is the vector of unknown parameters, $u(k) \in \mathfrak{R}^{n_u}$ is the control input vector, $y(k) \in \mathfrak{R}^{n_y}$ is the observation vector (system output), and $\xi(k) \in \mathfrak{R}^{n_\xi}$, $\varepsilon(k) \in \mathfrak{R}^{n_\varepsilon}$ and $\eta(k) \in \mathfrak{R}^{n_\eta}$ are vectors of independent random sequences with zero mean and known probability distributions; $\bar{f}_k$, v_k and h_k are known simple vector functions. The function v_k describes the stochastic time-varying parameters of the system. The probability density for the initial values $p(x(0), p(0))$ is assumed to be known.

The set of observations and control values available at time k is denoted as

$$\mathfrak{I}_k = \{y(k), \ldots, y(0), u(k-1), \ldots, u(0)\},$$
$$k = 1, \ldots, N-1, \quad \mathfrak{I}_0 = \{y(0)\} \qquad (4)$$

The performance index for control optimisation has the form

$$J = E\left\{\sum_{k=0}^{N-1} g_{k+1}(x(k+1), u(k))\right\} \qquad (5)$$

where g_{k+1} are known positive convex scalar functions. The expectation is taken with respect to all random variables $x(0)$, $p(0)$, $\xi(k)$, $\varepsilon(k)$ and $\eta(k)$, $k = 0, 1, \ldots, N-1$, which act upon the system.

The problem of optimal adaptive control consists in finding the control policy $u(k) = u_k(\mathfrak{I}_k) \in \Omega_k$, $k = 0$, $1, \ldots, N-1$, which minimises the performance index of eqn. 5 for the system described by eqns. 1–3, where Ω_k is the domain in the space $\mathfrak{R}^{n_u}$ that defines the admissible control values.

2.2 Formal solution using stochastic dynamic programming

Backward recursion on the following stochastic dynamic programming equations can generate the optimal stochastic (dual) control sought for the above problem:

$$J_{N-1}^{CLO}(\mathfrak{I}_{N-1}) = \min_{u(N-1) \in \Omega_{N-1}} [E\{g_N(x(N), u(N-1)) | \mathfrak{I}_{N-1}\}]$$
$$\qquad (6)$$

$$J_k^{CLO}(\mathfrak{I}_k) = \min_{u(k) \in \Omega_k} [E\{g_{k+1}(x(k+1), u(k))$$
$$+ J_{k+1}^{CLO}(\mathfrak{I}_{k+1}) | \mathfrak{I}_k\}],$$
$$k = N-2, N-3, \ldots, 0 \qquad (7)$$

where the superscript CLO denotes 'closed-loop optimal' according to the terminology suggested by Bar-Shalom and Tse [7].

It is known that the analytical difficulties in finding simple recursive solutions from eqns. 6 and 7, and the

numerical difficulties due to the dimensionality of the underlying spaces make this problem practically unsolvable, even for simple cases [7, 8]. But a detailed investigation of this problem enabled the main dual properties of the control signal in optimal adaptive systems to be found and used for other formulations of the adaptive dual control problems, which led to the elaboration of design methods for adaptive dual controllers and, practically, to the solution of the adaptive dual control problem.

3 How to classify adaptive controllers

The sheer quantity of different adaptive control approaches presented in the literature makes a survey of this field a cumbersome and formidable task. Before beginning to classify adaptive controllers, it is natural to give a definition of adaptive controllers and to differentiate between adaptive and non-adaptive controllers. Several attempts to strictly define adaptive control systems have been manifested by Saridis [16] and Åström and Wittenmark [17]. Preferred definitions of adaptive control and adaptation are also cited in the book of Tsypkin [18]. However, up to now no satisfactory definition has been available, by which one could strictly separate adaptive systems from non-adaptive ones. The difficulties lie in the yet undetermined separation of various systems with nonlinear and time-varying control laws from adaptive control ones. We suggest the following definition, which summarises what control engineers usually understand by an adaptive control system.

An *adaptive control system* is defined as a control system operating under conditions of *uncertainty* of the controller, which provides the desired system performance and changes its *parameters* and/or *structure* in order to reduce the *uncertainty* and to improve the approximation of the *desired system*.

Therefore, the presented notion of an adaptive control system stems from the notions of uncertainty, a desired system, parameters, and structure. In this definition the terms comprise the following:

(i) *Uncertainty:* Unknown parameters and characteristics of plant, environment or unknown controller.

(ii) *Parameters:* The values that determine system components, or connections between the components. Engineers separate parameters from states of the system: the states of the system are to be directly controlled and they can change more rapidly than parameters.

(iii) *Structure:* The structure of the system is the aggregate of the system components and connections between them.

(iv) *Desired system:* The system, which would be synthesised in the case of no uncertainty. The goal of adaptation is to reduce the uncertainty and to give the possibility of synthesising this desired system. For example, in the case of convergence, the desired system will be obtained as a system with fixed parameters after finishing the adaptation.

This definition allows a separation of the adaptive systems from systems with time-varying, nonlinear and robust controllers, and controllers with changing structure, where the structure and parameters are known. The yardstick for an adaptive system is the following. The parameters and/or structure of a controller that provides the desired performance for a given unknown plant are unknown, and a superimposed adaptive system tries to find these parameters and/or the structure of the controller during operation in real-time mode.

Therefore, the main goal of adaptation is to finish and switch off the adaptation as soon as possible, and to use the adjusted controller with a fixed structure and fixed parameters henceforth. The adaptive system aims at on-line identification and correction of the structure and the parameters of a standard control loop, and switching off the adaptation loop once convergence has occurred. However, this goal can be achieved only in cases of bounded variation of the plant or environment; otherwise the adaptation cannot be finished and the desired controller will be sought continuously, or the adaptation will be restarted after every change in the operating conditions. As pointed out by Unbehauen [19], there exist two kinds of adaptive control systems: (i) the systems in which one should apply the adaptation only once to reduce the *a priori* uncertainty, and (ii) the systems in which the parameters or structure is changing and the adaptation must be realised constantly. It should be noted that adaptive dual control allows one to accelerate the adaptation and to finish it earlier. The presented definition of an adaptive control system does not contradict the known and celebrated ones [16–18]. These also cover self-organising systems, which can be considered as complex cases of adaptive control systems. The introduction of this new definition of an adaptive control system, with the clearly indicated goal of adaptation, allows us to introduce a new cost functional for the deviation of the system output from an unknown nominal output of the system. This cost functional can be used to modify many control systems with direct and indirect adaptation by adding a dual control system component.

It should be noted that dual control systems can be adaptive as well as non-adaptive. For example, dual effects can be viewed in stochastic nonlinear systems, where the uncertainty consists of the inaccuracy of state estimation [7]. The definition of a dual control system is as follows.

A *dual control system* is a control system that operates in conditions of *uncertainty*, and where the control signal has the following properties: (i) it cautiously follows the control goal, and (ii) it excites the plant to improve the estimation.

The meaning of "cautiously follows the control goal" has been explained well, for example, in [7]. It means that in the case of uncertain parameters of the system, the control signal should be smaller (cautious) than the control signal in the system with known parameters and after adaptation.

The structures of a standard adaptive control system and adaptive dual control system are portrayed in Figs. 1–4. In the indicated structures (Figs. 1 and 3), the non-confined form of the controller block emphasises the goal of adaptation, which consists in determining the unknown parameters of the controller. The adaptive system tries to determine the controller parameters during the operation in real-time mode, whereas the adaptive dual control system realises this actively by means of optimal excitations added to the cautious control action. It should also be noted that the covariance matrix of the estimation error is transferred from the estimator to the controller design stage in the adaptive dual control systems (Figs. 3 and 4).

To summarise the above presented properties of dual control systems, the complete schemes of the conventional adaptive control system and adaptive dual control system are also shown in Figs. 2 and 4. Transferring the accuracy of the parameter estimates, by means of the covariance matrix P of the estimation error, from the estimator to the design stage is the main important difference between the presented structures. The utilisation of the accuracy of the estimation for the controller design allows the generation
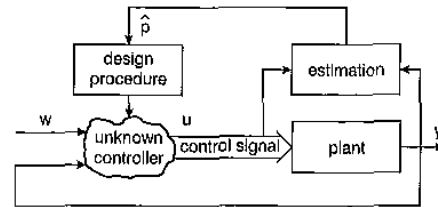

Fig. 1 *Non-dual standard adaptive control with goal to determine control law*
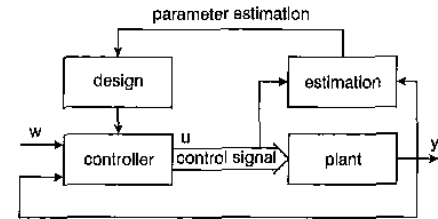

Fig. 2 *Complete standard adaptive control system based on CE assumption*
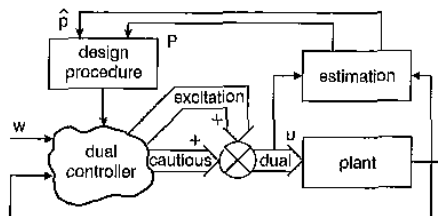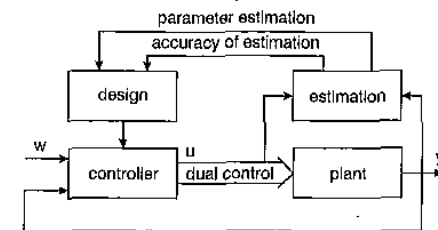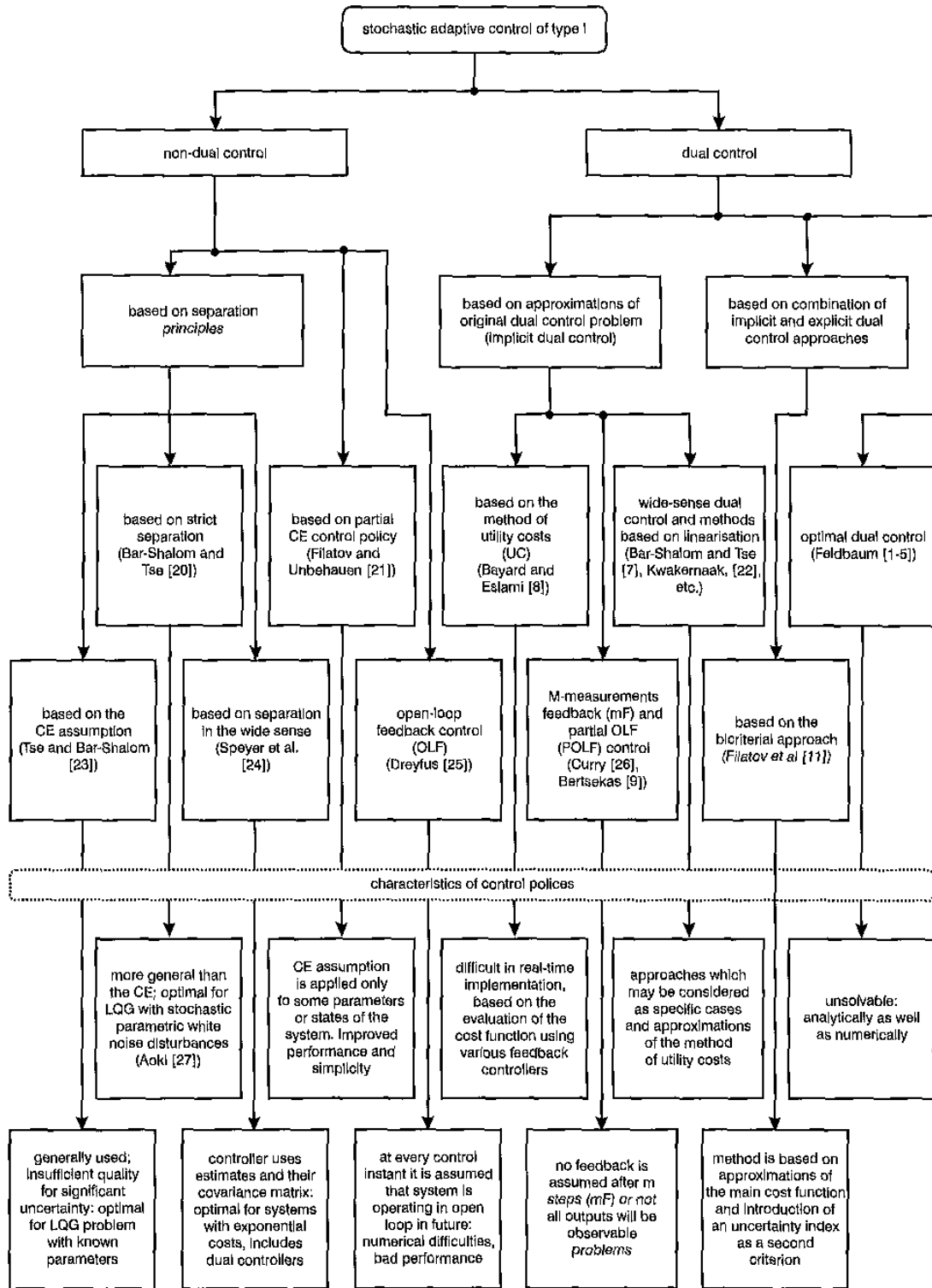

Fig. 3 *Adaptive dual control system*

Fig. 4 *Complete adaptive dual control system*

of the optimal excitation and cautious control signal for an adaptive dual controller and, furthermore, demonstrates a significant improvement in the control performance for the case of large uncertainties.

Control systems, adaptive and non-adaptive, may be classified in three large groups, which generate the manipulating signal in different ways, as indicated in Table 1. These types of control systems determine the corresponding control methods that have been developed for different groups of controllers. For example, almost all suboptimal stochastic approaches have appeared as a result of considering control problems for systems of type 1. Methods of predictive control take into consideration the systems of type II. Many controllers belong to type III. For example, the methods of implicit dual control were originally elaborated for systems of type I, and the explicit dual controllers for systems of type III.

The classification of stochastic control approaches of type I and their main characteristics are shown in Fig. 5. These approaches are based on various simplifications. Many approaches, indicated in Fig. 5, can also be applied for predictive control systems of type II. The detailed

stochastic adaptive control of type 1

non-dual control

dual control

based on separation *principles*

based on approximations of original dual control problem (implicit dual control)

based on combination of implicit and explicit dual control approaches

based on strict separation (Bar-Shalom and Tse [20])

based on partial CE control policy (Filatov and Unbehauen [21])

based on the method of utility costs (UC) (Bayard and Eslami [8])

wide-sense dual control and methods based on linearisation (Bar-Shalom and Tse [7], Kwakernaak, [22], etc.)

optimal dual control (Feldbaum [1-5])

based on the CE assumption (Tse and Bar-Shalom [23])

based on separation in the wide sense (Speyer et al. [24])

open-loop feedback control (OLF) (Dreyfus [25])

M-measurements feedback (mF) and partial OLF (POLF) control (Curry [26], Bertsekas [9])

based on the bicriterial approach (Filatov et al [11])

characteristics of control polices

more general than the CE; optimal for LQG with stochastic parametric white noise disturbances (Aoki [27])

CE assumption is applied only to some parameters or states of the system. Improved performance and simplicity

difficult in real-time implementation, based on the evaluation of the cost function using various feedback controllers

approaches which may be considered as specific cases and approximations of the method of utility costs

unsolvable: analytically as well as numerically

generally used; insufficient quality for significant uncertainty: optimal for LQG problem with known parameters

controller uses estimates and their covariance matrix: optimal for systems with exponential costs, includes dual controllers

at every control instant it is assumed that system is operating in open loop in future: numerical difficulties, bad performance

no feedback is assumed after m steps (mF) or not all outputs will be observable *problems*

method is based on approximations of the main cost function and introduction of an uncertainty index as a second criterion

Fig. 5 *Classification of stochastic adaptive control systems of type I. (Some of the considered methods of feedback and dual control can be directly applied to systems of type II)*

description of these methods, as simplified approaches for solving the stochastic control problem described in Section 2, will be given in Sections 5 and 6. The difference between stochastic control polices with the CE assumption, separation and wide sense separation will also be emphasised.

To find a CE control law for the problem described by eqns. 1–3 and 5, all stochastic variables should be replaced by their expectations. Let us denote the resulting deterministic feedback controller as

$$u(k) = \varphi_k(x(k)) \tag{8}$$

Table 1: Classification of discrete-time controllers for various types of control signals

Type	Description of group	Examples
I	a sequence of control signals $u(k),\ldots,u(N-1)$ or control policies $u_k(\Im_k),\ldots,u_{N-1}(\Im_{N-1})$ is generated, where $k=0$, $1,\ldots,N-1$; N can take values from the set $\{1,\ldots,\infty\}$	optimal control problems with finite and infinite horizon.
II	at every control instant k, a sequence of control signals $u(k),\ldots,u(k+N)$ is generated to optimise a cost function, but only $u(k)$ is applied, where $k=0, 1,\ldots,\infty,$; N can take values from the set $\{1,\ldots,\infty\}$	all predictive controllers. In the case of $N \to \infty$, the controllers coincide with type I.
III	at every control instant, only $u(k)$ is generated, where $k=0$, $1,\ldots,\infty$. Knowledge of the future reference signal is not required.	STR, GMV, various MRAC and APPC*, etc. Also controllers of type I, which generate constant feedback.

* STR = self-tuning regulators, GMV = generalised minimum variance controller, MRAC = model reference adaptive control, APPC = adaptive pole-placement controller

This CE controller is applied, replacing the unknown system state $x(k)$ by its estimate $\hat{x}(k)$:

$$u(k) = \varphi_k(\hat{x}(k)) \qquad (9)$$

The CE controller provides the LQG-control problem [7]. The notion of *separation* is more general than the CE and it will be given below. The control is generated using the state estimate as

$$u(k) = \psi_k(\hat{x}(k)) \qquad (10)$$

where the function ψ_k differs from the optimal deterministic feedback φ_k. Here, only the separation of estimator and controller is important. This control law is optimal for the linear quadratic control problem and for systems with stochastic parametric disturbances of white noise [27]. Sometimes, the definition of the control law with separation in the wide sense is used. The controller depends not only on the estimate, but also on the covariance matrix of the estimation $P(k)$:

$$u(k) = \phi_k(\hat{x}(k), P(k)) \qquad (11)$$

This control law is optimal for the linear stochastic system with known parameters and exponential cost function [19]. Many of the dual controllers are based on separation in the wide sense, and the parameter estimates and their covariance matrix are transferred from the estimator to the controller.

The classification of controllers of type III is presented in Fig. 6. It should be noted that the well known adaptive controllers such as STR, GMV, LQG, APPC, MRAC, etc. have originally been elaborated upon using the CE assumption. In a system with indirect adaptation the controller parameters are calculated using estimates of the plant parameters, whereas in the systems with direct adaptation, the controller parameters themselves are estimated directly from the input and output data without estimating the parameters of the plant model, as indicated in Figs. 7 and 8, respectively. The application of the bicriterial approach to the systems presented in Fig 6, with indirect as well as direct adaptation, allows one to design dual versions with improved control performance for all these systems, as will be shown later.
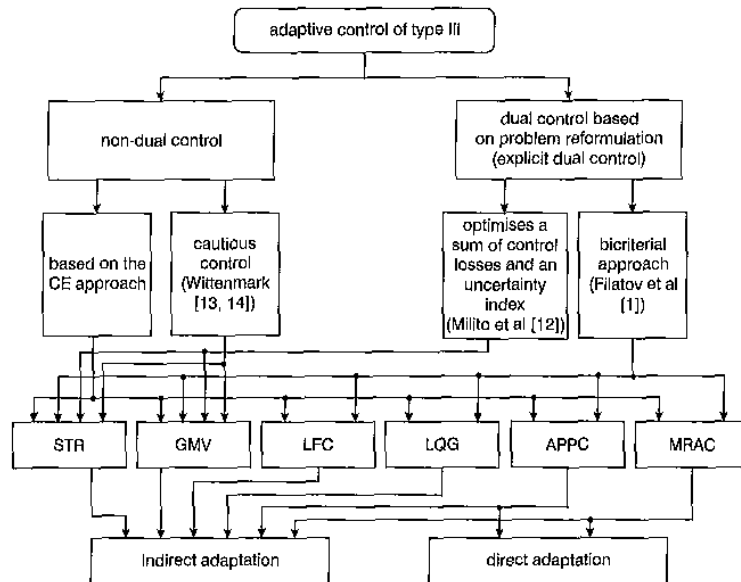
Fig. 6 *Classification of adaptive controllers of type III (Elaboration of the bicriterial approach allows the design of dual versions of all controllers of type III. The complexity of the controllers grows from left to right)*
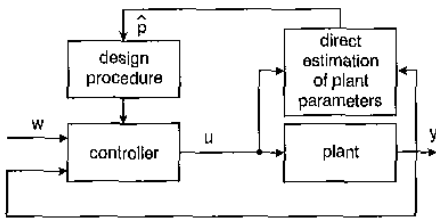
Fig. 7 *System with indirect adaptation (with direct identification)*
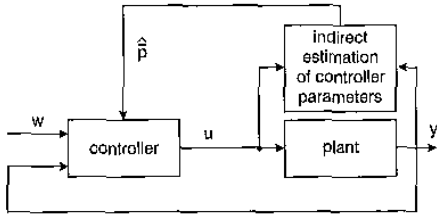

Fig. 8 *System with direct adaptation (with indirect identification)*

Adaptive control systems can also be classified by the types of models used. Likewise, various predictive controllers are based on nonparametric models. For example, dynamic matrix control (DMC) [28–30] uses a step response model, and model algorithmic control (MAC) [31] is based on an impulse response model. Various nonparametric controllers also use frequency-domain models of the plants. Many dual controllers have been elaborated for least squares (LS) and state-space models, but the results can be extended to CARMA and CARIMA models. This general classification is given in Fig. 9. Unbehauen [19] presented more general linear models.

4 Dual effect and neutral systems

In some systems, the accuracy of estimation is independent of the control action, and Feldbaum [1–4] has named them

neutral control systems or control systems with independent (passive) accumulation of information. The strict definition is given below.

A neutral control system is a control system that operates in conditions of *uncertainty*, and any excitations added to the control signal cannot improve the accuracy of the estimation.

It is necessary to point out that adaptive control systems are usually not neutral, and the 'additive' uncertainty of systems, which is attributed to such [1–5], can be compensated using an integral feedback control law without any adaptation.

Almost all adaptive systems have uncertain parameters or states, which are multiplicative to the control signal or state variables and, therefore, they are not neutral. The dual effect can be used to improve the performance of such control systems. Systems with additive uncertainty, like the system described by eqn. 8, are an exception. Therefore, one can conclude the above discussion with the following statement.

The performance of various *adaptive* control systems can be improved by applying dual control methods and replacing the CE controllers (or other non-dual controllers) with dual controllers, providing cautious behaviour with optimal excitations.

Some simple possibilities of replacing non-dual controllers with dual ones have been considered in [32] and are supported by examples.

Remark: To better understand the above statement, it is important to mention that the measure of performance can be understood in different ways in modern control system theory. For instance, if the tracking error is the measure of performance, then cautiousness can reduce tracking quality but increase robustness in the case of slight dynamical changes of the plant. Furthermore, there can be a mismatch between the estimated and actual stochastic uncertainty of the plant model, which might make the dual controller perform worse than CE control. At the same time, the choice of the parameters of the dual controller is very important for the improvement of the control performance. A dual controller
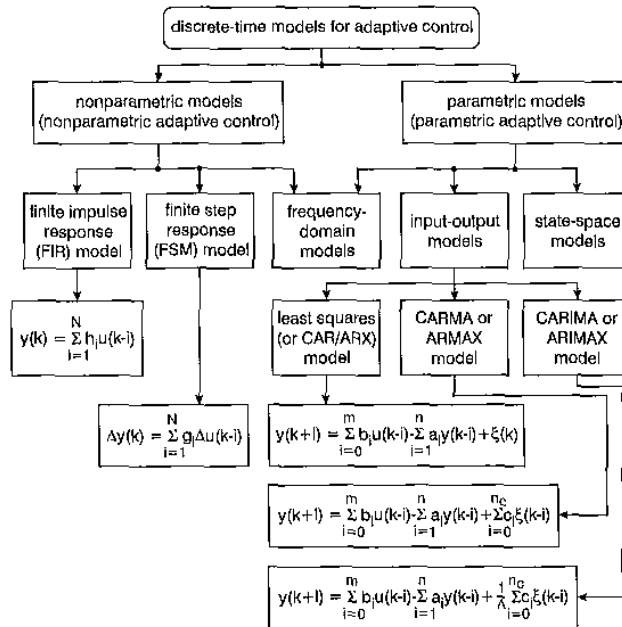

Fig. 9 *Classification of important linear discrete-time models used in adaptive control*

CAR = controlled autoregressive, ARX = autoregressive with auxiliary input, CARMA = controlled autoregressive moving-average, ARMAX = autoregressive moving-average with auxiliary input, CARIMA = controlled autoregressive integrated moving-average, ARIMAX = autoregressive integrated moving-average with auxiliary input, $\Delta x(k) = x(k) - x(k - 1)$

with uncommonly large excitation or too cautious behaviour might also perform poorer than a CE control.

5 Non-dual simplifications

The stochastic dynamic programming according to eqns. 6 and 7 gives, formally, a solution to the considered stochastic optimal control problem [5, 7, 9]. However, it is well known that the analytical difficulties in finding simple recursive solutions, and the numerical difficulties due to the dimensionality of the underlying spaces, make this problem practically unsolvable, even for simple cases [7, 8]. This fact has led to the development of various suboptimal stochastic adaptive control methods [7–9, 21, 25, 26, 33], which are based on different approximations and simplifying assumptions, and are classified in Fig. 5. Many of these simplifications can be interpreted as approximations of the probability measures of the unknown states and parameters of the system. Thus, the suboptimal adaptive control policies are based on the minimisation of the following remaining part (cost-to-go) of the general performance index, given by eqn. 5.

$$J_k(\mathfrak{I}_k, \rho_k) = \mathrm{E}_{\rho_k}\left\{\sum_{i=k}^{N-1} g_{i+1}(x(i+1), u(i))|\mathfrak{I}_k\right\} \quad (12)$$

$u_k(\mathfrak{I}_k)$ is to be found at sampling time k using the approximation ρ_k of the conditional probability densities of the system states and parameters for the future steps: $p(x(k+i), p(k+i)|\mathfrak{I}_{k+i})$, $i = 0, 1, \ldots, N-k-1$ (see also [21], where the notion of ρ-approximation has been introduced). In eqn. 12 the expectation E_{ρ_k} is calculated using the approximation ρ_k. The various suboptimal stochastic adaptive control approaches are based on different approximations ρ_k (ρ-approximations) in eqn. 12, as shown below:

(i) For the open-loop (OL) control policy, the system is assumed to be without feedback and the optimal control is found from *a priori* information about the system parameters and states. This simplifying assumption is equivalent to the following approximation of the probability densities for eqn. 12:

$$\rho_k = \rho_k^o = \{p(x(k+i), p(k+i)|\mathfrak{I}_{k+i})$$
$$= p(x(k+i), p(k+i)|\mathfrak{I}_0), \quad i = 0, \ldots, N-k-1\}$$
$$(13)$$

where $p(x(k+i), p(k+i)|\mathfrak{I}_{k+i})$ depends on the deterministic sequence $\{u(0), \ldots, u(N-1)\}$ for this case.

(ii) To find the control input for the open-loop feedback (OLF) control policy, the system is assumed to be without feedback in the future steps (from time k to N), but at every control time k the observation is used for the estimation of both parameters and states, and then the probability measures are corrected [9, 25]. This simplifying assumption can be described by the ρ-approximation in eqn. 12:

$$\rho_k = \rho_k^f = \{p(x(k+i), p(k+i)|\mathfrak{I}_{k+i})$$
$$= p(x(k+i), p(k+i)|\mathfrak{I}_k), \quad i = 0, \ldots, N-k-1\}$$
$$(14)$$

In the case of the approximate assumption described by eqn. 14, the information from feedback is used to improve the control quality. It is known that the OLF-control provides a superior control performance compared with the OL-control, using the ρ-approximation according to eqn. 13 [9].

(iii) The well known and generally used CE approach can also be interpreted for the considered control problem, using the following ρ-approximation of the probability densities for the 'cost-to-go' described by eqn. 12.

$$\rho_k = \rho_k^c = \{p(x(k+i), p(k+i)|\mathfrak{I}_{k+i})$$
$$= \delta(x(k+i) - \hat{x}(k+i))\delta(p(k+i)$$
$$- \hat{p}(k+i|k)), \quad i = 0, \ldots, N-k-1\} \quad (15)$$

where

$$\hat{x}(k+i) = \mathrm{E}\{x(k+i)|\mathfrak{I}_{k+i}\}, \quad \hat{p}(k+i|k) = \mathrm{E}\{p(k+i)|\mathfrak{I}_k\}$$
$$(16)$$

are the estimates, and δ is a Dirac function. The estimates are used here in the control law as if they were the real deterministic values of the unknown parameters.

It should be noted that only the OLF and the CE policies, with the approximations described by eqns. 13 and 15, are used in practice. The CE-control policy is simple in implementation but provides an insufficient control quality in many cases, because the inaccuracy of the estimates is not taken into account. The OLF-control policy gives better control quality but requires a numerical iterative optimisation procedure in real time. It should also be noted that the approximation using a Dirac function δ is actually a substitution of the deterministic values instead of the stochastic variables (CE assumption). Apparently, it can be named ρ-substitution instead of ρ-approximation.

(iv) A new ρ-approximation of the joint probability measures for both the system states and parameters has been suggested in [21]. This approach allows the derivation of adaptive control policies that are computationally simple, especially for linear systems, with improved control quality.

Consider the following extended state vector for the system described by eqns. 1–3:

$$z^T(k) = [x^T(k), p^T(k)] \quad (17)$$

The vector $z(k)$ is separated into two non-intersecting parts (vectors): $z_1(k)$ and $z_2(k)$. Introduce the following ρ-approximation of the extended state vector of eqn. 17, which will be used to design the control law via minimisation of the cost-to-go, according to eqn. 12

$$\rho_k = \rho_k^p = \{p(z_1(k+i), z_2(k+i)|\mathfrak{I}_{k+i})$$
$$= \delta(z_1(k+i) - \hat{z}_1(k+i))p(z_2(k+i)|\mathfrak{I}_k),$$
$$i = 0, \ldots, N-k-1\} \quad (18)$$

where $p(z_1(k+i), z_2(k+i)|\mathfrak{I}_{k+i}) = p(z(k+i)|\mathfrak{I}_{k+i}) = p(x(k+i), p(k+i)|\mathfrak{I}_{k+i})$. For the ρ-approximation, according to eqn. 13, the CE assumption (see eqn. 15) is applied to $z_1(k)$, the first part of the extended state vector, and the simplifying assumption, as for the OLF-control policy (see eqn. 14), to the second part $z_2(k)$. Thus it is assumed that at every sampling time k, the system operates in closed-loop feedback mode for the future time intervals with respect to the first part of the extended state vector $z_1(k+i)$, and in open-loop feedback mode for the second part $z_2(k+i)$, i.e., the information from the measurements is not used in the future to estimate $z_2(k+i)$. It is assumed at the same time that the CE assumption is applied for the first part of the extended state vector, but not for the second part. This partial certainty equivalence (PCE) approach, together with the assumption according to eqn. 18, allows one to design adaptive controllers that are simple in computation, especially for linear systems. Moreover, it is possible to estimate an upper bound of the criterion for this control [11], for the

case that the first part of the extended state vector $z_1(k)$ is exactly observable, and, as has been shown analytically [11], that the performance for the PCE policy is superior to that for the OL policy in this case. It should be mentioned that the suggested PCE approach proposes an informal separation of the extended state vector into its two parts, $z_1(k)$ and $z_2(k)$, which should be realised in accordance with the specific structure of the system, described in general by eqns. 1–3. Depending on this separation, the PCE-control policy can be a dual or non-dual one. An example of this separation for linear systems with unknown stochastic parameters has been given [11]. The PCE-control policy can be used together with the bicriterial approach for designing a dual controller, where both implicit and explicit dual control methods are applied together [22].

6 Implicit dual control

The implicit dual control methods are based on various approximations that maintain the dual properties of the system and, therefore, are very complex. Some of them are even unsolvable in spite of the approximations, such as the original dual control problem.

The partial open-loop feedback (POLF) policy [9] (see also Figs. 1 and 2) is based on the assumption that instead of full information $\Im_{k+i}$ in the future steps, $i = 0, \ldots, N - k - 1$, only incomplete information $\Im_{k+i}$ from the future measurements will be used. This assumption is equivalent to the following ρ-approximation:

$$\rho_k = \rho_k^l = \{p(x(k+i), p(k+i)|\Im_{k+i})$$
$$= p(x(k+i), p(k+i)|\bar{\Im}_{k+i}),$$
$$i = 0, \ldots, N - k - 1\} \qquad (19)$$

where $\bar{\mathcal{I}}_{k+i} = \{\bar{y}(k+i), \ldots, \bar{y}(k+1), \quad y(k), \ldots, y(0), u(k+i-1), \ldots, u(0)\}$, $\bar{y}$ is the partial observation vector, which does not contain all the elements of y and has the dimension $n_{\bar{y}} \le n_y$.

The m-measurement feedback (m-MF) control policy is based on the assumption that the system operates in feedback mode in the future m time steps, and without feedback mode after the time $k + m$ [26]. This assumption is equivalent to the approximation of the probability densities for eqn. 12 by

$$\rho_k = \rho_k^m = \{p(x(k+i), p(k+i)|\Im_{k+i})$$
$$= p(x(k+i), p(k+i)|\Im_{k+i}), \quad i = 0, \ldots, m;$$
$$p(x(k+m+j), p(k+m+j)|\Im_{k+m+j})$$
$$= p(x(k+m+j), p(k+m+j)|\Im_{k+m}),$$
$$j = 1, \ldots, N - k - m - 1\} \quad \text{for } k < N - m - 1 \quad (20)$$

and without any approximation for $k \ge N - m$. This assumption results in a suboptimal dual control scheme, which is very difficult to derive. The ρ-approximation, according to eqn. 20, also coincides with the one for the OLF-policy (eqn. 14) if $m = 0$.

The wide-sense dual (WSD) and the utility-costs (UC) control [7, 8] are based on other approximations. The WSD-control uses a linearisation of the system equations around the nominal trajectory of the system where the OLF-control is used. The utility costs approach [8] can be considered as a generalisation of the WSD-policy, where various control policies may be used as a nominal trajectory of the systems for further linearisation or other

approximations. Various implicit dual control policies have been suggested, where different kinds of approximations and linearisation have been used [10, 22, 34–37]. These approaches provide a dual control with improved quality, but significant computational requirements in real time restrict their practical applicability.

7 Explicit dual control

Various explicit dual control algorithms have received considerable attention [12, 14, 38–40]. They are based on the minimisation of cost functions of the following form

$$J_k^c = j_k^c + \lambda J_k^a, \quad \lambda \ge 0 \qquad (21)$$

where J_k^c characterises the control losses, and J_k^a is the uncertainty index. It should be noted that the requirement of the sufficiency condition for the optimum of the cost function according to eqn. 21, i.e.

$$\frac{\partial^2 J_k^c}{\partial u(k)^2} > 0 \qquad (22)$$

constrains the parameter λ as $0 \le \lambda < 1$ in many cases. A simple example for applying this approach has been given in [12]. These algorithms have been elaborated for very simple system models and a structure of the STR type. An application of these approaches in designing a dual version of the GMV controller was given by Chan and Zarrop [39]. The shortcoming of this approach consists in the fact that the magnitude of excitations cannot be controlled by means of the parameter λ. The amplitude of excitations varies significantly, and the control signal takes values in an interval between cautious and CE-control, as shown in [12]. The bicriterial approach, which has been introduced by Filatov and Unbehauen [21], is free from this shortcoming.

Some dual controllers, based on the minimisation of one cost function with constraints for other cost functions, have been presented by Alster and Belanger [41] and Bodyansky [42], but the advantages of these approaches over those with cost functions according to eqn. 21 are not clearly indicated.

It should be noted that dual controllers can be derived using various uncertainty indices J_k^a for the construction of the cost function according to eqn. 21. It has been suggested in [43] to use $P(k)$, the covariance of the unknown parameters. In the paper of Wittenmark [13], the cost function

$$J_k^a = \text{tr}\{P(k+1)\} \qquad (23)$$

is used, where $P(k)$ is the covariance matrix of the estimation error. Goodwine and Payne [44] applied

$$J_k^a = -\frac{\det\{P(k)\}}{\det\{P(k+1)\}} \qquad (24)$$

and for experiment design (not for dual control)

$$J_k^a = \log(\det\{P(k+1)\}) \qquad (25)$$

while Wittenmark and Elevitch [40], as well as Allison et al. [38], used the covariance of the estimate of the first parameter b_1 of the ARX model of the system. Instead of the well known innovational cost measure of Milito et al. [12], the following function can be used in the bicriterial design method

$$J_k^a = \text{tr}\{P^{-1}(k)P(k+1)\} \qquad (26)$$

or in the case of drift stochastic parameters

$$J_k^a = \text{tr}\{P^{-1}(k+1|k)P(k+1)\} \qquad (27)$$

8 Brief history of dual control systems and their applications

Important stages in the development of dual adaptive control systems and their applications are presented in chronological order in Figs. 10 and 11, respectively.

Naturally, the presented diagrams cannot include all results, and many different adaptive dual controllers have appeared in the literature since the first work of Feldbaum [1-4] on dual control. It should be noted that this contribution presents results in the development and application of adaptive dual control that have been obtained during the
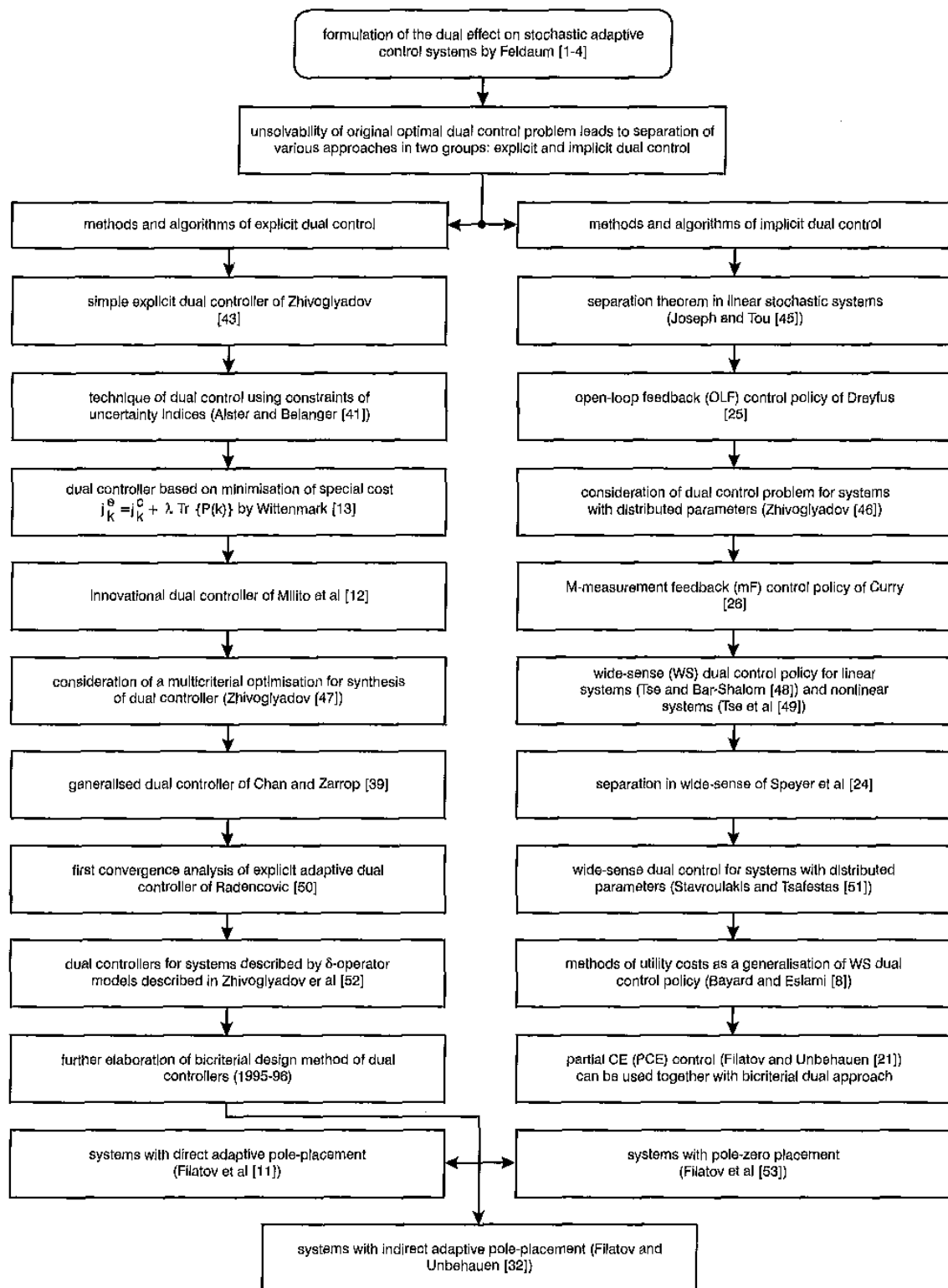
Fig. 10 *Historical stages in development of dual control theory*
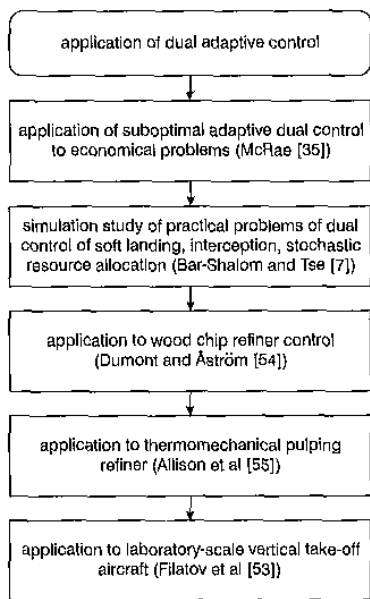
Fig. 11 *Historical stages in application of dual adaptive control*

last few years. Special attention is given to practical aspects of adaptive and dual control, as well as applications and development of modern software.

The most important stages in the development of adaptive dual control were: the discovery of the dual effect in stochastic adaptive systems; and the separation of the basic principles relating to the design of dual controllers into two groups: implicit and explicit dual control, which are based on approximations and reformulations of the dual control problem and the elaboration of the bicriterial design method. Also important was the development of dual adaptive control in systems with non-stochastic uncertainties [56, 57]. It should also be noted that the BPF and δ-operator models are very important for the development of modern software for real-time adaptive dual controllers with a high sampling rate.

9 Conclusions

A detailed classification of dual control methods and stochastic adaptive control approaches has been presented. The development of dual control theory and applications has been considered in chronological order from the early 1960s to present. Various simplified approaches for the design of dual control systems have been analysed and compared. The modern adaptive dual control methods, which are suitable for practical applications, have been reviewed.

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