Including Errata

CONTROL–THEORY AND ADVANCED TECHNOLOGY
Vol. 10, No. 4, Part 5, pp. 2083–2098, December, 1995

C93093RRRR

Contributed Paper

DERIVATION AND COMPUTATION OF THE DIGITAL LQG REGULATOR AND TRACKER FOR TIME-VARYING SYSTEMS IN THE CASE OF ASYNCHRONOUS AND APERIODIC SAMPLING*

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Abstract. In practice, the frequent, synchronous and periodic updating of controls and observations is often undesirable or impossible. As opposed to conventional digital control, in the case of asynchronous and aperiodic sampling, the frequent, synchronous and periodic updating of controls and observations is no longer assumed. In the case of asynchronous and aperiodic sampling, an *arbitrary* number of control variables is updated, and an *arbitrary* number of outputs is sampled at *arbitrary* time instants. This sampling scheme generalizes many deterministic sampling schemes considered in the control literature. The derivation and computation of the digital LQG regulator and tracker for time-varying systems in the case of asynchronous and aperiodic sampling is presented. The digital LQG regulator constitutes a truly implementable compensator for nonlinear systems having to track reference state-trajectories. It explicitly accounts for the inter-sample behavior since it is based on an integral cost functional. This also holds for the digital LQG tracker which applies to a linear system tracking a reference state-trajectory. The computation of the digital LQG regulator and tracker is considered and illustrated with a numerical example.

Key Words—Asynchronous aperiodic sampling, multi-rate sampling, sampled-data time-varying systems, digital LQG controllers.

1. Introduction

Most digital control system design procedures, put forward in the control literature, assume frequent, synchronous and periodic updating of controls and observations. In practice however, this is often undesirable or impossible. In the process industry, the economy and in the area of environmental control, this is due to different analyses and costs associated with measurements, different costs associated with updating control variables, actuator constraints and the locally distributed nature of the process. In the case of digital control of mechanical and electrical systems, this is due to limited I/O capabilities, limited computing power and limited computer memory of the digital controller.

Given these practical constraints, in general, the updating of an arbitrary number of control variables, as well as the sampling of an arbitrary number of outputs, may occur at arbitrary time instants. We will refer to this as asynchronous and aperiodic sampling. Because of analyses that may be involved, we also

* Received by the editors July 26, 1993 and in finally revised form July 25, 1995.

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consider the situation when observations, made at a certain time-instant, become available only sometime later on. This sampling scheme, we believe, generalizes many deterministic sampling schemes considered in the control literature. It, for instance, generalizes conventional sampling, multi-rate sampling, non-synchronous sampling and multiple-order sampling considered by Kalman and Bertram (1959) in their theory of sampling systems. Especially multi-rate digital control systems, characterized by multi-rate sampling, have received attention since then. Most analyses however, is restricted to the control of continuous time-invariant linear systems (Colaneri et al. (1992) and references therein).

To the best knowledge of the authors, digital LQG control, in the general case of asynchronous and aperiodic sampling and time-varying systems, has received no attention so far. It is however, of great practical importance, for instance to design and compute implementable digital compensators for asynchronous and aperiodically sampled nonlinear systems tracking (optimal) reference state-trajectories. Examples are a batch fermentation process and robot motion. The linearized dynamics about the trajectory in these cases constitute a *time-varying* system. Van Willigenburg (1995) treated the computation of digital optimal controls and associated state-trajectories for deterministic nonlinear systems, in the case of asynchronous and aperiodic sampling. Together with the results presented here, this allows for the design and computation of asynchronous and aperiodically sampled digital optimal control systems (Athans, 1971; Van Willigenburg, 1991).

In the case of aperiodic and synchronous sampling, where at arbitrary sampling instants all controls and observations are updated simultaneously, we have the digital LQG regulator (Halyo and Caglayan, 1976; De Koning, 1980; 1984) and tracker (Van Willigenburg and De Koning, 1992) for time-varying systems. Through a piecewise constant constraint on the control the digital nature of the controller is explicitly taken into account and, through the use of an integral cost functional, the inter-sample behavior is explicitly considered. Therefore a small sampling time is not required. This relaxes the computational burden on the computer and circumvents the demand to update controls and observations frequently. To solve digital LQ and LQG problems, they are generally transformed into unconstrained equivalent discrete-time problems (Levis et al., 1971; Dorato and Levis, 1971; Halyo and Caglayan, 1976). Until recently, computation of the equivalent discrete-time criterion matrices was only considered for time-invariant problems (Van Loan, 1978). The extension of this computation to the time-varying case is not straightforward. This was demonstrated by Van Willigenburg (1991; 1993) who resolved this problem.

In this paper the derivation and computation of the digital LQG regulator and tracker for time-varying systems in the general case of asynchronous and aperiodic sampling is presented. The paper is organized as follows. In Sec. 2 we present the digital LQG regulator and tracking problem, the first being a special case of the latter. In Sec. 3 the digital LQG tracking problem is converted into an unconstrained equivalent discrete-time problem. Then this discrete-time problem formulation is modified to describe the partial update of the controls. One of the characteristics of this modified problem formulation is that the equations obtain time-varying dimensions. In Sec 4 the solution, which has the certainty equivalence property, is presented. The feedback and feedforward, which can be computed *a priori*, follow from the modified equivalent discrete-time problem formulation, through iteration of matrix difference equations having time-varying dimensions. Extensions and simplifications of the problem and its solution are presented in Sec. 5. Section 6 presents a numerical example while Sec. 7 concludes the paper.

2. Problem Formulation

Consider the stochastic continuous time-varying linear system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t), \quad A(t) \in \mathscr{R}^{n\times n}, \quad B(t) \in \mathscr{R}^{n\times m}, \tag{1a}$$

where $\{v(t)\}$ is a continuous time zero-mean white noise process,

$$E\{v(t)\} = 0, \quad \operatorname{cov}(v(t), v(s)) = V(t)\delta_D(t-s), \quad V(t) \in \mathscr{R}^{n\times n}. \tag{1b}$$

In Eq. (1b), $\delta_D(\cdot)$ is the Dirac delta function. The *a priori* known initial state statistics are

$$E\{x(t_{u_0})\} = \bar{x}(t_{u_0}), \quad \operatorname{cov}(x(t_{u_0}), x(t_{u_0})) = G, \quad G \in \mathscr{R}^{n\times n}, \tag{1c}$$

where t_{u_0} denotes the first time instant at which some control variables are updated. The non-updated control variables at t_{u_0} are assumed to be deterministic and known. This will be described later on.

To describe the asynchronous and aperiodic sampling, we define an *a priori* known set of control instants and an *a priori* known set of observation instants, respectively:

$$T_u = \{t_{u_c}, \quad c = 0, 1, 2, \cdots, C-1, \quad t_{u_c} > t_{u_{c-1}}\}, \tag{1d}$$

$$T_m = \{t_{y_l}, \quad l = 1, 2, \cdots, L, \quad t_{y_l} > t_{y_{l-1}}\}. \tag{1e}$$

At each control instant t_{u_c}, $c = 0, 1, 2, \cdots, C-1$ one, several or all m control variables are updated while the others remain unchanged. The *a priori* known sets U_c, $c = 0, 1, 2, \cdots, C-1$ describe which control variables are updated at each control instant. They contain the m_c indices, $1 \le m_c \le m$ of updated control variables at t_{u_c}; i.e.,

$$\left.\begin{array}{c} \operatorname{card}(U_c) = m_c, \quad 1 \le m_c \le m \\ i \in U_c \Leftrightarrow u_i \text{ is updated at } t_{u_c}, \quad i = 1, 2, \cdots, m, \quad c = 0, 1, \cdots, C-1 \end{array}\right\}. \tag{1f}$$

In accordance with (1f) t_{u_C} is the final time involved in the digital LQG problem which satisfies,

$$t_{u_C} > t_{u_{C-1}}. \tag{1g}$$

After each control instant all control variables remain unchanged until the next control instant through the use of zero-order hold circuits,

$$u(t) = u(t_{u_c}), \quad t \in [t_{u_c}, t_{u_{c+1}}), \quad c = 0, 1, 2, \cdots, C-1. \tag{1h}$$

At each observation instant t_{y_l}, which may be equal to a control instant, one, several or all p outputs of the system are sampled. The *a priori* known sets Y_l describe which outputs are sampled at each observation instant. They contain the p_l indices, $1 \le p_l \le p$, of the sampled outputs at t_{y_l}; i.e.,

$$\left.\begin{array}{c} \mathrm{card}(Y_l) = p_l, \quad 1 \le p_l \le p \\ i \in Y_l \Leftrightarrow y'_i \text{ is sampled at } t_{y_l}, \quad i = 1, 2, \cdots, p, \quad l = 1, 2, \cdots, L \end{array}\right\}. \tag{1i}$$

In Eq. (1i) y' is given by the output equation,

$$y'(t_{y_l}) = C'(t_{y_l})x(t_{y_l}) + w'(t_{y_l}), \quad l = 1, 2, \cdots, L, \quad C'(t_{y_l}) \in R^{p \times n}, \tag{1j}$$

where $w'(t_{y_l})$ is a discrete-time zero-mean white noise process,

$$\left.\begin{array}{c} E\{w'(t_{y_l})\} = 0, \quad \mathrm{cov}(w'(t_{y_i}), w'(t_{y_l})) = W'(t_{y_l})\delta_{il} \\ W'(t_{y_l}) \in \mathscr{R}^{p \times p}, \quad i = 1, 2, \cdots, L, \quad l = 1, 2, \cdots, L \end{array}\right\}. \tag{1k}$$

In Eq. (1k) δ_{il} is the Kronecker delta. To obtain a causal and on-line computable control algorithm, we assume that the information available to compute the control updates at t_{u_c} consists of all observations and controls *preceding* t_{u_c}. In this case the time between t_{u_c} and the latest observation or control instant preceding t_{u_c} is available to compute the control update at t_{u_c}. Given this scheme, taking samples at or after the last control instant $t_{u_{C-1}}$ is useless. Information obtained from measurements before the initial control instant t_{u_0} can be incorporated in Eq. (1c). Therefore we assume

$$t_{y_l} \in [t_{u_0}, t_{u_{C-1}}), \quad l = 1, 2, \cdots, L. \tag{1l}$$

It may happen because of the analyses involved (for instance, given certain chemical or economical measurements), that past observations are *not yet* available at t_{u_c}. This case will be treated separately in Sec. 5. Finally we assume that $x(t_{u_0})$, $\{v(t)\}$ and $\{w'(t_{y_l})\}$ are independent.

The digital LQG regulator problem for the system (1) is to minimize the cost function,

$$J = E\left\{x^T(t_{u_C})Hx(t_{u_C}) + \int_{t_{u_0}}^{t_{u_C}} x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)dt\right\}. \tag{2}$$

The digital LQG tracking problem for the system (1) is to minimize the cost function,

$$E\left\{(x(t_{u_C}) - x_r(t_{u_C}))^T H(x(t_{u_C}) - x_r(t_{u_C})) \right.$$
$$\left. + \int_{t_{u_0}}^{t_{u_C}} (x(t) - x_r(t))^T Q(t)(x(t) - x_r(t)) + u^T(t)R(t)u(t)dt\right\}, \tag{3a}$$

where

$$x_r(t), \quad t \in [t_{u_0}, t_{u_C}] \tag{3b}$$

is the reference state-trajectory to be tracked. In Eqs. (2) and (3a), $Q(t)$, H and $R(t)$ are symmetric matrices that satisfy,

$$R(t) \geq 0,\ Q(t) \geq 0,\ H \geq 0,\ R(t) \in \mathscr{R}^{m\times m},\ Q(t) \in \mathscr{R}^{n\times n},\ H \in \mathscr{R}^{n\times n}. \quad (4)$$

From Van Willigenburg and De Koning (1992), observe that the assumption $R(t) \geq 0$ is sufficient *in general* for the results to hold while $R(t) > 0$ is *strictly* sufficient. Since the digital LQG regulator problem is a special case of the digital LQG tracking problem, i.e. the case where $x_r(t) = 0$, from now on only the digital LQG tracking problem will be considered.

3. Equivalent Discrete-time Control Problem Formulation

The equivalent discrete-time system, which describes the state transitions of the system (1) from each control instant to the next is represented by

$$x_{c+1} = \Phi_c x_c + \Gamma_c u_c + v_c, \quad c = 0, 1, 2, \cdots, C-1, \quad (5a)$$

where the index c refers to t_{u_c} and v_c is a discrete-time zero mean white noise process; i.e.,

$$E\{v_c\} = 0, \mathrm{cov}(v_i, v_c) = V_c\delta_{ic}, i = 0, 1, 2, \cdots, C-1, c = 0, 1, 2, \cdots, C-1. \quad (5b)$$

The equivalent discrete-time cost function, which describes the costs (3) as a function of the state and control at the control instants, is represented by

$$J = E\left\{ \sum_{c=0}^{C-1} x_c^T Q_c x_c + 2x_c^T M_c u_c + u_c^T R_c u_c - 2\lambda_c x_c - 2\tau_c u_c \right\}$$
$$+ E\{x_C^T H x_C - 2x_r^T(t_{u_C}) H x_C\} + \sum_{c=0}^{C-1} \xi_c + \gamma_c. \quad (6)$$

The system and criterion matrices in Eqs. (5), (6) can be computed numerically from those in (1a)–(1c) and (2) (Van Willigenburg and De Koning, 1992; Van Willigenburg, 1993).

In the equivalent discrete-time tracking problem (5), (6) u_c appears as the control. From the Eq. (1f) observe that in general not all of the control variables are updated at each control instant. The actual control at each control instant consists of only the updated control variables. Therefore a problem formulation is required in which only the updated control variables appear as the control. This problem formulation is obtained in two steps. First we rearrange the control variables u_c into u_c', which separates into a first part u_c^u, containing the updated control variables and a second part u_c^0, containing the unchanged control variables,

$$u_c' = \begin{bmatrix} u_c^u \\ u_c^0 \end{bmatrix}. \quad (7a)$$

For each control instant t_{u_c}, $c = 0, 1, \cdots, C-1$, this rearrangement is defined by two one to one mappings, $U_{u_c}(\cdot)$ and $U_{0_c}(\cdot)$.

$$U_{u_c}(i) = j, \quad i \in \{1, 2, \cdots, m_c\}, \quad j \in U_c, \quad c = 0, 1, 2, \cdots, C-1 \quad (7b)$$

indicates that the updated control variable $u^u_{c_i}$ corresponds to u_{c_j} and

$$U_{0_c}(i) = j, \quad i \in \{1, 2, \cdots, m - m_c\}, \quad j \in M \backslash U_c, \quad c = 0, 1, 2, \cdots, C-1 \quad (7c)$$

indicates that the unchanged control variable $u^0_{c_i}$ corresponds to u_{c_j}. Here i and j are indices of vector elements. The set U_c defined by (1f) contains the m_c indices of updated control variables at t_{u_c}, and the set $M \backslash U_c$ contains the indices of the unchanged control variables at t_{u_c} with

$$M = \{1, 2, \cdots, m\}. \quad (7d)$$

Given this rearrangement of u_c into u'_c, we can reformulate the equivalent-discrete time problem (5), (6). Equation (5) converts into

$$x_{c+1} = \Phi_c x_c + \Gamma'_c u'_c + v_c, \quad c = 0, 1, 2, \cdots, C-1, \quad (8a)$$

while Eq. (6) becomes

$$J = E\left\{ \sum_{c=0}^{C-1} x_c^T Q_c x_c + 2x_c^T M'_c u'_c + u'^T_c R'_c u'_c - 2\lambda_c x_c - 2\tau'_c u'_c \right\}$$
$$+ E\{x_C^T H x_C - 2x_r^T(t_{u_C}) H x_C\} + \sum_{c=0}^{C-1} \xi_c + \gamma_c \quad (8b)$$

with

$$\Gamma'_{c_i} = \Gamma_{c_j}, \quad M'_{c_i} = M_{c_j}, \quad \tau'_{c_i} = \tau_{c_j}, \quad (8c)$$

where i and j are column indices related through the mappings (7b), (7c). Finally,

$$R'_{c_{i,r}} = R_{c_{j,s}}, \quad (8d)$$

where the pair i, r and the pair j, s point to matrix elements and i and j, like r and s, are related through the mappings (7b), (7c).

From (8) we proceed to obtain the equivalent discrete-time problem formulation which contains the actual control u^u_c, given by (7a). Through augmentation of the state x_c, $c = 0, 1, \cdots, C-1$, with the unchanged control variables u^0_c, we are able to describe their influence properly. The augmented discrete-time system thus becomes

$$x^a_{c+1} = \Phi^a_c x^a_c + \Gamma^a_c u^u_c + v^a_c, \quad c = 0, 1, 2, \cdots, C-1, \quad (9a)$$

where

$$x^a_c = \begin{bmatrix} x_c \\ u^0_c \end{bmatrix}, \qquad x^a_c \in \mathscr{R}^{(n+m-m_c)\times 1}, \quad (9b)$$

$$\Phi^a_c = \begin{bmatrix} \Phi_c & \Gamma^2_c \\ 0 & I^2_c \end{bmatrix}, \qquad \Phi^a_c \in \mathscr{R}^{(n+m-m_{c+1})\times(n+m-m_c)}, \quad (9c)$$

$$\Gamma^a_c = \begin{bmatrix} \Gamma^1_c \\ I^1_c \end{bmatrix}, \qquad \Gamma^a_c \in \mathscr{R}^{(n+m-m_{c+1})\times m_c}, \quad (9d)$$

$$v_c^a = \begin{bmatrix} v_c \\ 0 \end{bmatrix}, \quad v_c^a \in \mathscr{R}^{(n+m-m_{c+1})\times 1}. \tag{9e}$$

Since the controls u_c^0 are deterministic, v_c^a is a discrete-time zero-mean white noise process characterized by

$$E\{v_c^a\} = 0, \quad \operatorname{cov}(v_i^a, v_c^a) = \begin{bmatrix} V_c & 0 \\ 0 & 0 \end{bmatrix} \delta_{ic} = V^a \delta_{ic} \in \mathscr{R}^{(n+m-m_{c+1})\times(n+m-m_{c+1})},$$
$$i = 0, 1, 2, \cdots, C-1, \quad c = 0, 1, 2, \cdots, C-1. \tag{9f}$$

Γ_c^1 and Γ_c^2 describe the influence of the updated control variables and the unchanged control variables at t_{u_c} respectively, and given the division (7a) of u_c', are obtained from the corresponding division of Γ_c' given by

$$\Gamma_c' = [\Gamma_c^1 \;\; \Gamma_c^2], \quad \Gamma_c^1 \in \mathscr{R}^{n\times m_c}, \quad \Gamma_c^2 \in \mathscr{R}^{n\times(m-m_c)}. \tag{9g}$$

The matrices, I_c^1 and I_c^2, are such that the unchanged control variables u_c^0 in the augmented state x_c^a are properly updated,

$$I^1_{c_{i,j}} = 1, \quad \text{if } U_{0_{c+1}}(i) = U_{u_c}(j) \text{ else } I^1_{c_{i,j}} = 0, \quad I_c^1 \in \mathscr{R}^{(m-m_{c+1})\times m_c}, \tag{9h}$$

$$\left.\begin{aligned} I^2_{c_{i,j}} &= 1, \quad \text{if } U_{0_{c+1}}(i) = U_{0_c}(j) \text{ else} \\ I^2_{c_{i,j}} &= 0, \quad I_c^2 \in \mathscr{R}^{(m-m_{c+1})\times(m-m_c)} \end{aligned}\right\}, \tag{9i}$$

where the pair i, j, points to matrix elements, and the mappings $U_{u_c}(\cdot)$, $U_{0_c}(\cdot)$ are given by (7b), (7c). Given the description (9a) for the initial state we obtain

$$x_0^a = \begin{bmatrix} x_0 \\ u_0^u \end{bmatrix}, \tag{9j}$$

where the unchanged control variables u_0^u at the initial control instant t_{u_0} are assumed to be deterministic and known. Together with (9a) and (9f) we obtain for G^a, the covariance of the initial augmented state,

$$G^a = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}, \quad G^a \in \mathscr{R}^{(n+m-m_0)\times(n+m-m_0)}. \tag{9k}$$

Since the controls after the final time play no part in the problem for the final state we obtain,

$$x_C^a = x_C. \tag{9l}$$

Equation (9) describes a discrete-time system with a state x_c^a and a control u_c^u which have dimensions that vary with c. The time evolution of this system matches that of the original equivalent discrete-time system (5) for corresponding sequences of u_c and u_c' related by (7).

Finally, the equivalent discrete-time cost function (8b)–(8d) has to be adjusted in accordance with (9) so that it generates the same costs when the evolutions of

(5) and (8a) coincide. Therefore the partitioning of the matrices R_c', M_c', and τ_c', given by (8c), (8d), is introduced which corresponds to the partitioning (7a) of u_c',

$$\left.\begin{aligned} &R_c' = \begin{bmatrix} R_c^1 & R_c^2 \\ R_c^{2^T} & R_c^3 \end{bmatrix}, \quad R_c^1 \in \mathscr{R}^{m_c \times m_c} \\ &R_c^2 \in \mathscr{R}^{m_c \times (m-m_c)}, \quad R_c^3 \in \mathscr{R}^{(m-m_c)\times(m-m_c)} \end{aligned}\right\}, \tag{10a}$$

$$M_c' = [M_c^1 \quad M_c^2], \quad M_c^1 \in \mathscr{R}^{n \times m_c}, \quad M_c^2 \in \mathscr{R}^{n \times (m-m_c)}, \tag{10b}$$

$$\tau_c' = [\tau_c^1 \quad \tau_c^2], \quad \tau_c^1 \in \mathscr{R}^{1 \times m_c}, \quad \tau_c^2 \in \mathscr{R}^{1 \times (m-m_c)}. \tag{10c}$$

Given these partitionings, the cost function corresponding to (9) becomes

$$J = E\left\{ \sum_{c=0}^{C-1} x_c^{a^T} Q_c^a x_c^a + 2x_c^{a^T} M_c^a u_c^u + u_c^{u^T} R_c^a u_c^u - 2\lambda_c^a x_c^a - 2\tau_c^a u_c^u \right\}$$
$$+ E\{x_c^{a^T} H x_c^a - 2x_r^T(t_{uc}) H x_c^a\} + \sum_{c=0}^{C-1} \xi_c + \gamma_c, \tag{10d}$$

where

$$Q_c^a = \begin{bmatrix} Q_c & M_c^2 \\ M_c^{2^T} & R_c^3 \end{bmatrix}, \quad Q_c^a \in \mathscr{R}^{(n+m-m_c)\times(n+m-m_c)}, \tag{10e}$$

$$M_c^a = \begin{bmatrix} M_c^1 \\ R_c^{2^T} \end{bmatrix}, \quad M_c^a \in \mathscr{R}^{(n+m-m_c)\times m_c}, \tag{10f}$$

$$R_c^a = R_c^1, \quad R_c^a \in \mathscr{R}^{m_c \times m_c}, \tag{10g}$$

$$\lambda_c^a = [\lambda_c \quad \tau_c^2], \quad \lambda_c^a \in \mathscr{R}^{1\times(n+m-m_c)}, \tag{10h}$$

$$\tau_c^a = \tau_c^1, \quad \tau_c^a \in \mathscr{R}^{1 \times m_c}. \tag{10i}$$

The discrete-time control problem (9), (10) again is equivalent with the original digital LQG tracking problem (1), (3), (4). Now, in (9), (10), the actual control u_c^u appears as the control.

4. Solution to the Equivalent Discrete-time Control Problem

Inspection of the solution to the problem (5), (6) (Van Willigenburg and De Koning, 1992) reveals that although the matrices appearing in (9), (10) have time-varying dimensions, as opposed to those in (5), (6), the part of the solution based on stochastic dynamic programming still holds. Therefore, if we have the minimum variance estimator $\hat{x}_c^a$ of the augmented system state x_c^a in (9) at each time t_{u_c}, $c = 0, 1, 2, \cdots, C-1$, where the available information consists of all observations and controls preceding t_{u_c}, and the estimator has a conditional covariance P_c^a that is independent of previously applied controls, then the solution to (9), (10) is given by (Van Willigenburg and De Koning, 1992)

$$K_c = (R_c^a + \Gamma_c^{a^T} S_{c+1} \Gamma_c^a)^{-1} (\Gamma_c^{a^T} S_{c+1} \Phi_c^a + M_c^{a^T}), \quad c = 0, 1, 2, \cdots, C-1, \tag{11a}$$

$$K_c^1 = (R_c^a + \Gamma_c^{a^T} S_{c+1} \Gamma_c^a)^{-1} \Gamma_c^{a^T}, \quad c = 0, 1, 2, \cdots, C-1, \tag{11b}$$

$$K_c^2 = (R_c^a + \Gamma_c^{a^T} S_{c+1} \Gamma_c^a)^{-1}, \quad c = 0, 1, 2, \cdots, C-1, \tag{11c}$$

$$u_c^u = -K_c \hat{x}_c^a + K_c^1 N_c + K_c^2 \tau_c^a, \quad c = 0, 1, 2, \cdots, C-1, \tag{11d}$$

$$S_c = Q_c^a + \Phi_c^{a^T} S_{c+1} \Phi_c^a - K_c^T (R_c^a + \Gamma_c^{a^T} S_{c+1} \Gamma_c^a) K_c, \quad S_C = H, \quad c = 0, 1, 2, \cdots, C-1, \tag{11e}$$

$$N_c = (\Phi_c^a - \Gamma_c^a K_c)^T N_{c+1} - K_c^T \tau_c^{a^T} + \lambda_c^{a^T}, \quad N_C = H x_r(t_{u_C}), \quad c = 0, 1, 2, \cdots, C-1, \tag{11f}$$

$$\begin{aligned} J = {} & \bar{x}_0^{a^T} S_0 \bar{x}_0^a - 2\bar{x}_0^{a^T} N_0 + x_r^T(t_{u_C}) H x_r(t_{u_C}) \\ & + \sum_{c=0}^{C-1} [\xi_c - (K_c^1 N_{c+1})^T (2\tau_c^{a^T} + \Gamma_c^{a^T} N_{c+1}) - \tau_c^{a^T} K_c^2 \tau_c^a] + \mathrm{tr}(S_0 G^a) \\ & + \sum_{c=0}^{C-1} [\mathrm{tr}(V_c^a S_{c+1}) + \gamma_c] + \sum_{c=0}^{C-1} \mathrm{tr}(K_c^T (R_c^a + \Gamma_c^{a^T} S_{c+1} \Gamma_c^a) K_c P_c^a. \end{aligned} \tag{11g}$$

The first four terms on the right side of Eq. (11g) can be compared to the cost in the deterministic, or LQ case, where we have complete state information and no system noise. The fifth term on the right is due to uncertainty with respect to the initial state while the sixth is due to disturbances acting on the system. The last term is caused by uncertainty with respect to the state estimation.

The remaining problem is to find the linear minimum variance estimator $\hat{x}_c^a$ of the augmented system state x_c^a, and to see if it has a conditional covariance P_c^a independent of previously applied controls. Note that we have perfect state information about the part u_c^0 of x_c^a, given by (9b), because u_c^0 consists of previously applied controls. Therefore only $\hat{x}_c$ is required.

Consider the Eqs. (1i)–(1k) and the following *a priori* known one to one mappings,

$$Y_{r_l}(i) = j \Leftrightarrow y_i(t_{y_l}) = y'_j(t_{y_l}), \quad i = 1, 2, \cdots, p_l, \quad j \in Y_l, \quad l = 1, 2, \cdots, L. \tag{12}$$

The mappings Y_{r_l} in Eq. (12) specify how the sampled elements of the full output vector $y'(t_{y_l})$ are mapped on the actual output vector $y(t_{y_l})$. From (1i)–(1k) and (12), we obtain the following actual output equations:

$$y(t_{y_l}) = C(t_{y_l}) x(t_{y_l}) + w(t_{y_l}), \quad l = 1, 2, \cdots, L, \tag{13a}$$

where

$$C_i(t_{y_l}) = C'_j(t_{y_l}), \quad C(t_{y_l}) \in \mathscr{R}^{p_l \times n}. \tag{13b}$$

In Eq. (13b) i and j are row indices related through the mappings Y_{r_l}, given by

Eq. (12). Furthermore $w(t_{y_l})$ is a discrete-time zero-mean white noise process characterized by

$$E\{w(t_{y_l})\}=0, \quad \mathrm{cov}(w(t_{y_i}), w(t_{y_l}))=W(t_{y_l})\delta_{il}, \qquad i=1,2,\cdots,L, \quad l=1,2,\cdots,L. \tag{13c}$$

In Eq. (13c) $W(t_{y_l})$ is given by

$$W_{i,r}(t_{y_l})=W'_{j,s}(t_{y_l}), \quad W(t_{y_l})\in \mathscr{R}^{p_l\times p_l}, \tag{13d}$$

where i, r, like j, s, point to matrix elements and i and j, like r and s, are related through the mapping Y_{r_l}, given by Eq. (12).

Consider the set T_{um} of all observation and control instants ordered by magnitude, its elements being denoted by t_k,

$$T_{um}=T_u\cup T_m=\{t_k,\ k=0,1,2,\cdots,C+L-1-e\,|\,t_{k+1}>t_k\}. \tag{14}$$

In Eq. (14) e is the number of control and observation instants that are equal. Equations (1d)–(1h), (11) and (14) imply that

$$u(t)=u(t_k)=u_k\in\mathscr{R}^m, \quad t\in[t_k,t_{k+1}), \quad k\in\{k: t_k\in T_{um}\}. \tag{15a}$$

The equivalent discrete-time system which describes the state transitions of the system (1) from t_k to t_{k+1} is given by

$$x_{k+1}=\Phi_k x_k+\Gamma_k u_k+v_k, \quad k\in\{k: t_k\in T_{um}\}, \tag{15b}$$

where v_k is a discrete-time zero mean white noise process characterized by

$$E\{v_k\}=0, \quad \mathrm{cov}(v_i,v_k)=V(t_{k+1},t_k)\delta_{ik}=V_k\delta_{ik}, \qquad i,k\in\{k: t_k\in T_{um}\}, \quad V_k\in\mathscr{R}^{n\times n}. \tag{15c}$$

The system matrices in Eqs. (15b), (15c) can be computed numerically from those in (1a)–(1c) (Van Willigenburg, 1993). For each t_k that is an observation instant, we can rewrite the output equations (13),

$$y_k=C_k x_k+w_k, \quad k\in\{k: t_k\in T_m\}, \tag{16a}$$

where

$$y_k=y(t_k)=y(t_{y_l})\in\mathscr{R}^{p_l}, \qquad k\in\{k: t_k\in T_m\}, \qquad l=1,2,\cdots,L, \tag{16b}$$

$$C_k=C(t_k)=C(t_{y_l})\in\mathscr{R}^{p_l\times n}, \qquad k\in\{k: t_k\in T_m\}, \qquad l=1,2,\cdots,L, \tag{16c}$$

$$w_k=w(t_k)=w(t_{y_l})\in\mathscr{R}^{p_l}, \qquad k\in\{k: t_k\in T_m\}, \qquad l=1,2,\cdots,L. \tag{16d}$$

According to (16d) and (13c), (13d), w_k is a discrete-time zero-mean white noise process characterized by

$$\left.\begin{array}{l} E\{w_k\} = 0, \quad \operatorname{cov}(w_i, w_k) = W_k\delta_{ik} = W(t_k)\delta_{ik} = W(t_{y_l})\delta_{ik} \\ W(t_{y_l}) \in \mathscr{R}^{p_l \times p_l}, \quad k \in \{k: t_k \in T_m\}, \quad l = 1, 2, \cdots, L \end{array}\right\}. \tag{16e}$$

From (15), (16), and according to well known results from discrete-time Kalman filtering (Lewis, 1986), the *a priori* linear minimum variance estimator at all control and observation instants $\hat{x}(t_k)^- = \hat{x}_k^-$, $k \in \{k: t_k \in T_{um}\}$ is given by

$$\hat{x}_{k+1}^- = \Phi_k \hat{x}_k + \Gamma_k u_k, \qquad k \in \{k: t_k \in T_{um}\}, \tag{17a}$$

$$P_{k+1}^- = \Phi_k P_k \Phi_k^T + V_k, \qquad k \in \{k: t_k \in T_{um}\}. \tag{17b}$$

If t_k is not an observation instant only the time update (17a), (17b) of the discrete-time Kalman filter is performed, and we have

$$\hat{x}_k = \hat{x}_k^-, \qquad k \in \{k: t_k \in T_{um} \backslash T_m\}, \tag{17c}$$

$$P_k = P_k^-, \qquad k \in \{k: t_k \in T_{um} \backslash T_m\}. \tag{17d}$$

If t_k is an observation instant, which may be equal to a control instant, the time update (17a), (17b) is followed by a measurement update of the discrete-time Kalman filter,

$$K_k = P_k^- C_k^T (C_k P_k^- C_k^T + W_k)^{-1}, \qquad k \in \{k: t_k \in T_m\}, \tag{17e}$$

$$P_k = (I - K_k) P_k^- (I - K_k)^T + K_k W_k K_k^T, \qquad k \in \{k: t_k \in T_m\}, \tag{17f}$$

$$\hat{x}_k = \hat{x}_k^- + K_k(y_k - C_k \hat{x}_k^-), \qquad k \in \{k: t_k \in T_m\}. \tag{17g}$$

The recursions are initiated with

$$\hat{x}_0^- = \bar{x}_0, \quad P_0^- = G. \tag{17h}$$

Obviously from (14) and (1d), (1e), with $\hat{x}_k^-$ we have obtained $\hat{x}_c$, and its conditional covariance P_c is independent of previously applied controls. Finally consider the conditional covariance P_c^a of the minimum variance estimator $\hat{x}_c^a$ of the augmented state x_c^a. From (9b) and the fact that we have perfect information about the controls u_c^0, we have

$$P_c^a = \begin{bmatrix} P_c & 0 \\ 0 & 0 \end{bmatrix}, \quad P_c^a \in \mathscr{R}^{(n+m-m_c)\times(n+m-m_c)}, \tag{18}$$

and so indeed P_c^a is independent of previously applied controls. This concludes the derivation of the digital LQG tracker in the case of asynchronous and aperiodic sampling. The digital LQG tracker is given by Eqs. (11), (14)–(18).

Equations (15)–(18) are stated in terms of the conventional equivalent discrete-time system (15b), (15c) corresponding to the system (1), which describes the state transitions from each control and/or observation instant to the next. The time-varying dimension of the output equation (16) does not affect the discrete-time Kalman filter results. Because of the partial update of controls at the control instants, Eq. (11) is stated in terms of the modified equivalent discrete-time system (9) corresponding to (1a)–(1h), and the modified equivalent discrete-time cost-function (10), corresponding to (3). The modified equivalent discrete-time system describes the state transitions from each control instant to the next and has as controls only the updated control variables. It is obtained through rearrangement of matrices and vectors which describe the conventional equivalent discrete-time system (5). Likewise, the modified cost-function (10) is obtained through rearrangement of matrices and vectors which describe the conventional equivalent discrete-time cost-function (6). The rearrangements are dictated by the mappings (7b), (7c).

5. Extension and Simplification of the Digital LQG Tracker

Using the result of Engwerda and Van Willigenburg (1992), we may extend the system equation (1a) to become

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t) + v(t),\ A(t) \in \mathscr{R}^{n\times n},\ B(t) \in \mathscr{R}^{n\times m} \quad (19a)$$

and extend the cost functional (3) to become

$$\left.\begin{aligned} &E\left\{(x(t_{u_C}) - x_r(t_{u_C}))^T H (x(t_{u_C}) - x_r(t_{u_C}))\right.\\ &\quad + \int_{t_{u0}}^{t_{u_C}} (x(t) - x_r(t))^T Q(t)(x(t) - x_r(t))\\ &\quad \left.+ (u(t) - u_r(t))^T R(t)(u(t) - u_r(t))dt\right\}\\ &Q(t) \geq 0,\ H \geq 0,\ R(t) \geq 0,\ Q(t) \in \mathscr{R}^{n\times n},\ H \in \mathscr{R}^{n\times n},\ R(t) \in \mathscr{R}^{m\times m} \end{aligned}\right\}. \quad (19b)$$

In Eq. (19a) $d(t)$ is an *a priori* known deterministic exogenous input, and in Eq. (19b) $u_r(t)$ an *a priori* known control reference. Both of them are often involved in economic control policy problems. If the observation $y(t_k)$ becomes available sometime *after* t_{k+1}, say t_i, then at t_k we can only perform a time update which we have to store. Next we have to store all consecutive measurements until t_i and after t_i rerun Eq. (17), where we start with the measurement update at t_k and finish at the earliest measurement or control instant after t_i.

If the time necessary to compute the control updates at the control instants t_{u_c} is negligible, we may consider the available information at t_{u_c} to be all observations and controls up to *and including* t_{u_c}. In this case all our results hold if we replace the estimator $\hat{x}_k^-$ with $\hat{x}_k$ and the covariance P_k^- with P_k which are also given by Eq. (17).

The augmentation of the state described by Eq. (9a) at the initial time t_{u_0} is unnecessary if all unchanged control variables at t_{u_0} are zero; i.e. $u_0^u = 0$. In this case in Eqs. (9) and (11) x_0^a reduces to x_0, G^a to G, and in Eqs. (10) and (11) Q_0^a reduces to Q_0, M_0^a to M_0^1 and λ_0^a to λ_0^1.

6. A Numerical Example

The computation of the equivalent discrete-time system and criterion matrices (5), (6) and (15b), (15c) constitutes the main difficulty in computing the digital LQG tracker. Van Willigenburg (1991; 1993) presented a method to compute these. The following numerical example is chosen to contain all key features of the digital LQG tracker. Consider the digital LQG problem (1), (3), (4), where

$$A(t)=\begin{bmatrix}-2-3\sin(0.5\pi t) & 0\\ 5 & -4-5^{*}\cos(0.5\pi t)\end{bmatrix}, \tag{20a}$$

$$B(t)=\begin{bmatrix}\sin(3t) & 1\\ -1 & \cos(3t)\end{bmatrix}, \tag{20b}$$

$$V(t)=0.5\begin{bmatrix}1.5+\cos(2\pi t) & 0.2\\ 0.2 & 1.3+\sin(\pi t)\end{bmatrix}, \tag{20c}$$

$$\bar{x}_0=\begin{bmatrix}1\\1\end{bmatrix},\quad G=\begin{bmatrix}0.2 & 0.1\\ 0.1 & 0.3\end{bmatrix}, \tag{20d}$$

$$Q(t)=\begin{bmatrix}2+\sin(2t) & 0\\ 0 & 2+\sin(2t)\end{bmatrix}, \tag{20e}$$

$$R(t)=0.1\begin{bmatrix}2+\cos(2t) & 0\\ 0 & 2+\cos(2t)\end{bmatrix}, \tag{20f}$$

$$H=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, \tag{20g}$$

$$x_r(t)=(\sin(t)\ \ \cos(t))^T, \tag{20h}$$

$$C'(t_{y_l})=\begin{bmatrix}-\sin(2\pi t_{y_l}) & 1\\ -2 & 3\cos(\pi t_{y_l})\end{bmatrix}, \tag{20i}$$

$$W'(t_{y_l})=\begin{bmatrix}0.7+0.5\cos(\pi t_{y_l}) & 0.15\\ 0.15 & 1+0.5\cos(4\pi t_{y_l})\end{bmatrix}, \tag{20j}$$

$$t_{u_0}=0.0,\quad t_{u_C}=t_{u_4}=2.1. \tag{20k}$$

The control updating is characterized by

$$M_{\text{control}}=\begin{bmatrix}0.0 & 0.5 & 0.8 & 1.5\\ 1 & 2 & 1 & 1\\ 1 & 1 & 2 & 1\\ 2 & 2 & 1 & 2\end{bmatrix}. \tag{20l}$$

Each column of the matrix M_{control} refers to the control instant specified by the first element, i.e. to t_{u_c}, $c=0,1,2,\cdots,C-1$. The second element defines the number of updated control variables, i.e. m_c in Eq. (1f). The remaining elements define how u_c is mapped on u_c'. They equal $U_{u_c}(i)$, $i = 1, 2, \cdots, m_c$ and $U_{0_c}(i)$, $i = 1, 2, \cdots, m - m_c$, given by (7b), (7c) respectively. This for example implies that at $t_{u_2} = 0.8$, only the second control variable is updated.

The sampling of the output is described by

$$M_{\text{output}} = \begin{bmatrix} 0.2 & 0.5 & 0.9 & 1.4 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}. \tag{20m}$$

Each column of the matrix M_{output} refers to the observation instant specified by the first element, i.e. to t_{y_l}, $l = 1, 2, \cdots, L$. The second element specifies the number of outputs p_l, given by (1i), that are sampled. The first p_l elements that follow specify which outputs of the full output vector are sampled and the order in which they appear in the actual output vector; i.e. they equal $Y_{r_l}(i)$, $i = 1, 2, \cdots, p_l$, given by Eq. (12). For example, from (1e), (1i), (1j), (12), (13) and (20i), (20j), (20m) for the actual output equation (13) at $t_{y_1} = 0.2$ we obtain,

$$C(t_{y_1}) = [-2 \quad 3\cos(0.2\pi)], \tag{21a}$$

$$W(t_{y_1}) = 1 + 0.5^{*}\cos(0.8\pi). \tag{21b}$$

The time step involved in the numerical integration of Eqs. (5), (6) and (15b), (15c) (Van Willigenburg, 1991; 1993) was chosen to be 0.01 everywhere. We confine ourselves to mentioning the minimum costs of the problem (1), (3), (20) computed from the algorithm since it requires all computations within the algorithm to be performed. The minimum is computed to be ~~[illegible]~~. 5.2565

The first four terms on the right of Eq. (11g) were computed to be 3.3658. They represent the cost of the corresponding digital LQ tracking problem. These costs may be verified against an alternative computation which constitutes a function minimization (Van Willigenburg, 1991; 1993). The minimum costs and optimal control obtained from this alternative computation matched those computed from our algorithm within 0.1[%]. This verifies the LQ part of our algorithm.

7. Conclusions

In practice conventional sampling is often undesirable or impossible. The development of digital control system design procedures for asynchronous and aperiodically sampled continuous-time systems circumvents the requirement for frequent, synchronous and periodic updating of controls and observations. It therefore is of great practical importance. The derivation and computation of the digital LQG tracker for time-varying systems was extended from the case of synchronous sampling to the more practical case of asynchronous (and aperiodic) sampling. In this case, the updating of an arbitrary number of control variables and the sampling of an arbitrary number of outputs occurs at arbitrary time instants. Because of analyses that may be involved, we also dealt with the situation where some observations, made at a certain time-instant, become available only some time later on.

To obtain the solution, the system state at each control instant was augmented by the unchanged part of the control. As a result the equivalent discrete-time system and criterion matrices had to be modified and obtained time-varying dimensions. Despite these modifications the solution still is certainty equivalent. The feedback, feedforward and the estimator gains can still be computed *a priori* through recursion. The linear feedback of the estimated, augmented state implies that the optimal control at each control instant is a linear function of the estimated state and the unchanged part of the control.

As a special case of the digital LQG tracker, the digital LQG regulator, together with results presented by Van Willigenburg (1995), permits the design and computation of digital optimal controllers for asynchronous and aperiodically sampled nonlinear systems (Athans, 1971; Van Willigenburg, 1991).

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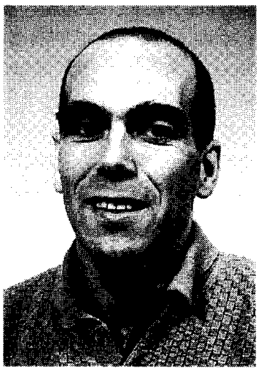

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