Randomized Digital Optimal Control (RDOC)

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Digital Control Systems

\[ H(t_i) \xrightarrow{\text{Cont. syst.}} c(t_i) \xrightarrow{C} \text{Sample} \xrightarrow{S(t_{i+1})} \]

\[ u(t_i) \xrightarrow{\text{Dig. Comp.}} y(t_i) \]

\[ T_i \]

\[ \text{Computation} \]

Sampling

- Det. \((t_i \text{ known})\)
- Stoch. \((p(t_i) \text{ known})\)

Periodic \((t_i = iT)\)

Non-periodic \((T_{i+1} \neq T_i)\)

Periodic \((t_{i+1} = iT + T_i)\)

Additive \((t_{i+1} = t_i + T_i)\)

In interval \([t_i, t_{i+1}]\)

Non-instant. \((t_{i+1} \text{ known})\)

Instant. \((t_{i+1} \text{ not known})\)

Non-instant. \((t_{i+1} \text{ not known})\)

Infinite. \((t_{i+1} \text{ not known})\)
Digital Optimal Control Problem

Digital systems (DS):

\[ C_8: \quad \dot{x}(t) = Ax(t) + Bu(t) + b(t), \quad t \geq 0 \]

\[ H: \quad u(t) = u(t_i), \quad t_i \leq t < t_{i+1}, \quad i = 0, 1, \ldots \]

\[ S: \quad y(t_i) = Cx(t_i) + w(t_i), \quad i = 0, 1, \ldots \]

\[ x(t) \in \mathbb{R}^n \]

\[ u(t) \in \mathbb{R}^m \]

\[ y(t_i) \in \mathbb{R}^l \]

\[ b(t) \in \mathbb{R}^n \]

\[ w(t_i) \in \mathbb{R}^e \]

\[ A, B, C \text{ real matrices of appropriate dimensions.} \]

\[ A, B, C \text{ known} \]

\[ x(t_0) = x_0, \quad u(t), \quad w(t_i) \text{ independent with} \]

\[ E \{x_0\} = \overline{X}_0, \quad \text{cov}(x_0) = \Sigma \]

\[ E \{u(t)\} = \mathbf{0}, \quad \text{cov}(u(t)) = \mathbf{V} \delta(t - s) \]

\[ E \{w(t_i)\} = \mathbf{0}, \quad \text{cov}(w(t_i)) = \mathbf{W} \delta_{ij} \]

\[ G, V, W \text{ real symmetric and } > 0. \]

\[ x_0, G, V, W \text{ known} \]
Continuous-Time Criterion (CC):

}\begin{align*}
\mathcal{O}_{\text{CC}}(u(t)) &= \lim_{t_f \to \infty} \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \mathbb{E}[x^2(t)q(x(t)) + u^2(t)R(u(t))\mathcal{J}(t)] dt \\
\end{align*}\]

\text{QR Real symmetric and } > 0.

\text{QR Known}.

\text{Ult} = \{u(t_0), \ldots, u(t_i)\}.

\text{Sampling:}

\begin{align*}
T_i &= t_i + t_{i-1}, \quad i = 0, 1, \ldots \\
\{T_i\} &\text{ a sequence of } \text{i.i.d.} \text{ independent and identically distributed (IID) stochastic variables} \\
\text{indep. of } x_0, \{u(t_i)\}, \{w(t_i)\}.
\end{align*}

\text{p}(T_i) \text{ Known}.

\begin{align*}
0 < \alpha &\leq T_i \leq \beta < \infty, \quad i = 0, 1, \ldots, \alpha, \beta \text{ pos. real scalars.}
\end{align*}

Discrete Opt. Control Prob. (DOCP)

\begin{align*}
Y_i &= \{y(t_0), \ldots, y(t_i)\} \\
\text{Ult} &= \Phi(Y_{i-1}, U_{i-1}), \quad i = 0, 1, \ldots, \Phi \text{ def.}.
\end{align*}

Given DS and CC, find \(U^*_0 = \{u^*(t_0), u^*(t_i), \ldots\} \) which minimizes \(\mathcal{O}_{\text{CC}}(U_0)\) and find the min. sol. \(\mathcal{O}_{\text{CC}}^*\).
Equivalent Discrete-Time Opt. Control Problem

\[ t_i \rightarrow s \iff x_i \rightarrow x_s \]
\[ x_i = x(t_i), \quad u_i = u(t_i), \quad y_i = y(t_i) \]
\[ p(x_i) \text{ independent of } i, \text{ then } s_i \text{ denoted by } \bar{s} \]

Equivalent Discrete-Time System (EDS):

\[ x_{i+1} = \Phi(x_i) x_i + \Gamma u_i + \eta_i, \quad i = 0, 1, \ldots \]
\[ y_i = C x_i + \omega_i, \quad i = 0, 1, \ldots \]
\[ \Phi_i = \Phi(T_i) = e^{A T_i} \]
\[ \Gamma_i = \Gamma(T_i) = \int_0^{T_i} \Phi(s) B ds \]
\[ V_i = V(T_i) = \int_0^{T_i} \Phi(s) V \Phi^T(s) ds \]
\[ x_0, \{ u_i \}, \{ w_i \} \text{ independent with } \]
\[ E(x_0) = \bar{x}_0, \quad \text{cov}(x_0) = \Sigma \]
\[ E(u_i) = 0, \quad \text{cov}(u_i) = \Sigma_{u_i} \]
\[ E(w_i) = 0, \quad \text{cov}(w_i) = \Sigma_{w_i} \]

\( u_i, \Gamma_i, \{ v_i \}, \{ w_i \} \text{ are i.i.d. } \]
\( x_i, \{ u_i \}, \{ w_i \} \text{ independent with } \bar{s}, \Sigma \)
\( \Phi, \Gamma_i \text{ independent of } s_i, i \in J, \text{ and uncorrelated with } \bar{s} \).
\[ U = B \sum_{i} w_i \bar{y}_i, \quad U > 0 \]
\[ \bar{y}_i, p(\Phi_{\eta_i}), p(T_i) \text{ known} \]

(asymptotic) behaviour of the DS at the sampling instants is identical to the (asymptotic) behaviour of the EDS.

**Equivalent Discrete-Time Criterion (EDC):**

\[
\delta_{\infty}(U_0) = \lim_{T \to \infty} \frac{1}{N} \left[ \sum_{i=1}^{N} x_i \bar{y}_i + 2 \sum_{i=1}^{N} u_i + \sum_{i=1}^{N} \bar{y}_i \right]
\]

\[ + \frac{1}{T} \tilde{y} \]

\[
Q_2 = Q(T_i) = \int_{0}^{T_i} \Phi^T(s) \Psi \Phi(s) ds
\]
\[ M_2 = M(T_i) = \int_{0}^{T_i} \Phi^T(s) \Psi M(s) \Phi(s) ds \]
\[ R_2 = R(T_i) = \int_{0}^{T_i} \left[ R + \Psi^T(s) \Psi M(s) \right] ds \]
\[ \delta_2 = \delta(T_i) = \int_{0}^{T_i} 2 \left[ \Psi(s) \Phi(s) \right] ds \]
\[ \bar{Q}, \bar{M}, \bar{R}, \bar{\delta} \text{ known} \]

\[ U_0 = \phi \left( \bar{y}_{i-1}, U_{i-1} \right), \text{ } \phi \text{ Cet } \]

Value of CC = Value of EDC.

Using Lebesgue Dom.-Conv. Th.: \[ \lim_{N \to \infty} \frac{1}{T_i} \tilde{y} \]

strong law of large numbers: \[ \frac{1}{T_i} \tilde{y} \xrightarrow{a.s.} 0 \text{ as } i \to \infty \]

\[ u_t = \phi (Y_{t-1}, u_{t-1}), \quad t = 0, 1, \ldots \]

Given EDS, EOC,
Find \( u^*_0 = [u^*_0, u^*_1, \ldots] \) which minimizes \( J(u_0) \) and find the min. vol. \( \overline{\text{Vol}} \).

Solution DOCP \( \equiv \) Solution EDOCP.
System Theory

\[ x_{i+1} = \Phi_i x_i, \quad i=0,1, \ldots \quad (\Phi_i) \]
\[ \Phi_i \in \mathbb{R}^{n \times n} \]
9. \( \Phi_i \) is a seq. of indep. rand. matz's with const. stats.
\[ \text{x}_{\text{det}} - \text{ms}: \text{mean square} \]

(\( \Phi_i \)) \text{ ms-stable} if \( \|x_i\|_2 \to 0 \) as \( i \to \infty \), \( \forall x_0 \).

\[ x_{i+1} = \Phi_i x_i + \eta_i u_i, \quad i=0,1, \ldots \quad (\Phi_i, \eta_i) \]
9. \( \Phi_i, \eta_i \) is a seq. of indep. rand. matz's with
\[ \text{const. statistics} \]
\[ \text{x}_{\text{det}} - \text{Suppose } u_i = -L x_i, \quad L \text{ real matz} \]
\[ x_{i+1} = (\Phi_i - \eta_i L) x_i, \quad i=0,1, \ldots \]
(\( \Phi_i, \eta_i \)) \text{ ms-stabilizable if } L \in (\Phi_i - \eta_i L)
\[ \text{ms-stable}. \]

\[ x_{i+1} = \Phi_i x_i, \quad y_i = C_i x_i, \quad i=0,1, \ldots \quad (\Phi_i, C_i) \]
9. \( \Phi_i, C_i \) is a seq. of indep. rand. matz's with
\[ \text{const. stats} \]
\[ \text{x}_{\text{det}} - \quad (\Phi_i, C_i) \text{ ms-detectability} \text{ if } \|y_i\|^2 = 0, \quad i=0,1, \ldots \quad \Rightarrow \]
\[ \|x_i\|_2 \to 0 \text{ as } i \to \infty \].
There are explicit conditions for ms-stably and ms-detly which are easy to calculate.

\((\Phi_i)\) ms-stable \(\Rightarrow\) \((\Theta_i, \Pi_i)\) ms-stabilizable.
\((\Phi_i, C_i)\) ms-detetable

\((\Theta_i, \Pi_i, C_i)\) def. and constant \(\Rightarrow\)

ms-stability
ms-stabilizability

\[\#\] 

in the usual sense.
Solution eq. discr. time opt. cont. probl.

$S^n$: lin. sp. of real symm. $n \times n$ matrs $\Phi^n$. Generalized discr. time Rec. trans. $B_i S^n \to S$

$B_i x = \Phi_i x + \bar{\Phi}_i - L_x (\Phi_i x + R)L_x, x \in S^n$

$L_x = (\Phi_i x + R)^{-1}(\Phi_i x + \bar{\Phi}_i + L_x^T), x \in S^n$

$\Phi_i' = \Phi_i - P_i R M^T$

$\bar{\Phi}_i' = \bar{\Phi}_i - M \bar{R}^{-1} M^T$

$\hat{x}_i$: min. error var. opt. estimator of $x_i$ given $y_{i-1}$ and $u_{i-1}$. Error variance is $E[(x_i - \hat{x}_i)^T (x_i - \hat{x}_i)]$

$P_i = E[(x_i - \hat{x}_i)(x_i - \hat{x}_i)^T]$ is the estimator error covariance.

$\hat{x}_i$ is a det. funct. of $y_{i-1}$ and $u_{i-1}$.

An arbitrary, not nec. opt. estimator $\hat{x}_i$ is variance neutral if $P_i$ is not a function of $u_{i-1}$. 
Solution to DDCP

\[ \tilde{R} > 0 \]
\[ \tilde{z}_1 \] \text{ is neutral, } \tilde{z} = \tilde{z}_1 \text{ --- } \\
\[ P = \lim_{\varepsilon \rightarrow \infty} \frac{1}{\varepsilon} \sum_{k=0}^{\infty} P_k \text{ exists } \]

\[ (\Phi_{\tilde{z}}^1, \tilde{R}) \text{ is stable, } (\Phi_{\tilde{z}}^1, \tilde{Q}_1) \text{ is det. } e \Rightarrow \\
S = \lim_{\varepsilon \rightarrow \infty} B_{\tilde{z}}^\varepsilon \text{ exists } \]

S is the unique nonneg. def. sol. of
\[ S = B_{\tilde{z}}^\varepsilon S. \]

\[ \tilde{u}_1 = [-L_0 \tilde{x}_0, -L_0 \tilde{x}_1, \ldots] \]
\[ (\Phi_{\tilde{z}} - \tilde{R}_L) \text{ is ms-stable } \]
\[ C^* = \frac{1}{\varepsilon} [VS + (\Phi^* S \Phi + \tilde{Q} - S)P] + \frac{1}{\varepsilon} \]

\[ \tilde{u}_1 \text{ is a det. funct. of } \tilde{z}_1 \]
\[ \tilde{z}_1, \tilde{z} = \tilde{z}_1 \text{ ---, var. neutral } \Rightarrow \text{DDCP separable. } \]

Stability of total syst. depends also on the stability of the estimator.
Now suppose \( \varphi_2(T_2) = 0 \), and the opt. LQG gain is \( L^D \), and the opt. estimator is \( \hat{x}_i^D \).

**Eq. discr.-time subopt. cont.-prob. (EDSOCP):**

\[
 u_i = -L^D \hat{x}_i^D, \quad i = q_i, \quad \text{--- Find the min. val. } P_{S^D}.
\]

**Define**

\[
 B_{LD} : S^n \rightarrow S^n \quad (B_{x^*} \text{ with } L^D):
\]

\[
 B_{LD} X = \Omega^T \Phi + \Omega - (\Phi^T T + R) L^D - L^T (\Omega^T \Phi + \Omega^T T + R) L^D, \quad X \in S^n.
\]

**Solution EDSOCP**

\[
 \bar{R} > 0
\]

\[
 P = \lim_{i \to \infty} \frac{1}{i} \sum_{k=0}^{i} P_k \quad \text{exits}
\]

\[
 U_{LD} = \{ -L^D \hat{x}_0, \quad -L^D \hat{x}_1, \quad \ldots \}
\]

\[
 (\Phi_1 - R_i L) \text{ms- stable} \Rightarrow
\]

\[
 S_{LD} = \lim_{i \to \infty} B_{LD}^i \Theta \quad \text{exits}
\]

\[
 S_{LD} > 0
\]

\[
 S_{LD} \text{ is the unique sol. of } S_{LD} = B_{LD} S
\]

\[
 \theta_{LD} = \frac{1}{2} \left[ v S_{LD} + (\Phi^T S_{LD} \Phi - S_{LD}) P + \frac{1}{2} \right]
\]
Optimal Estimation

Suppose $\hat{x}_i$ an arbitrary, not nec. opt.,
estimator of $x_i$.

$B = 0, \hat{x}_0, ..., \hat{x}_{i+1}$ \textit{w.r.} \text{r.p.} \Rightarrow T_0, ..., T_i$ \text{det.}

So we have to know the realization of $T_i$ before determining $\hat{x}_{i+1}$.

Therefore we assume \textit{intentional stock sampling}\ (ISS), i.e. the comp. generates $T_i$ first in the interval $[t_i, t_{i+1})$ and then determines $x_{i+1}$ on the basis of $Y_i, U_i, T_0, ..., T_i$.

Thus

$x_{i+1} = \Phi_i x_i + \Gamma_i U_i + \epsilon_i, \quad i = 0, 1, \ldots$

$Y_i = C X_i + W_i, \quad i = 0, 1, \ldots$

$\Phi_i = \Phi(T_i), \quad \Gamma_i = \Gamma(T_i), \quad V_i = V(T_i)$ \text{known}.

Now we have a discrete-time syst. with
\text{time-dep. p.a.}'s.
\[ \hat{x}_i = \text{lin. min. var. (opt. lin.) estimator of } x_i \text{ given } y_{i-1}, u_{i-1} \]
\[ P_{\hat{x}}: \text{estimator error cov.} \]

**Optimal linear ISS estimator:**
\[ \hat{x}_{i+1} = \Phi_{i-1} \hat{x}_i + \Gamma_i u_i + K_i (y_i - C \hat{x}_i), \quad \hat{x}_0 = x_0, \]
\[ K_i = \Phi_i P_{\hat{x}}^{-1} (CP_i C^T + W)^T \]
\[ P_{i+1} = \Phi_i P_{\hat{x}}^{-1} \Phi_i^T + V_i - K_i (CP_i C^T + W)K_i^T, \quad P_0 = \mathbf{I} \]

+ Moore - Penrose pseudo inverse
\[ \hat{x}_i \text{ var. neutral} \]

**Syst.** \((\Phi_i, \gamma_i, C)\) a realization of the EDS, dep. on a real. of \(\gamma_i\), and \(x_i\) given by the opt. lin. ISS estimator.

\(W_{70}, (\Phi_i, C)\) a.s. uniformly det., \((\Phi_i, V_i)^{\frac{1}{2}}\)

**Estimator:** \(\hat{x}_i\) a.s. exponentially stable,
\(P_{\hat{x}}\) a.s. bounded,
\(P = \lim_{k \to \infty} \sum_{i=0}^{k} P_{\hat{x}} \text{ a.s. exists} \)
Suppose $T_x^1 = T$, thus $\Phi_x^2 = \Phi(T) = \Phi_x$, $\Pi_x = \Pi(T_x) = \Pi$.

- $(A, B)$ stable, $(A, C)$ det'ed, $d(A)$ real $\Rightarrow$
  $\Phi, \Pi$ stable, $(\Phi, C)$ det'ed.

- $(A, B)$ stable, $(A, C)$ det'ed, $d(A)$ complex, $T \neq \frac{kT}{|\text{Din}(g(A))|}$, $k = 0, 1, \ldots$ $\Rightarrow$
  $(\Phi, \Pi)$ stable, $(\Phi, C)$ det'ed.

$d(A)$ complex: we may lose stability of $(\Phi, \Pi)$ and det'ed of $(\Phi, C)$, thus sol. of LQG prob. may break down.

$\text{det}(T_x) > 0 \Rightarrow$ no loss of unsur. stability of $(\Phi_x, \Pi_x)$ and unsur. det'ed of $(\Phi_x, C)$.

Thus sol. of LQG prob. does not break down.
$T_i$ uniformly distributed:

\[ P(T_i) \]

\[ 0 \rightarrow T \rightarrow T_i \]

Mean $\bar{F}$

\[ \text{Var}(T_i) = E[ (T_i - \bar{F})^2 ] \]

$T_i > 70 \Rightarrow \text{Var}(T_i) < \frac{1}{3} \bar{F}^3$

Denote $\theta_0$ with $\frac{\text{Var}(T)}{\bar{F}} = \theta$ as $\theta_0(\bar{F})$.

Consider:

$\hat{\theta}_1^* (0)$ Opt., no stock $T_i$.

$\hat{\theta}_3^* (0)$ Opt., stock $T_i$ taken into account.

$\hat{\theta}_2^* (0)$ Subopt., stock $T_i$ not taken into account, indep. of $\bar{F}$.
Assume

\[ A = \begin{bmatrix} 0.01 & 1 \\ 0 & 0.01 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \]

\[ V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0.01 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [0]. \]

\((A, B)\) stable, \((A, C)\) detectable

\(\lambda(A) = 0.01, 0.01\), thus, \(\lambda(A)\) and unstable

\(\text{Var}(T) < 0 \Rightarrow (\Phi, \eta)\) stable, \((\Phi, \sigma)\) detectable

\(\Rightarrow \theta_0^*\) ex. and LP.

\(\text{Var}(T) > 0 \Rightarrow \text{R}\theta_0, (\eta_x, \eta^*_x)\) ms-stable, \((\Phi, \sigma)\)

a.s.univ. detectable

\(\Rightarrow \theta_0^*\) ex.

\(\text{Var}(T) > 0 \Rightarrow (\phi_{x, \eta_{x}(0)}^L)\) ms-stable

\(\Rightarrow \theta_0^L\) ex.

\[ \text{Graph} \]

\[ \begin{array}{c}
\text{Y-axis: 500} \\
\text{X-axis: 25} \\
\text{Other axes: 0}
\end{array} \]

\[ q \rightarrow T \]
Assume

\[ A = \begin{bmatrix} 0.01 & -1 \\ 1 & 0.01 \end{bmatrix}, \quad B, C, U, W, P, R \text{ the same.} \]

(A, B) reachable, \((A, C)\) det\(C\).

\[ A(A) = 0.01 \pm i, \quad \text{thus complex and unstable.} \]

\[ \text{Var}(T) = 0 \implies (B, P) \text{ not stable, } (B, C) \text{ not det} C. \]

\[ \text{Var}(T) > 0 \implies \mathbb{R} > 0, \text{(}\Phi, \mathbb{P}^\infty\text{) ms-stable, } (B, C) \text{ a.s. det} C. \]

\[ \implies \Phi^*(C) \text{ exists}. \]

\[ \text{Var}(T) > 0 \implies (\Phi, -\mathbb{P}^\infty \mathbb{L}^D) \text{ ms-unstable. For } \]

\[ T = k\pi, \quad k = 1, 2, \ldots \]

then \(\Phi^*(C)\) does not exist.
Conclusions

Named effects arise in general for cont. - time syst. with \((A,B)\) stabilizable, \\
\((A,C)\) detectable, and general distirb's of \(T_i\)

Especially if \(\text{var}(T) > 0\):
- Take stock. samp. into account. Then no asympotes, furthermore almost the same behav.
- Take stock. not into account. Then yes asympotes, furthermore almost the same behav.

With stock samp. more infor. comes out of the syst. Ms-obs' of system increases.

USE IT OR LOSE IT principle:
- Stock samp. may incr. or restore stab.'y \\
  if taken into account in determining the opt. cont.'s
- Stock samp. may decr. or even destroy stab.'y \\
  if not taken into account in determining the opt. cont.'s

Incomplete state infor.:
- Stock samp. intentional, then opt. cont.' problems is separable.
- Stock samp. not intentional, then opt. cont.' \\
  prob. is not separable. This problem is much more difficult.
Compulsory Exercises

Pick up from the course internet site the paper:
W.L. de Koning, L.G. van Willigenburg
Randomized digital optimal control
Nonuniform sampling: Theory and Practice,

Exercises

1. Suppose the state inform. is incompl.
   and the shoot. sampl. is not intentional.
   Then the EDCOP is not separable.
   Give suggestions to solve this 
   problem in an optimal way.

2. Reproduce fig.3 and fig.5 from 
   the above mentioned paper in Matlab,
   using the software package on the 
   course internet site.
Solution compulsory exercise 1.

What to do and how to solve the problem when the stochastic sampling is not intentional is described in the conclusion of the paper where the reader is referred to reference [28] and [29]. Reference [28] and [29] solve the associated non-separable control problem in the infinite horizon time-invariant and finite horizon time-varying case respectively.

Solution compulsory exercise 2.
See the Matlab solution m-files.