DUAL-CONTROL THEORY. I

A. A. Fel'dbaum

Moscow

Translated from Avtomatika i Telemekhanika, Vol. 21, No. 9, pp. 1240-1249, September, 1960

Original article submitted March 23, 1960

Some fundamental problems in communication theory and control theory are compared. The problem of designing an optimum (in the statistical sense) closed-loop dual-control system, is formulated. Its solution, as well as examples and some generalizations, will be given in parts II, III, and IV.

Introduction

A general block diagram of signal transmission, as investigated in communication theory is shown in Fig. 1. The transmitted signal x* proceeds from the transmitting device A to the communication channel H*. The mixing of signal and interference (noise) h* now takes place. The resultant signal y* represents a mixture of the transmitted signal and the interference. The resultant signal proceeds to the input of receiver B. The optimum receiver problem consists in obtaining the signal x̲ at its output, such that it is, in a specified sense, closest to the transmitted signal x* or to some transformation of the signal x*. The mathematical side of the problems related to such systems has been the subject of important investigations by A. N. Kolmogorov [1], N. Wiener [2], C. E. Shannon [3], and A. Wald [4]. This type of system was investigated in the works on communication theory by V. A. Kotel'nikov [5], D. Middleton, D. Van Meter [6], and others. The cited works differ in their various approaches to the problem, but are all basically concerned with the investigation of the scheme represented by the block diagram in Fig. 1. The results obtained in the above-cited works, and in particular the Kolmogorov-Wiener theory, have proved useful in formulating the statistical theory of automatic-control systems. This theory has been expounded in the books of V. S. Pugachev [7], J. H. Laning, Jr., and R. H. Battin [8], and others. The fullest consideration has been given to the theory of linear systems. If a system is linear, then whatever the closed-loop system, it is easy to obtain an open-loop system equivalent to it. That is why the automatic-control systems, which, as a rule, are closed-loop systems, enable one to use the scheme as in Fig. 1, provided that the system is linear. Some complications and difficulties arise when the interference does not appear at the input of the system, but at the input of the controlled object – the latter being inside the closed-loop network. This also creates difficulties which are not, however, of a fundamental nature. More serious difficulties arise due to the bounds to which the power of the system's signals are subjected. The problem becomes more involved when the controlled object is nonlinear or it is required that an optimum control system, which often proves to be nonlinear, is designed. It is not always possible to proceed from an open nonlinear system to an equivalent closed-loop one; furthermore, this is an extremely involved process. In such a case, the open-loop scheme depicted in Fig. 1 cannot be used in practice. A number of attempts have been made to reduce approximately nonlinear systems to equivalent linear ones (see, for example, the paper of I. E. Kazakov [9]). Such studies are of considerable practical value but do not, in their present state, provide a means of estimating how close the obtained approximation is to the true solution; neither do they enable one to synthesize the optimum system.

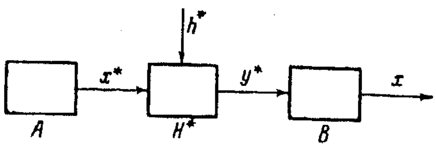

Fig. 1.

In order to be able to solve optimum problems of the control theory, a fundamentally different approach is required. Firstly, a different block diagram replacing that depicted in Fig. 1 is needed. Before selecting a common scheme to be used in automatic-control theory, it seems advisable to have a preliminary survey of certain basic concepts of the theory.

Figure 2, a shows the controlled object B with x̲ as its output, u̲ as the controller, and z̲ as the disturbance (interference). When the system has several inputs and outputs, one can regard x̲, u̲, and z̲ as vector quantities.

The output x̲ depends on u̲ and z̲. This dependence can be described either by a linear or a nonlinear operator, and in a particular case of memoryless systems, only by a function. The interference z̲ is generally a function of time. Thus, since a change in the system's characteristics can be considered a particular result of interference (e.g., the parametric effect), then hereafter everything in the system's characteristics that changes with time will be attributed to interference. If, for example, x depends on u as in

$$x = [a_0 + f_0(t)]\, u^2 + [a_1 + f_1(t)]\, u + a_2 + f_2(t), \quad (1)$$

then the vector

$$z = \{z_0, z_1, z_2\},$$
$$z_0 = f_0(t), \quad z_1 = f_1(t), \quad z_2 = f_2(t) \quad (2)$$

gives the interference or disturbance, and the formula

$$x = [a_0 + z_0]\, u^2 + [a_1 + z_1]\, u + a_2 + z_2 \quad (3)$$

represents a particular operator.

If the object has memory and if, for example, its motion can be described by a differential equation of the nth order, then its state is considered as one of the characteristics of the object as described by the value of the vector x in the n-dimensional phase space.

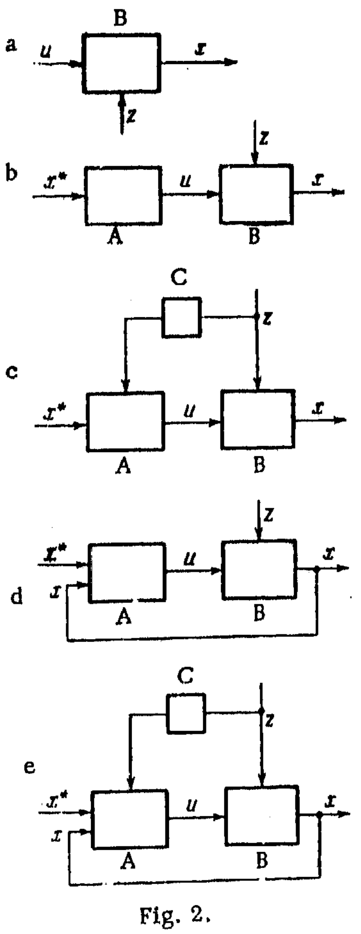

Fig. 2.

Complete information about a system thus consists of information about its operator, interference (noise), and the state of the system. The controlling device may be given and is considered as known. The open-loop systems are automatic-control systems of a simple type. A block diagram of an open-loop system is shown in Fig. 2, b and is of the same character as the one given in Fig. 1. The exciting quantity x* enters the input of the regulating member A, determining how the output quantity x should vary. The output u of the regulating device enters the input of the object B under control. The output x of the controlled object B does not proceed in this case to the regulating member A.

The required rule of change in x can only be implemented when full information about the controlled system is available, i.e., when its operator and state x are known, at least at the initial moment of time, as well as the interference z. The latter should be known a priori at all moments of time, including future moments. The required rule of change in x must be one of the admissible ones, such that it can be implemented for the given class of initial states of the system and for a class of controlling motions u staying within acceptable bounds.

The above conditions, and, in particular, the a priori full knowledge of the controlled system, cannot be satisfied in practice. This is why an accurate implementation of the required control rule cannot be obtained. Sometimes the interference z is not known a priori, but it is possible to measure it with a device C (see Fig. 2, c) and to introduce the outcome of the measurement into the controlling member A. One can then find in the latter the required controlling rule u. Such a scheme is also an open-loop one. But the scheme depicted in Fig. 2, c differs in some ways from the scheme with complete a priori information about the system, as now future magnitudes of the interference z remain unknown. Because of this, the exact implementation of the required rule of variation of the controlling quantity x is not always feasible.

When the state x of the system is not known, then, generally speaking, it is not possible to implement the required rule of the change in x. To be able to attain the required variation in x, or one near to it, a feedback network is needed to feed the output quantity x to the input of the controlling member A (see Fig. 2, d). Having compared x and x*, the controlling member generates the regulating action u, bringing x to its required value. The block diagram of Fig. 2, d is a closed-loop scheme, and is of the utmost importance in the automatic-control theory.

A closed-loop network offers far-reaching possibilities not available in an open-loop system. For example, it may be possible for a class of objects B of control to obtain a process x close to the one required even when the interference z remains unknown and incapable of measurement. Let, for example, the interference z, together with the controlling action u, be applied at the input of the controlled object B, the latter representing an inertia member. If the quantity $x^* = x$ is to be attained, the regulating member can be implemented in the form of an amplifier of high gain $k >> 1$, the difference $x^* - x$ being sent to its input.

It is not difficult to see that the requirement $x^* = x$ will be satisfied approximately, whatever the continuously varying interference z, provided k is sufficiently high and the bounds of variability of u are such that the interference z can be compensated by u.

This principle of neutralizing the interference can be generalized and applied to cases considerably more involved, combining the system's accuracy with its stability. A detailed analysis of the applicability of this principle was carried out by M. V. Meerov in his monograph [10].

When the interference z can be measured, it is possible to implement a combined system (see Fig. 2, e) of measurement of the state of the controlled system x as well as of its interference. Such systems are of considerable practical value. We shall not, however, concern ourselves with them but shall limit the study to the "pure" type of closed-loop systems, considering them as being of primary importance.

The input quantity x^* may be previously unknown; usually, neither do we have any prior knowledge of the interference z. Consequently, these processes become random, and, in a favorable case, the a priori information is limited to our knowledge of their statistical characteristics. Such processes may be regarded as belonging to a class of curves $x^*(\lambda)$ and $z(\mu)$ where λ and μ are parameter vectors $(\lambda_1, \ldots, \lambda_q)$ and $(\mu_1, \ldots, \mu_m)$, respectively, with their probability distributions either known or unknown.

In communication channels connecting the blocks of a system, the errors of measurement or noise can be regarded as subsidiary random processes as well, with either known or unknown characteristics. Thus, the analysis of a control system and the synthesis of the regulating member can be regarded as problems of a statistical nature. The problem should be solved for an over-all block diagram in which all the above features of an automatic-control system are reflected. Such a block diagram is depicted in Fig. 3; it is the subject of the present paper as well as that of further papers in this series.

The input quantity x^* proceeds to the input of the controlling member A through channel H^* where it becomes mixed with noise h^*. Thus, the quantity y^* entering the input of A is generally not equal to the actual value of the input quantity x^*. There also exists a class of systems with the external input x^* altogether absent. Generally speaking, however, it cannot be neglected. A similar mixing takes place of the state x of controlled object B and noise h in channel H; quantity y entering A will not, as a rule, be equal to x. The regulating action u proceeds next from A to controlled object B having previously passed through channel G where it was mixed with noise g. The quantity v proceeding to the controlled object is not, as a rule, equal to u.

Dual Control

One cannot neutralize, in a general case, the interference z by a regulation u if the interference z is not known. Its direct measurement is not, as a matter of fact, often possible. In such a case, an open-loop system is useless. But the closed system in Fig. 3 shows how z can be indirectly determined by measuring the input and the output, in and out of object B, by studying its characteristics. The input of controlling member A enters both the input v and the output x of the object or, in any case, the quantities u and y related to v and x. The examination of the quantities u and y provides information on the characteristics of object B. It should be understood that this information is never complete, as the noises g and h render an exact measurement of B's characteristics impossible; if the actual form of the object's operator is not known either, a full determination of its characteristics would not be possible even in the absence of noise, unless the determination time is infinitely great. The lack of complete information on the disturbance z can assume the form of an a posteriori probability distribution of its parameters. Although the latter does not provide precise values of the parameters, it is more accurate than an a priori distribution, as the former reflects the real character of the interference.

If the random process can be measured directly, one is able eventually to specify its statistical characteristics more accurately. The method which provides such improvement with the aid of dynamic programming was discussed in examples by R. Bellman and R. Kalaba [11, 12] and also by M. Freimer [13]. One is able to find the characteristics of the process x^* more accurately in the open part of the block diagram in Fig. 3, or in a similar scheme in Fig. 1.

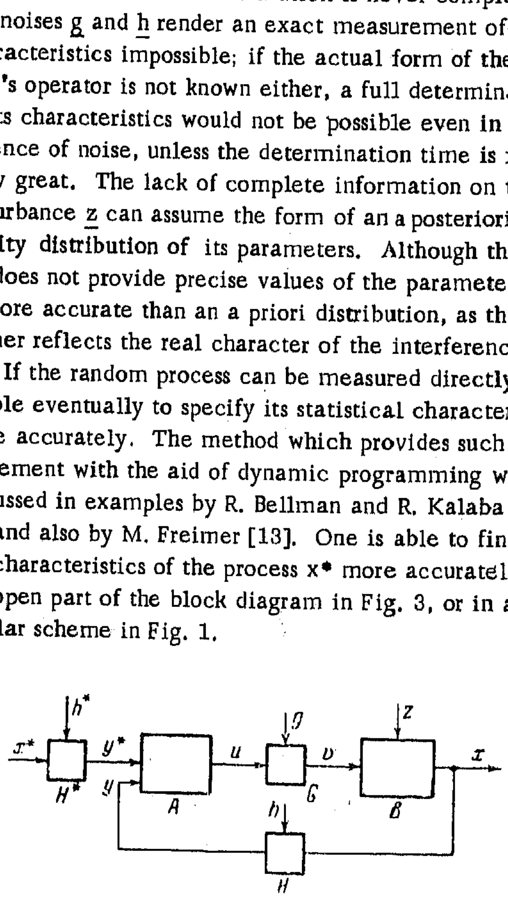

Fig. 3.

This formulation of the problem is characteristic for an open system. In a closed system its formulation becomes totally different. It is shown that some processes in the system of Fig. 3 may occur which have no counterpart in open-loop systems. Whereas open systems can only be studied by passive observations, the study may develop into an active one in closed systems. In order to improve the investigation one may vary the signals $\underline{u}$ (or $\underline{v}$) which act on the controlled object B. The object is, as it were, "reconnoitered" by signals of an enquiring character whose purpose it is to promote a more rapid and more accurate study of the object's characteristics and of the methods of controlling it.

However, the controlling movements are necessary not only to study or to learn the characteristics of the object or the ways of controlling it, but also to implement the regulation, to direct the object to the required state. Thus, the controlling effects in the general block diagram in Fig. 3 must be twofold: they must, to a certain extent, be investigating as well as directing.

The control whose regulating effects are of this twofold character will in the sequel be called dual control; the papers in the present series will be devoted to the theory of dual control.

Dual control is particularly useful and even indispensible in cases where the operator and the interference $\underline{z}$ in the object B are complex, and the object is thus distinguished either by its complexity or by the variability of its characteristics. Some typical examples of systems with dual control are to be found in automatic search systems, in particular, in automatic optimization systems (see, for example, [14 and 15]). In these systems, the investigating or "trial" part can usually be separated easily from thecontrolling or "operating" part of the signal, either by the difference in their frequency ranges or because they interweave in time. Such a separation, however, need not always take place; an effect can be twofold in character by virtue of being partly diagnostic and partly regulating.

Thus, in dual-control systems, there is a conflict between the two sides of the controlling process, the investigational and the directional. An efficient control can only be effected by a well-timed action on the object. A delayed action weakens the control process. But the control can only be effective when the properties of the object are sufficiently well known; one needs, however, more time to become familiar with them. A too "hasty" controlling member will carry out the operational movement without making proper use of the results of trial investigations performed on the object. A too "cautious" system will bide its time unnecessarily long and process the received information without directing the object to its required state at the right time. In each case, the control process may not prove the best one and may not even prove to be up to the mark. Our problem is to find out, one way or another, which combination of these two sides of the regulation would prove to be most suitable. The operations must be so selected as to maximize a criterion of the control's quality.

As shown above, the incomplete information about the object will be expressed by the presence of the probability distributions of potentially possible characteristics of the object. The regulating member compares, as it were, the various hypotheses on the object, with probability of its occurrence being attached to each hypothesis. These probabilities vary with time. There may be a control method such that the most probable hypothesis will always be selected and, therefore, assuming that it is valid, the optimum control method will be attained. Such a control system is not generally optimum in the absolute meaning of the word as the complete information on the object has not been utilized. The probability distribution of the different hypotheses extracted from the experiments is distorted as the probability 1 was ascribed to one of them and the probability 0 to others. A better control method will be one whereby the probabilities of all the hypotheses would be taken into account.

The probability distribution of hypotheses will vary with time, the higher probabilities concentrating more and more in the region of those hypotheses which approach the true characteristics of the object. The pace of concentration and, therefore, the success of the subsequent regulating movements, depends on the character of the preceding regulating movements, on how well they have "sounded" the object. Thus, two factors should be taken into account by the controlling member which decides the specific amount of regulating movement at any given moment of time:

(a) The loss occurring in the value of the quality criterion due to the fact that the outcome of the operation at a given moment, and at subsequent moments of time, will cause a deviation of the object either from the required state or from the best attainable one. The average value of this loss shall be called the action risk.

(b) The loss occurring in the value of the quality criterion due to the fact that the magnitude of the controlling action has not proved the best to obtain information on the characteristics of the object; in view of this, the subsequent actions will not be the best posssible ones either. The average value of this loss shall be called the investigation risk.

It will be shown that for a certain class of systems, the total risk will be equal to the sum of the action and investigation risks.

All systems of automatic search (see [14]) are characterized by trial actions. Dual control, therefore, is applicable to all systems of automatic search and, in particular, to automatic optimization systems. It can also be applied to other types of closed-loop systems which do not belong to the automatic search class at all. To illustrate the difference between the two types of dual-control systems, a few examples will be given.

Figure 4,a shows a system which operates as follows: the main regulating member A implements the control of object B, either in an open- or in a closed-loop network (the closed one is indicated by a dashed line). The

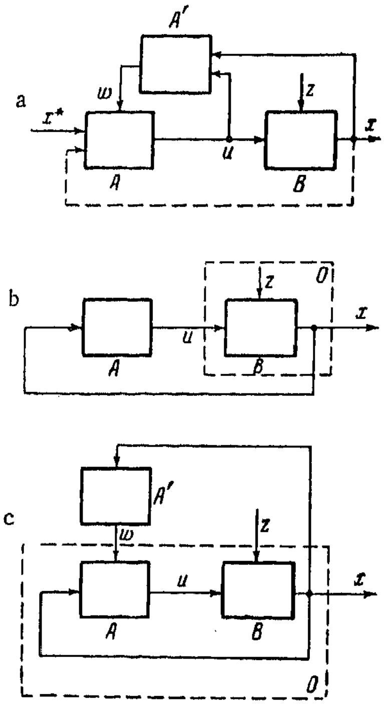

Fig. 4.

regulating movements u are of the investigational and action type simultaneously. The quantities u and x, from the input and the output of object B, respectively, enter an additional controlling member A'. The latter receives the characteristics of object B from the results of the investigations; subsequently, in accordance with an algorithm given in advance and fed into the device from outside, the parameters of controlling member A are so computed that its controlling action is optimal. Having the results of the computation, the additional regulating action w establishes the computed optimum parameters in the main controlling member A. This process may repeat itself periodically.

Such systems contain investigational movements; however, the automatic search is absent. The parameters of member A are established, not via automatic search, but from an algorithm given in advance, from a function of the determined characteristics of object B. There is no investigational component in the operation w. The channel w is not usually found in a closed network, as the change in the coefficients of A has no effect on the coefficients of B.

A block diagram of an automatic optimization system is presented in Fig. 4,b. Here the action u is dual in character, investigating object B as well as directing it to its optimum mode of action. The latter corresponds to an extremum of quantity Q dependent on x. The optimum mode is found by means of automatic search. The latter is conducted in such a way that the information received from the investigating action u and from the output x of the system is analyzed in controlling member A. This permits determination of the regulating part of the same action u on the same object B whose input and output were investigated, so that the same quantity x which was being investigated can be changed in the right direction.

The combination of the investigating and directing operations has, so far, not constituted the whole search, but only one of its distinct features. There is no search in the system depicted in Fig. 4,a; it takes place, however, in the system of automatic optimization shown in Fig. 4,b. But both systems are of the dual-control type.

In Fig. 4,b the controlled object is inside a dashed rectangle denoted by O. In this case, it does not differ from object B. The dual control, however, can also be applied to control the entire automatic system, considered as a complex object. For example, in Fig. 4, c the complex object O inside the dashed rectangle comprises the controlling member A and the object B of control. The auxiliary controlling member A' investigates the process x and, with the aid of the controlling process w, can vary the algorithm of regulation implemented by member A. The w processes are twofold in character. The investigation of changes in the algorithm of member A and their effect on the process x results in regulating processes w, bringing the algorithm of member A to such a form that the process x will either prove admissible, favorable, or optimum, depending for what purpose the system will be used. Here an automatic search takes place in the closed network of processes w → x → u.

Statement of the Synthesis Problem of an Optimum System of Dual Control

The problem of designing an optimum, in the specified meaning of the word, controlling member A, as shown in Fig. 3, is formulated below. It is advisable when formulating the problem to make use of certain concepts of the theory of games and of A. Wald's [4] theory of statistical decisions (see D. Blackwell and M. A. Girshick [16], and also Chow [17]). In solving the variational problem as stated later in the present series of papers, use is made of the concepts of R. Bellman's dynamic programming (see, for example, [18]). In the subsequent parts of this series, the mathematical exposition may appear somewhat cumbersome but is actually quite simple. The main contents of the papers deals with further development of the concepts of automatic control briefly described above.

Consider the scheme presented in Fig. 3. The following limitations of the statement of the problem are introduced.

1) A discrete-continuous system is investigated in which the time but not the level is quantized. All magnitudes occurring in the system are considered at discrete moments of time $t = 0, 1, 2, \ldots, n$ only. Any

magnitude at the sth moment of time will carry the index s. Thus, the considered quantities are x^*_s, x_s, y_s, v_s, g_s, etc.

Such limitation enables one to simplify the computation. Moreover, in many cases this actually occurs. The transition to the continuous time can in some cases be accomplished in an intuitive manner by making the time interval between the discrete values approach zero (see Part III). One meets with considerable difficulties in more fully examining the passage to the limit.

2) The time interval, or the number of cycles n within which the process is being investigated, is assumed to be a fixed constant. In certain cases no major difficulties arise when proceeding to the limit with $n \to \infty$. A wider generalization relating to a variable number n of cycles not known beforehand would be of interest, but will not be tackled in the present paper.

3) A Bayesian problem, in which a priori densities of random variables are given, is considered. Other formulations, for example, minimax, are also of considerable interest, but far more difficult to solve. This problem could also be formulated in relation to the concept of the so-called "inductive probability" (see, for example, the paper of L. S. Schwartz, B. Harris, and A. Hauptschein [19]).

We assume that h^*_s, h_s, g_s, are sequences of independent random variables with identical distribution densities $P(h^*_s)$, $P(h_s)$, $P(g_s)$. Further, let $z_s = z(s, \mu)$, and $x^*_s = x(s, \lambda)$, where μ and λ are random parameter vectors with coordinates μ_i and λ_i, respectively:

$$\mu = (\mu_1, \ldots, \mu_m), \qquad \lambda = (\lambda_1, \ldots, \lambda_q). \tag{4}$$

The a priori probability densities $P(\mu)$ and $P(\lambda^*)$ are given.

4) The object B is assumed to be memoryless; in other words, the values x_s of its output depend only on the values of the input quantities z_s and v_s at the same moment of time:

$$x_s = F_0(z_s, v_s). \tag{5}$$

The functions F_0 and z_s are assumed to be finite and single-valued, continuous and differentiable.

A generalization relating to objects with memory and with x_s depending on x_r, z_r, v_r $(r < s)$ will be given in Part IV. It should be pointed out that memoryless objects are of great practical value. Namely, if the input data (initial conditions or values of parameters) are given for a certain model, and one is able to carry out experiments using this model and also to register the results, then such an object becomes equivalent to a memoryless one.

5) A simple criterion W of quality is introduced.

Let the partial loss function corresponding to the sth time moment be of the form

$$W_s = W(s, x_s, x^*_s). \tag{6}$$

Moreover, let the total loss function W for the total time be equal to the sum of partial loss functions (such a criterion shall be called a simple one):

$$W = \sum_{s=0}^{s=n} W(s, x^*_s, x_s). \tag{7}$$

The smaller the mathematical expectation of W, the better is the system. It shall be called optimum when its average risk R (i.e., the mathematical expectation M of the quantity W) is minimal. The amount of risk is given by the formula

$$R = M\{W\} = \tag{8}$$

$$= M\left\{\sum_{s=0}^{s=n} W(s, x^*_s, x_s)\right\} = \sum_{s=0}^{s=n} M\{W_s\} = \sum_{s=0}^{s=n} R_s.$$

Each $R_s = M\{W_s\}$ will be called a partial risk due to the sth cycle.

There may be many types of simple criteria, for example,

$$W_s = \alpha(s)[x_s - x^*_s]^2. \tag{9}$$

Criteria of practical importance need not always be simple, and generalizations relating to other criteria would therefore be of interest.

The formulation of the optimum strategy problem in terms of risks is not the only one in existence. There exist a number of studies in which closed systems are investigated from the point of view of the information theory (see, for example, R. L. Dobrushin's paper [20]). As the primary aim of a control system does not lie in transmitting information but in designing required processes, the formulation of the problem in the language of statistical decisions fits in better with the intrinsic nature of the problem.

6) All the quantities occurring in the sth cycle will be regarded as scalar. Our object, therefore, has only a single input v and a single output x. The exposition becomes more involved with generalizations relating to objects with several inputs and outputs (see Part IV).

7) We assume that the manner by which the signal and the noise are combined in H*, H, or G blocks is

known and invariable, and that the blocks are memoryless. Thus,

$$v = v(u, g),$$
$$y^* = y^*(h^*, x^*), \qquad (10)$$
$$y = y(h, x).$$

Therefore, the conditional probabilities $P(y^*|x^*)$ and $P(y|x)$ and $P(v|u)$ make sense.

8) We assume that the controlling member A generally possesses a memory and that, moreover, for the sake of generality, the algorithm of its action is a random one, i.e., the part A exhibits random strategy.

We introduce the vectors $(0 \leq s \leq n)$:

$$\mathbf{u}_s = (u_0, u_1, \ldots, u_s),$$
$$\mathbf{y}_s^* = (y_0^*, y_1^*, \ldots, y_s^*), \qquad (11)$$
$$\mathbf{y}_s = (y_0, y_1, \ldots, y_s).$$

The controlling member can now be characterized by the probability densities

$$P_s(u_s) = \Gamma_s(u_s, \mathbf{u}_{s-1}, \mathbf{y}_s^*, \mathbf{y}_{s-1}) \quad (0 \leqslant s \leqslant n). \qquad (12)$$

The problem consists in finding a sequence of functions F_s such that the average risk R (see [8]) becomes minimal.

LITERATURE CITED

1. A. N. Kolmogorov "Interpolation and extrapolation of stationary random sequences," Izvest. AN SSSR Ser. Matem. 5, 1 (1941).
2. N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series (J. Wiley and Sons, New York, 1949).
3. C. E. Shannon, "A mathematical theory of communication," Bell System Techn. J. 27, 3 (1948).
4. A. Wald, Statistical Decision Functions (J. Wiley and Sons, New York; Chapman and Hall, London, 1950).
5. V. A. Kotel'nikov, Theory of Potential Noise Stability [in Russian] (Gosénergoizdat, 1956).
6. D. Van Meter and D. Middleton, Modern Statistical Approaches to Reception in Communication Theory, Trans. IRE, IT-4 (Sept., 1954).
7. V. S. Pugachev, Theory of Random Functions and Its Applications to Automatic Control [in Russian] (Gostekhizdat, 1957).
8. J. H. Laning, Jr. and R. H. Battin, Random Processes in Automatic Control (McGraw-Hill, New York, 1956).
9. I. E. Kazakov, "An approximate statistical analysis of accuracy of essentially nonlinear systems," Avtomat. i Telemekh. 17, 5 (1956).*
10. M. V. Meerov, The Synthesis of Networks of Automatic Control Systems of High Accuracy [in Russian] (Fizmatgiz, 1959).
11. R. Bellman and R. Kalaba, On Communication Processes Involving Learning and Random Duration. IRE National Convention Record, Part 4 (1959).
12. R. Bellman and R. Kalaba, On Adaptive Control Processes. IRE National Convention Record, Part 4 (1959).
13. M. Freimar, A Dynamic Programming Approach to Adaptive Control Processes. IRE National Convention Record, Part 4 (1959).
14. A. A. Fel'dbaum, Computers in Automatic Systems [in Russian] (Fizmatgiz, 1959).
15. A. A. Fel'dbaum, Problems of statistical theory of automatic optimization, Proc. of the 1st. International Congress of Automatic Control (IFAC) [in Russian] (Moscow, 1960).
16. D. Blackwell and M. A. Girshick, Theory of Games and Statistical Decisions [Russian translation] (IL, 1959).
17. C. K. Chow, An Optimum Character Recognition System Using Decision Functions. IRE Trans. EC-6, No. 4 (1957).
18. R. Bellman, "Dynamic programming and stochastic control processes," Information and Control 1, 3 (Sept., 1958).
19. L. S. Schwartz, B. Harris, and A. Hauptschein, Information Rate from the Viewpoint of Inductive Probability. IRE National Convention Record, Part 4 (1959).
20. R. L. Dobrushin, "Transmission of information in channels with feedback," Teor. Ver. i ee Prim. 3, 4 (1958).

* See English translation.