Computational Economics **8:** 107–125, 1995.

Optimal Sampling-Rates and Tracking Properties of Digital LQ and LQG Tracking Controllers for Systems with an Exogenous Component and Costs Associated to Sampling

JACOB ENGWERDA* and GERARD VAN WILLIGENBURG**
**Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg; **Department of Agricultural Engineering and Physics, Wageningen Agricultural University, Bomenweg 4, 6703 HD Wageningen, The Netherlands*

(Accepted: October 1994)

Abstract. In this paper it is shown that if costs are associated to sampling operations which are added to a performance criterion, the minimization of this new performance criterion results in a controller operated at an optimal sampling rate. This, under the assumptions that the system is periodically sampled, the applied control is kept fixed between every two sampling instances and some technical conditions are met. In case the considered planning horizon in the performance criterion is finite an algorithm is devised which calculates in a finite number of steps the optimal sampling period. It is shown that the technical conditions mentioned above are satisfied by the finite planning horizon time-varying LQG tracking problem. Since stability is a major requirement in controller design we also consider the case of an infinite planning horizon. This analysis is focused on the time-invariant digital LQ tracking problem. Given some mild regularity conditions a numerical algorithm is presented that approximates the optimal solution within any prespecified error norm. It is shown that also in this case an optimal sampling-rate exists. The algorithm for determining the optimal sampling period if the planning horizon is finite is illustrated in an economic example.

1. Introduction

Dealing with practical problems in engineering and economics the natural questions arise how often the system should be sampled and how the performance is affected if only sampled datasets are available.

Macro-economic systems, for example, evolve continuously in time while economic data from the system are gathered only at certain sampling instants. Increasing the sampling period of a macro-economic system brings on additional costs of data gathering, whereas the additional information that can be extracted from this data will, in general, decrease on an increasing sampling period (since e.g. the data is corrupted by noise). Therefore, the problem arises to weigh out the advantage of an increase in the performance of the system and the additional cost of data gathering.

In this paper we present sufficient conditions for the existence of an optimal sampling period for an economic system. We assume that the underlying economy is described by a linear continuous-time system containing an exogenous component, and that the policymakers want to minimize a social welfare function.

There is an extensive ongoing literature dealing with various problems concerning sampled data systems (see e.g. Aström et al. (1990) and Engwerda et al. (1992) for references). Main research performed in this area has been done in the field of control engineering, in particular from a digital control point of view. In particular the so-called digital LQG tracking controllers have been studied. They are characterised by sampled data, piecewise constant controls and a quadratic integral costfunctional which includes a reference for the state only. However, the design does not take into account the influence of an exogenous component, a reference for the control and costs associated to sampling. Characteristics, which are typical for example many economic and chemical systems.

In this paper the analysis of the finite planning horizon time-varying digital LQG tracking problem and the infinite planning horizon time-invariant LQ tracking problem and their numerical computation, are extended to deal with problems where an exogenous component and a reference for the control are involved. The cost associated to sampling operations are added to the quadratic performance index. It is demonstrated that minimising this new performance index results in a digital LQG (LQ, respectively) tracking controller operated at an optimal sampling-rate.

The outline of the paper is as follows. First, we consider the finite planning horizon problem in a general setting. It is assumed that the goal is to choose the sampling period such that the sum of sampling cost and social welfare cost is minimized. It is shown, by making a reasonable choice for the sampling cost as a function of the number of samples, that under some technical conditions on the welfare function, also viewed as a function of the sampling period, there exists a sampling period minimizing this sum. A numerical algorithm is given which computes this optimal sampling period in a finite number of steps. Then, it is shown that if the welfare function is given by a quadratic tracking criterion and the underlying system is described by a linear time-varying differential equation corrupted by white noise the above mentioned technical conditions are satisfied. This is achieved by deriving, for an a priori specified sampling period, an explicit formula for the minimal value of this social welfare function. This value can be split into four terms. One term which can be compared to the costs of the corresponding LQ problem, a second term that is caused by the initial state uncertainty, a third term which is caused by disturbances acting on the system and a fourth term caused by uncertainty of the state estimation. Ideally, it would be desirable to analyze this minimal value as an explicit function of the sampling period and then investigate the dependency of this value on the sampling period (i.e. evaluate the derivative). However, this value is a highly nonlinear function of the sampling period which has as a consequence that the resulting expression for the derivative requires more computational effort than the actual calculation of the minimal value for a number of appropriate sampling frequencies (see Powell (1967)). But, fortunately, it can be easily verified from this value that the technical conditions mentioned above are satisfied, which yields the above mentioned conclusion.

Since robustness for unmodeled disturbances and staying close to prespecified policy paths is a major requirement in economic controller design too (see e.g. Pitchford et al. (1977), Preston et al. (1982) and Engwerda (1990)), we consider in a seperate section the case where the social welfare function has an infinite planning horizon. To simplify this analysis we restrict it to time-invariant digital LQ tracking problems. Again, first, the existence of an optimal sampling period in case costs associated to sampling operations are added to the welfare function is treated in a general context. Technical conditions are presented again from which one can conclude that this optimization problem has a solution. Then, these results are used to derive an explicit expression for the minimal value of the LQ performance criterion, given a fixed sampling period. This theoretical expression for the minimal value is used to verify the above technical conditions and thus to conclude the existence of an optimal sampling period for the problem.

To calculate this optimal sampling period one may expect that one has to calculate the minimal welfare for a number of fixed sampling periods. Since exact calculation of the minimal welfare is an, in general, impossible job we present in a seperate section a numerical algorithm which computes (under some conditions) this value within any prespecified error norm together with some numerical considerations. Finally, we illustrate how the optimal sampling-rate can be computed numerically for an economic control policy problem.

2. Optimal Sampling Rates

In this section we discuss the problem how to determine the optimal sampling rate in a general finite planning horizon setting.

Assume that a government likes to minimize a social welfare function J_W which is defined over a planning interval $[0, t_f]$, and that it reconsiders its policy only at discrete points z_i in time based on new information that it receives at points t_i on the economy. Moreover, **assume** that the policy variables remain constant in between every two successive timepoints z_i and z_{i+1}. To avoid unnecessary complications **assume** that the timepoints t_i and z_i coincide. A point t_i will be called a *sampling point*. Additionally **assume** that the time that elapses between any two consecutive sampling points is constant, and that this *sampling period* equals T. Finally, **assume** that t_f is an integer multiple of this sampling period and that for every fixed sampling period the minimum of the welfare function, viewed as a function of the policy variables, exists and is denoted by J_W^*. Now, one might expect that the more frequent the economy is sampled, the smaller the welfare cost will be. This, however, is not always the case as shows the following example:

Example 1:
Consider the scalar system: $\dot{x} = x + u$, and the corresponding social welfare function: $J_W = \{\int_0^1 (u-1)^2 dt + \int_1^2 (u-2)^2 dt\}$. Then, it is clear that if the policy variable u may be changed at the sampling points $0, (2/3)$ and $(4/3)$, the minimal

welfare cost $J_W^* > 0$, whereas if the system is only sampled twice (at the points 0 and 1) the minimal welfare costs are zero. □

Since the set of admissible policies for a sampling period T_1 contains the set of admissible policies for a period T_2, whenever T_2 is an integer multiple of T_1, it is obvious that a relationship which does hold is that $J_W^*(T_1) \leqslant J_W^*(T_2)$. This implies in particular that

Lemma 2:
Assume that $\lim_{T \downarrow 0} J_W^*(T) = \bar{J}_0$. Then, $J_W^*(T) \geqslant \bar{J}_0$ for every admissible sampling period T.

Proof:
(by contradiction) Assume that there exists a sampling period T_0 such that $J_W^*(T_0) < \bar{J}_0$. Then (see the above argument) also $J_W^*(T_0/N) \leqslant J_W^*(T_0), \forall N = 0, 1, 2, \ldots$ So, in particular we get $\bar{J}_0 = \lim_{N \to \infty} J_W^*(T_0/N) \leqslant J_W^*(T_0) < \bar{J}_0$. □

In the sequel we make the **assumption** that $\bar{J}_0 := \lim_{T \downarrow 0} J_W^*(T)$ exists as a finite number.

As motivated in the introduction it seems reasonable to assume that the collection of information brings on costs with it. Again for simplicity reasons, we make the **assumption** that these costs, denoted by J_S, consist of fixed costs, $\bar{J}_f$, and variable costs which are the product of some fixed amount c and the number of samples. That is:

$$J_S(T) = \bar{J}_f + c * \frac{t_f}{T}. \tag{1}$$

Obviously under the above assumptions the sum of the welfare cost and sampling cost $J^*(T) = J_W^*(T) + J_S(T)$, goes to infinity if the sampling period T goes to zero. Therefore a sampling period T^* bounded away from zero exists such that the sum of the welfare and sampling cost $J^*(T^*)$ is minimal, i.e. $J^*(T^*) \leqslant J^*(T)$ for all admissible T. This inequality together with equation (1) and lemma 1 gives rise to the inequality $(c * t_f/T^*) \leqslant J^*(T)_{-\bar{J}_f} - \bar{J}_0$, from which we obtain $T^* \geqslant c * t_f/(J(t_f) - \bar{J}_f - \bar{J}_0)$. This inequality forms the basis of the following recursive algorithm that computes the optimal sampling rate T^* together with the minimum cost $J^*(T^*)$ in a finite number of steps.

Algorithm 3:

- initialization step.
  $T^* := t_f$, $J^* := J_W^*(T^*) + J_S(T^*)$; Number of samples $:= 1$.
- updating the sampling period.
  Increase the number of samples by 1. Calculate the corresponding sampling period $(T := (t_f/\text{number of samples}))$.

- stopping rule.
  If $T \leqslant c * t_f/(J^* - \bar{J}_f - \bar{J}_0)$ then the algorithm stops: the optimal sampling period is T^* and the corresponding minimal cost is J^*.
- minimality test.
  If $J^*(T) < J^*(T^*)$ then $T^*{:=}T$, and $J^*(T^*){:=}J(T)$.
  Goto step 2 of the algorithm. □

In the next section we show that the above mentioned assumptions on the welfare function are satisfied if we consider a time-varying quadratic welfare function containing a reference trajectory for the control and the economy is described by a time-varying linear differential equation containing an exogenous component. Furthermore we present explicit formulas to calculate the optimal control policy for the optimal sampling period.

3. The Optimally Sampled Time-Varying Digital LQG Tracker

First, we formulate and solve the digital LQG tracking problem for an arbitrarily chosen admissible sampling period. Consider a system described by the following linear, finite-dimensional differential equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t) + v(t), \tag{2}$$

with $Ex(t_0) = \bar{x}(t_0)$, and $\mathrm{cov}(x(t_0), x(t_0)) = G$. Here $x(t)$ is an n-dimensional state vector, $A(t)$ and $B(t)$ are the system matrices, $d(t)$ is a deterministic exogenous component and $v(t)$ is a white noise process with, $Ev(t) = 0$, and $\mathrm{cov}(v(t), v(s)) = V(t)\delta_D(t-s)$, where $\delta_D(t-s)$ is the Dirac delta function. The covariance matrices $V(t)$ and G are assumed to be positive semidefinite ($\geqslant 0$).

Now, assume that measurements on the system are made at the sampling points $t_i, i = 1, \ldots, N$ as follows

$$y(t_k) = C(t_k)x(t_k) + w(t_k), k = 0, 1, \ldots, N,$$

where $w(t_k)$ is a discrete-time white noise process with $E\{w(t_k)\} = 0$, and $E\{w(t_k)w^T(t_l)\} = W(t_k)\delta_{kl}$, where δ_{kl} is the Kronecker delta. The objective is to let the state variables track a prespecified trajectory $x^*(\cdot)$ by using a piecewise constant control path $u(\cdot)$ that does not diverge too much from a prespecified control path $u^*(\cdot)$. This idea of tracking can be formalized by considering a quadratic cost-functional of the form:

$$\begin{aligned} J_W(u(\cdot), x(t_0), t_0, t_N) &{:=} E\{(x(t_N) - x^*(t_N))^T H(x(t_N) - x^*(t_N))\} \\ &+E\left\{\int_{t_0}^{t_N} (x(t) - x^*(t))^T Q(t)(x(t) - x^*(t)) \right. \\ &\left. +(u(t) - u^*(t))^T R(t)(u(t) - u^*(t))\right\}, \end{aligned} \tag{3}$$

where $x^*(\cdot)$ and $u^*(\cdot)$ are references for the state and control, respectively, and $Q(t) > 0, R(t) > 0$ and $H \geqslant 0$.

The digital LQG problem consists of minimizing (3) subject to (2). It is well known that digital LQG problems can be transformed into *equivalent* discrete-time problems which consist of an equivalent discrete-time system and an equivalent discrete-time costfunction (see e.g. Levis et al. (1971)). The equivalent discrete-time system corresponding to (2) is given by

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + d_k + v_k \tag{4}$$

$$y_k = C_k x_k + w_k,$$

where the index k refers to values at times t_k. Here $\Phi_k = \Phi(t_{k+1}, t_k)$, where $\Phi(t, s)$ is the state transition matrix of system (2) from time s to time t; $\Gamma_k = \Gamma(t_{k+1}, t_k)$, where $\Gamma(t, t_k) = \int_{t_k}^{t} \Phi(s, t_k)B(s)ds$; $d_k = d(t_{k+1}, t_k)$, where $d(t, t_k) = \int_{t_k}^{t} \Phi(s, t_k)d(s)ds$; and the white noise v_k is characterised by $Ev_k = 0, Ev_k^T v_k = V_k = V(t_{k+1}, t_k)$, where $V(t, t_k) = \int_{t_k}^{t} \Phi(s, t_k)V(s)\Phi^T(s, t_k)ds$.

The equivalent discrete-time costfunction corresponding to (3) is given by

$$\begin{aligned} J_W(u(\cdot), x(t_0), t_0, t_N, T) &= E\{(x_N - x_N^*)^T H(x_N - x_N^*)\} \\ &+ \sum_{k=0}^{N-1} E\{x_k^T Q_k x_k + 2x_k^T M_k u_k + u_k^T R_k u_k + 2x_k^T q_k + 2r_k^T u_k\} \\ &+ f_k + z_k, \end{aligned} \tag{5}$$

where $R_k = \int_{t_k}^{t_{k+1}} R(t) + \Gamma^T(t, t_k)Q(t)\Gamma(t, t_k)dt$; $Q_k = \int_{t_k}^{t_{k+1}} \Phi^T(t, t_k)Q(t)\Phi(t, t_k)dt$; $M_k = \int_{t_k}^{t_{k+1}} \Phi^T(t, t_k)Q(t)\Gamma(t, t_k)dt$; $r_k = \int_{t_k}^{t_{k+1}} \Gamma^T(t, t_k)Q(t)d(t, t_k) - x^*(t) - R(t)u^*(t)dt$; $q_k = \int_{t_k}^{t_{k+1}} \Phi^T(t, t_k)Q(t)(d(t, t_k) - x^*(t))dt$; $f_k = \int_{t_k}^{t_{k+1}} tr(V(t, t_k)Q(t))dt$; and $z_k = \int_{t_k}^{t_{k+1}}(x^*(t) - d(t, t_k))^T Q(t)(x^*(t) - d(t, t_k)) + {u^*}^T(t)R(t)u^*(t)dt$. Note that, since $t_{k+1} - t_k = T$, all these matrices and variables depend on the sampling period T. For notational simplicity this dependency is omitted.

So, solving the digital LQG tracking problem (2,3) is equivalent to solving the discrete-time problem (4, 5). The solution to this problem is summarized in the next theorem.

Theorem 4:
The control sequence minimizing (5) subject to (4) is given by

$$u_k = -G_{k;N}\hat{x}_k - g_{k,N}, \tag{6}$$

where

$$G_{k,N} := (R_k + \Gamma_k^T K_{k+1,N}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1,N}\Phi_k + M_k^T) \tag{7}$$

$$g_{k,N}:=(R_k + \Gamma_k^T K_{k+1,N}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1,N}d_k - \Gamma_k^T h_{k+1,N} + r_k), \tag{8}$$

and $K_{k,N}$ and $h_{k,N}$ are given by the recursions

$$\begin{aligned} K_{k,N} &:= Q_k + \Phi_k^T K_{k+1,N}\Phi_k \\ &\quad -(\Phi_k^T K_{k+1,N}\Gamma_k + M_k)(R_k + \Gamma_k^T K_{k+1,N}\Gamma_k)^{-1} \\ &\quad \times(\Gamma_k^T K_{k+1,N}\Phi_k + M_k^T), \\ K_{N,N} &= H; \end{aligned} \tag{9}$$

$$\begin{aligned} h_{k,N} &:= (\Phi_k - \Gamma_k G_{k,N})^T(h_{k+1,N} - K_{k+1,N}d_k) + G_{k,N}^T r_k - q_k, \\ h_{N,N} &= Hx^*(t_n). \end{aligned} \tag{10}$$

Moreover, the minimum cost over the time interval $[t_k, t_N]$ equals:

$$\begin{aligned} J_W^*(t_k, t_N, T) &:= \hat{x}_k^T K_{k,N}\hat{x}_k - 2\hat{x}_k^T h_{k,N} + x_{t_N}^{*T} H x_{t_N}^* \\ &+ \sum_{i=k}^{N-1} \{(K_{i+1,N}d_i - h_{i+1,N})^T (R_i + \Gamma_i^T K_{i+1,N}\Gamma_i)^{-1}\Gamma_i(\Gamma_i^T \\ &\times(h_{i+1,N} - K_{i+1,N}d_i) - 2r_i) \\ &-2d_i^T h_{i+1,N} + d_i^T K_{i+1,N}d_i \\ &-r_i^T(R_i + \Gamma_i^T K_{i+1,N}\Gamma_i)^{-1}r_i + z_i\} \\ &+tr(K_{k,N}P_k) + \sum_{i=k}^{N-1}\{tr(V_i K_{i+1,N}) + f_i\} \\ &+ \sum_{i=k}^{N-1} tr(G_i(R_i + \Gamma_i^T K_{i+1,N}\Gamma_i)G_i P_i). \end{aligned} \tag{11}$$

In equations (6, 11) $\hat{x}_k, P_k, k = 0, 1, \ldots, N-1$ are generated by the well known Kalman one step ahead predictor for the discrete-time system (4) given by,

$$\hat{x}_{k+1} = (\Phi_k - L_k C_k)\hat{x}_k + L_k y_k + \Gamma_k u_k + d_k, \hat{x}_0 = \bar{x}_0,$$

where $L_k = \Phi_k P_k C_k^T(C_k P_k C_k^T + W_k)^{-1}$, $P_{k+1} = (\Phi_k - L_k C_k)P_k(\Phi_k - L_k C_k)^T + L_k W_k L_k^T + V_k$ with $P_0 = G$, and $W_k = W(t_k)$.

Proof:
The proof follows from results presented by Engwerda and Van Willigenburg in (1992), who consider the corresponding LQ problem, and results presented by Van Willigenburg and De Koning in (1992), who consider the LQG problem without an exogenous component and a reference for the control. □

Taking a closer look at the cost function (11) shows that in this equation the first four terms can be compared to the costs of the corresponding LQ problem;

the fifth term is caused by the initial state uncertainty; the sixth term is caused by disturbances acting on the system; and the last is caused by uncertainty of the state estimation. Furthermore, it is easily seen that $J^*(T)$ is bounded for every admissible sampling period chosen in the interval $(0, t_N - t_0]$ and that $\lim_{T\downarrow 0} J_W^*(T)$ exists. So, we can apply algorithm 3 to find the optimal sampling period.

Corollary 5:
If the welfare cost are given by (3) and sampling cost by (1), then there exists an optimal sampling period for system (2). This period can be calculated using algorithm 3. Furthermore, the control minimizing the welfare cost (3) w.r.t. system (2) and the corresponding welfare cost can, for any admissible sampling period, be calculated from theorem 4. □

We conclude this section by noting that the numerical computation of the solution presented in theorem 4 can be performed using the results presented by Van Willigenburg in (1992).

4. The Infinite Planning Horizon Case

In addition to optimality usually robust performance with respect to unmodeled disturbances is desired. It is well known that if one considers a quadratic performance criterion over an infinite planning horizon and a system described by a linear differential equation like (2) controllers are obtained which, under some weak additional conditions, stabilize the closed-loop of the system. Therefore, in this section we consider the existence question of an optimal sampling period in case the considered social welfare function is defined over an infinite planning horizon $[t_0, \infty)$. To be more specific, we consider the existence of an optimal sampling period for system (2) in case the welfare function is given by (3), with the planning horizon t_f extended to infinity, and the sampling cost given by (1). To simplify the analysis throughout this section, we will assume all matrices occurring in (2) to be time-invariant, the system is not to be corrupted by noise and full state observations (i.e. $C = I$). So, the system under consideration is

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t); y(t_k) = x(t_k) \tag{12}$$

In principle now any sampling period between 0 and ∞ is a candidate for being the optimal sampling period. On the other hand, if $J_W^*(T)$ is well defined for every finite positive sampling period T, in general one may expect that $\lim_{T\to\infty} J_W^*(T) = \infty$. Making these last **assumptions** we have the following general observation.

Theorem 6:
Let $J^*(T) = J_W^*(T) + J_S(T)$, where J_S is given by (1) and $J_W^*(T)$ is an arbitrarily continuous welfare function defined on $(0, \infty)$ for which $\lim_{T\downarrow 0} J_W^*(T)$ exists

and $\lim_{T\to\infty} J_W^*(T) = \infty$. Then, there exists an optimal sampling period T^*, minimizing J^*.

Proof:
Due to our assumptions on J_S and J_W^*, it is easily seen that both $\lim_{T\downarrow 0} J^*(T) = \infty$ and $\lim_{T\to\infty} J^*(T) = \infty$. Using the continuity of $J^*(T)$ elementary analysis shows the existence of an optimal sampling period. □

So the problem is to find conditions on the system and social welfare function such that the conditions in theorem 6 are satisfied. To that end we first present conditions guaranteeing that the optimization problem

$$\text{minimize} \lim_{t_N\to\infty} J_W(u(\cdot), x(t_0), t_0, t_N, T) \text{ w.r.t. (12)}, \tag{13}$$

under the constraint that the closed-loop system is stabilized, has a proper solution for every chosen sampling period T. In fact two kind of problems can occur. First of all, the reference paths for the control and policy variables must be such that for any sampling period T the solution has a finite welfare function value. It is clear that to satisfy this condition necessarily there must exist a control sequence $u(\cdot)$ such that the difference between the actual and desired state variables, $x - x^*$, converges to zero if time goes to infinity. If the sampling period approaches zero it seems reasonable (compare Engwerda (1990, theorem 6)) to require that the desired state variables satisfy a differential equation which corresponds with the system (12). On the other hand, if the sampling period differs from zero, then the desired control variables should ultimately become periodically constant with a period which equals the sampling period. Since this property should hold for any sampling period this implies that the desired control variables must ultimately converge. Based on these considerations we make the following assumption:

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t) + d(t) + \bar{v}(t); x^*(t_0) = \bar{x}(t_0) \tag{14}$$

where both $u_d(t) := u^*(t) - u^*$ and $\bar{v}(t)$ converge exponentially fast to zero. (Here u^* is a constant vector).

The second problem, already mentioned by Kalman et al. in (1963) (see also Levis et al. (1971)), is that although the continuous time system (12) may be controllable, in general the equivalent discrete-time system will not have this property. This has immediate consequences in case one considers infinite planning horizon problems, since this loss of controllability may cause the problem to have no proper solution. So, we have to deal with this problem too. Now, in general we have a rough idea about the magnitude of the optimal sampling period T^*. Therefore throughout this section we assume that $T^* \in [0, T_U]$, for some known upperbound T_U. Conditions that guarantee the stabilizability of the sampled system for any $T \in [0, T^U]$ are as follows.

Lemma 7:
Let (A, B) be stabilizable.
Then $(\Phi(T), \Gamma(T))$ is stabilizable for all $T \in [0, T_U]$ if for all eigenvalues $\lambda, \mu \in \sigma(A) \cap \mathbb{C}_b$, $\lambda - \mu \neq (2\pi i/\theta), \forall |\theta| < T_U$. Here $\mathbb{C}_b := \{z \in \mathbb{C} \mid \text{Re } z \geqslant 0\}$.

Proof:
Let $T \in (0, T_U]$ be any sampling period. Then λ is an eigenvalue of A if and only if $e^{\lambda T}$ is an eigenvalue of $\Phi(T)$. Consequently, if the real part of λ is smaller than zero then $|e^{\lambda T}| < 1$. That is, $e^{\lambda T}$ is a stable eigenvalue of $\Phi(T)$. Next consider the case that $\lambda \in \sigma(A) \cap \mathbb{C}_b$. Then, by assumption, λ will be controllable, i.e. rank$[A - \lambda I \; B] = n$. We will show that rank$[\Phi(T) - e^{\lambda T} I \; \Gamma(T)] = n$ too. That is, $e^{\lambda T}$ is a controllable eigenvalue of the sampled system.

To that end we first note that $e^{zT} \neq e^{\lambda T}$ for all $z \in \sigma(A)/\{\lambda\}$. So, $q(z) := (e^{zT} - e^{\lambda T})/(z - \lambda) \neq 0 \; \forall z \in \sigma(A)/\{\lambda\}$ and, moreover, $\lim_{z \to \lambda} q'(z) \neq 0$.

From both these observations we conclude that $e^{zT} - e^{\lambda T} = (z - \lambda) g(z)$, where $g(z) \neq 0 \; \forall z \in \sigma(A)$. So, using the spectral mapping theorem, we have that rank $[\Phi(T) - e^{\lambda T} I \; \Gamma(T)] =$ rank$[(A - \lambda I) g(A) \; \Gamma(T)]$, where $g(A)$ is invertible. Using standard arguments the result follows immediately. □

To answer the existence question of an optimal sampling period for the infinite planning horizon LQ problem we first consider the solution to the equivalent discrete-time LQ optimization problem for a fixed sampling period. It is well known that (see e.g. Kwakernaak et al. (1972)) if (Φ, Γ) is stabilizable, then both $K := \lim_{N \to \infty} K_{k,N}$ and, consequently, $G := \lim_{N \to \infty} G_{k,N}$ exist, where $K_{k,N}$ and $G_{k,N}$ are as in theorem 4. Furthermore the spectral radius of the matrix $\Phi - \Gamma G$, denoted by ζ, is smaller than one, and K can be found as the unique positive definite solution satisfying the algebraic Riccati equation:

$$K = \bar{Q} + \Phi^T K \Phi - (M + \Gamma^T K \Phi)^T (\bar{R} + \Gamma^T K \Gamma)^{-1} (M + \Gamma^T K \Phi), \quad (ARE) \tag{15}$$

where the time-invariant matrices Q_k and R_k are denoted by $\bar{Q}$ and $\bar{R}$, respectively. With this result one can prove analogous to corollary 2 in Engwerda (1990) that:

Theorem 8:
Let (Φ, Γ) be stabilizable and d_k, q_k, r_k be such that for all $k \geqslant 0$ $h_k := \lim_{N \to \infty} h_{k,N}$ exists. Then, the optimal control minimizing $\lim_{t_N \to \infty} J_W(u(\cdot), x(t_0), t_0, t_N, T)$ w.r.t. (12), is given by:

$$\begin{aligned} u_k &= -(\bar{R} + \Gamma^T K \Gamma)^{-1} (M + \Gamma^T K \Phi) x_k \\ &\quad -(\bar{R} + \Gamma^T K \Gamma)^{-1} (M + \Gamma^T K d_k - \Gamma^T h_{k+1} + r_k), \end{aligned} \tag{16}$$

where K satisfies (ARE). Moreover, this controller stabilizes the closed-loop system. The minimum welfare cost equals

$$J_W^*(t_0, T) := x_0^T K x_0 - 2 x_0^T h_0 + x_{t_N}^{*T} H x_{t_N}^*$$

$$+\sum_{i=k}^{\infty}\{(Kd_i - h_{i+1})^T(\bar{R}+\Gamma^T K\Gamma)^{-1}\Gamma(\Gamma^T(h_{i+1}-Kd_i)-2r_i)$$
$$-2d_i^T h_{i+1} + d_i^T K d_i - r_i^T(\bar{R}+\Gamma^T K\Gamma)^{-1} r_i + z_i\} \qquad \square$$

Sufficient conditions for the existence of h_k are (see Engwerda (1990, theorem 3)) that the growth rates of the deterministic variables d_k, q_k and r_k are smaller than $(1/\zeta)$, i.e. $\|d_{k+1}\| \geqslant \gamma\|d_k\|$, $\|q_{k+1}\| \geqslant \gamma\|q_k\|$ and $\|r_{k+1}\| \geqslant \gamma\|r_k\|$ for some $\gamma < (1/\zeta)$.

In practice only an approximate solution can be calculated. This brings on some specific approximation problems which are discussed in the next section.

Theorem 9:
Let (A, B) be stabilizable and $\lambda - \mu \neq (2\pi i/\theta)\forall|\theta| < T_U$; $\lambda, \mu \in \sigma(A)\cap\mathbb{C}_b$. Then, if $x^*(\cdot)$ satisfies (14), $J_W^*(T)$ is a continuous function on $(0, T_U]$ and $\lim_{T\downarrow 0} J_W^*(T)$ exists.

Proof:
First note that due to our assumptions, for any sampling period $0 \leqslant T \leqslant T_U$, the sampled system will be stabilizable on $(0, T_U]$ (see lemma 7).

Now, define the output error and control error as $e(t){:=}x(t) - x^*(t)$ and $\Delta u(t){:=}u(t) - u^*$, respectively. Then the optimization problem (12) can be rewritten as

$$\min \lim_{N\to\infty}\left\{e(t_N)^T H e(t_N) + \sum_{k=0}^{N-1}\left[\int_{t_k}^{t_{k+1}} e^T(t)Qe(t) + \Delta u^T(t)Ru(t)\right.\right.$$
$$\left.\left.-2\Delta u^T(t)Ru_d(t) + u_d^T(t)Ru_d(t)\}dt\right]\right\}$$

subject to

$$\dot{e}(t) = Ae(t) + B\Delta u(t) + \bar{v}(t) - Bu_d(t); e(t_0) = 0.$$

Since $e(t) = \Phi(t, t_k)e(t_k) + \Gamma(t, t_k)\Delta u(t_k) + \bar{d}(t, t_k)$, where $\bar{d}(t, t_k){:=} \int_{t_k}^{t} \Phi(s, t_k)(\bar{v}(s) - Bu_d(s))ds$ the problem is equivalent to:

$$\min \lim_{N\to\infty}\left\{e(t_N)^T He(t_N) + \sum_{k=0}^{N-1}[e_k^T\bar{Q}e_k + 2e_k^T M\Delta u_k\right.$$
$$\left.+\Delta u_k^T\bar{R}\Delta u_k + 2e_k^T\bar{q}_k + 2\bar{r}_k^T\Delta u_k + \bar{z}_k\right\}$$

subject to

$$e_{k+1} = \Phi e_k + \Gamma\Delta u_k + \bar{d}_k,$$

where $\bar{d}_k := \bar{d}(t_{k+1}, t_k)$, $\bar{r}_k := \int_{t_k}^{t_{k+1}} \Gamma^T(t, t_k)Q\bar{d}(t, t_k) - 2R(t)u_d(t)dt$; $\bar{q}_k = \int_{t_k}^{t_{k+1}} \Phi^T(t, t_k)Q\bar{d}(t, t_k)dt$; and $\bar{z}_k = \int_{t_k}^{t_{k+1}}(\bar{d}^T(t, t_k)Q\bar{d}(t, t_k) + u_d^T(t)Ru_d(t)dt$.

Obviously, all conditions of theorem 8 are satisfied, so that the minimal cost for this problem equals:

$$\begin{aligned} &J_W^*(t_0, T) \\ &:= \sum_{i=k}^{\infty} \{(K\bar{d}_i - h_{i+1})^T(\bar{R} + \Gamma^T K\Gamma)^{-1}\Gamma(\Gamma^T(h_{i+1} - K\bar{d}_i) - 2\bar{r}_i) \\ &\quad - 2\bar{d}_i^T h_{i+1} + \bar{d}_i^T K\bar{d}_i - \bar{r}_i^T(\bar{R} + \Gamma^T K\Gamma)^{-1}\bar{r}_i + \bar{z}_i\} \end{aligned}$$

To prove the continuity of $J_W^*(T)$, note that $K_{k,N}(T)$ is a uniform continuous function, for every N, which is bounded independently of N. Consequently $K(T)$ and, thus, G are bounded continuous functions too. Using (10) it can be shown that h_i satisfies (see Engwerda (1990, theorem 3)):

$$\begin{aligned} h_k(T) &:= \sum_{i=k}^{\infty} s_i(T), \text{ where} \\ s_i(T) &:= (\{(\Phi - \Gamma G)^T\}^{i-k}\{G^T\bar{r}_i - \bar{q}_i - (\Phi - \Gamma G)^T K\bar{d}_i\})(T) \end{aligned}$$

Now, $s_i(T)$ is a continuous function in T, which converges exponentially fast to zero if i tends to infinity (the spectral radius of $(\Phi - \Gamma G)$ is smaller than one, and $\bar{r}_i, \bar{q}_i$ and $\bar{d}_i$ converge exponentially fast to zero). So, $h_k(T)$ is also a bounded continuous function. Furthermore, it is easily seen that also h_k converges exponentially fast to zero if k tends to infinity. The same arguments as we used to show that $h_k(T)$ is a bounded continuous function, show that $J_W^*(T)$ is a bounded continuous function on $(0, T_U]$, which completes the proof. □

Corollary 10:
Assume that the welfare cost is given by $\lim_{N\to\infty} J(u(\cdot), x(t_0), t_0, t_N, T)$ and sampling cost by (1). Then, under the conditions of theorem 9, there exists an optimal sampling period for the control problem. □

5. Computational Remarks

Based on slight modification of results presented by Van Willigenburg (1992) we are able to numerically compute the solution (6) to the LQG-problem (2, 3) if the planning horizon is finite.

If the planning horizon is extended to infinity the computation of (10) in principle requires an infinite number of computations which in turn require an infinite number of data concerning the exogenous variables of the system and reference variables in the cost functional. Therefore an algorithm is needed to approximate this solution. Loosely speaking, we will show that under the growth rate conditions mentioned after theorem 8 the outcome of the backward recursion (10), i.e. $h_{k,N}$, is hardly influenced by $h_{k',N}$ when $k' - k$ is large. Or, in other words, that the outcome of the backward recursion is hardly influenced by previous far distant outcomes of the recursion. Therefore, taking a sufficiently large horizon the recursion (10) will approximate the solution h_k arbitrarily close. More formal, the result reads as follows.

Lemma 11:
Let i be an arbitrary positive integer. Consider

$$\hat{h}_i^N := \sum_{j=1}^{N-1} (\phi - \Gamma G)^{T^{j-i}} \{-(\phi - \Gamma G)^T K d_j + G w_j - v_j\} \tag{17}$$

Then, under the assumptions of theorem 4, for any $\epsilon > 0$ there exists an N such that $||h_i - \hat{h}_i^N|| < \epsilon$. Moreover, N can be calculated from either (ii) or (iii) in the appendix. □

Since the proof of this lemma is rather technical it is deferred to the appendix of this paper.

This lemma gives rise to the following algorithm for calculating an approximate optimal control in theorem 8. Starting point for the algorithm are the cost criterion (13) and system (12), and a fixed sampling period T. The algorithm assumes that all reference and exogenous variable paths are known.

Algorithm 12:
1a) Check whether $Q > 0$ and $R > 0$.
 b) Check whether (A, B) is stabilizable
2a) Calculate the equivalent discrete time system matrices Φ, Γ, and d_k
 b) the weight matrices $\bar{Q}, M$ and $\bar{R}$, and the
 c) vectors q_k, r_k, and z_k in the equivalent cost criterion (5).
3a) Calculate the positive definite solution K of the algebraic Riccati equation (ARE)
 b) Calculate the spectral radius ζ of the closed-loop matrix $\phi - \Gamma G$.
4a) Calculate the maximal growth rate γ of the variables d_k, q_k and r_k.
 b) Check whether $\gamma < 1/\zeta$.
5a) Choose an approximation error ϵ for h_i, i.e. $\hat{h}_i^N$ will be constructed such that $||h_i - \hat{h}_i^N|| < \epsilon$.
 b) Choose an N that satisfies inequality (ii) of the appendix.
 c) Calculate

$$\hat{h}_i^N := \sum_{j=1}^{N} (\phi - \Gamma G)^{T^{j-i}} \{G^T r_j - q_j - (\Phi - \Gamma G)^T K d_j\}. \tag{18}$$

6a) Implement the optimal control (16), with h_i replaced by $\hat{h}_i^N$
 b) increment i by 1, and return to 5). □

Note that in step 2b) the vectors v_k, w_k and z_k only need to be calculated up to time N, under the assumption that the growth rate assumption is satisfied. (One may expect that in practice the verification of this last condition will not be too difficult.) Consequently in actual situations the order of computations is: 1), 2a), 3), 4), 5a), 5b), 2b), 5c), 6), 5a), 5b), 2b), 6) etc.

Another point worth mentioning is that if matrix B is full column rank, $\phi - \Gamma G$ is invertible (see Engwerda (1990, theorem 5)). This allows recursive calculation of h_{i+1} as $(\phi - \Gamma G)^{-T}\{h_i + v_i - Gw_i\} + d_i$. This formula might be useful in implementing the algorithm. However, one has to be very careful in using this scheme, since all the eigenvalues of $(\phi - \Gamma G)^{-T}$ are outside the unit circle. So any error in h_i is exponentially forwarded to h_{i+1}.

Finally we note that to find the optimal sampling period in the infinite planning horizon case is much more involved than in the finite planning horizon case. Two remarks which may be helpful are: 1) one may proceed similarly as in algorithm 3 to find a lower-bound for this period, and 2) the solution K of the algebraic Riccati equation is a monotonically increasing function of the sampling period T (See Levis et al. (1971)).

6. An Economic Example

To illustrate some of our results, we consider the following deterministic macro economic multiplier-accelerator model (Turnovsky (1972)),

$$\begin{aligned} Y &= \gamma C + I + G + D \\ I &= \alpha \dot{C} - i\dot{I} \\ \dot{C} &= \delta(Y - C), \end{aligned}$$

where Y denotes national income, C consumption, I investment, G government expenditure and D autonomous expenditure. This model can be written as:

$$\begin{pmatrix} \dot{C} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} \delta(\gamma - 1) & \delta \\ \frac{\alpha\delta}{i}(\gamma - 1) & \frac{\alpha\delta - 1}{i} \end{pmatrix} \begin{pmatrix} C \\ I \end{pmatrix} + \begin{pmatrix} \delta \\ \frac{\alpha\delta}{i} \end{pmatrix} G + \begin{pmatrix} \delta \\ \frac{\alpha\delta}{i} \end{pmatrix} D.$$

Choosing $\alpha = 0.0789, \delta = 0.6068 + 0.2\sin(2\pi t), \gamma = 0.4171, i = 0.2782$ and $D(t) = -100e^{0.02t}$, this equation constitutes a deterministic linear time-varying system with state vector $x(t) := (C\, I)^T$, control vector $u(t) := G$ and exogenous vector $d(t) := (\delta\, (\alpha\delta/i))^T D$. We assume that only consumption, i.e. the first state variable, is measured at the sampling times. To describe the uncertainty of the economic model and the measurements we use the system description (2) with,

$$\begin{aligned} A(t) &= \begin{pmatrix} \delta(\gamma - 1) & \delta \\ \frac{\alpha\delta}{i}(\gamma - 1) & \frac{\alpha\delta - 1}{i} \end{pmatrix}; B(t) = \begin{pmatrix} \delta \\ \frac{\alpha\delta}{i} \end{pmatrix}; \\ d(t) &= \begin{pmatrix} \delta \\ \frac{\alpha\delta}{i} \end{pmatrix} D; V(t) = \begin{pmatrix} 2.25 & 0 \\ 0 & 0.09 \end{pmatrix}; \end{aligned}$$

$$x(t_0) = \begin{pmatrix} 350 \\ 70 \end{pmatrix}; G = \begin{pmatrix} 300 & 0 \\ 0 & 300 \end{pmatrix} \text{ and}$$
$$C(t_k) = (1 \quad 0), k = 0, 1, \ldots, N-1; W(t_k) = 3.0, k = 0, 1, \ldots, N-1.$$

Given this system, the economic control policy is to minimise the welfare cost (3) with $t_N = 5; Q(t) = \begin{pmatrix} 0.065 & 0 \\ 0 & 0.2 \end{pmatrix}; R(t) = 0.001; u^*(t) = 300 + 0.1t$ and $x^*(t)$ generated similar to $x(t)$, but with δ replaced by 0.6, $u(t)$ given by $u^*(t)$ and $x^*(t_0) = (100 \quad 400)^T$. Note that the system generating the reference state trajectory is time-invariant and that it may be regarded as the "average" of the original system, i.e. when the periodicity of δ is disregarded. The construction of this economic control policy is partially based on work presented by Turnovsky (1972) and Kendrick (1981). It only serves to demonstrate the possible application of our results in economic control policy problems. We assume the economy to be periodically sampled i.e. $t_{k+1} - t_k = T, k = 0, 1, \ldots, N-1$, and that the cost for gathering one measurement are 1.0 (so $J_S = (5/T)$).

Figure 1 shows the total costs as a function of the sampling period T for this example. From this figure it is clear that algorithm 3 yields an optimal sampling period of $T^* = 1/3$. The corresponding minimum total costs are 79.7974.

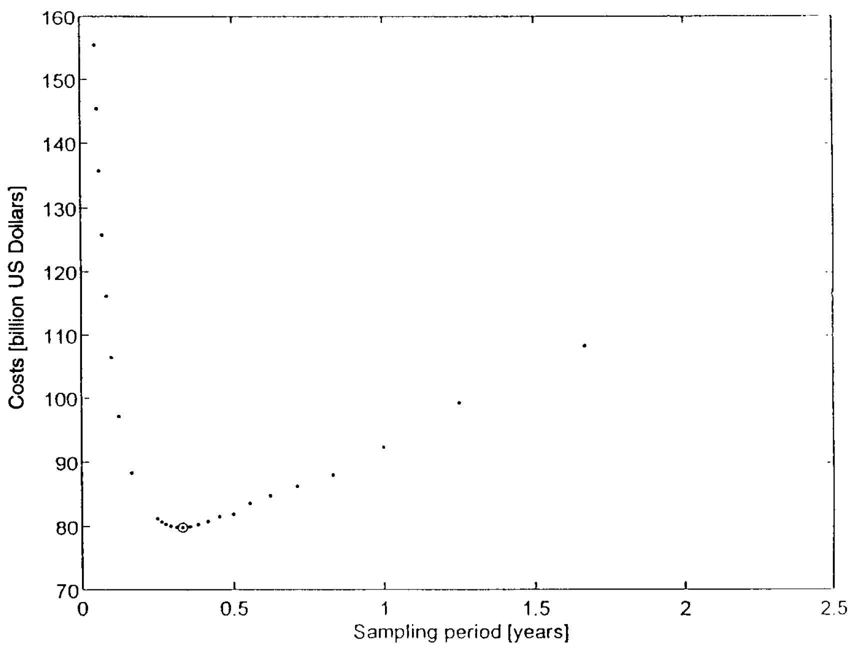

Fig. 1. Total costs versus the sampling period in case of digital LQG control.

7. Conclusions

In this paper we considered the question whether there exists an optimal sampling rate for systems that are periodically sampled. Optimal, in the sense that this sampling rate is such that the sum of a performance measure and the cost of data gathering is minimized. Under the basic assumption that the policy variables do not change in between every two successive sampling instances, it is shown that the answer to this question is affirmative under some technical conditions.

In case the performance is considered over a finite planning horizon, we presented an algorithm which calculates the optimal sampling rate in a finite number of steps. It is shown that if the system is described by a linear time-varying differential equation, containing an exogenous component, which is corrupted by noise and the performance criterion is a quadratic tracking equation containing a reference signal for as well the state as the policy variables, the above mentioned technical conditions are satisfied. So, under these conditions an optimal sampling rate exists and can be computed in a finite number of steps. Explicit formulas are given to accomplish this computation.

Since for robustness reasons stabilizing controllers are desired the existence question of an optimal sampling rate was also raised in an infinite planning horizon setting. Again, some technical conditions were presented under which an optimal sampling rate exists. The analysis for LQG systems was extended from a finite to an infinite planning horizon while some simplifying assumptions were made with respect to the system and performance criterion. Conditions on the system and target variables guaranteeing that the minimal value of the performance criterion remains bounded for any sampling rate have been presented. Provided these conditions are satisfied an optimal sampling rate exists.

We showed that there exists a compact interval where the optimal sampling period is situated. By calculating the minimal performance for different sampling periods in this interval one may get an idea of the location of the optimal sampling period. A problem with this approach is the possible existence of local minima. To carry out this idea, one has to calculate (for fixed different sampling periods) the minimal performance criterion. Since the considered planning horizon is infinite, this is not a trivial job. Some numerical remarks were made which may help to accomplish this task.

From the analysis of the finite planning horizon LQG and infinite planning horizon LQ case it is clear that the optimal sampling period depends on the initial state of the system and the prespecified output and control paths. So a basic problem, left for future research, is to investigate how robust the design is for changes in the initial state and reference paths. Another open problem is to what extent our basic assumption that the policy variables remain fixed in between every two successive sampling instances influences the optimal sampling rate. One might expect that if the optimal sampling rate is relatively small this assumption does not affect the outcome too much.

Appendix

Proof of lemma 11:
In this appendix we show how large the planning horizon N in

$$\hat{h}_i^N := \sum_{k=i}^{N-1} (\Phi - \Gamma G)^{T^{k-1}} \{-(\Phi - \Gamma G)^T K d_k + G w_k - v_k\} \tag{19}$$

must be chosen to have an estimate for h_i that satisfies the inequality

$$\|\hat{h}_i^N - h_i\|_2 \leqslant \epsilon\{\|(\Phi - \Gamma G)K\|_2 \|d_i\|_2 + \|G\|_2 \|w_i\|_2 + \|v_i\|_2\} =: \epsilon \bar{d}. \tag{20}$$

for a prespecified estimation error ϵ.

To that end, we make a Jordan-decomposition of matrix $\Phi - \Gamma G : \Phi - \Gamma G =: D + J$, where D is a diagonal matrix, J is a nilpotent matrix and $DJ = JD$. Moreover, we denote $-(\Phi - \Gamma G)^T K d_k + G w_k - v_k$ by $\bar{d}_k$. Then,

$$\|h_i - \hat{h}_i^N\|_2 = \left\| \sum_{k=N}^{\infty} (D + J)^k \bar{d}_k \right\|_2 = \left\| \sum_{k=N}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} D^{k-j} J^j \right) \bar{d}_k \right\|_2 \tag{21}$$

Under the assumption that N is larger than the dimension n of matrix $\Phi - \Gamma G$, this sum equals

$$\left\| \sum_{k=N}^{\infty} \left(\sum_{j=0}^{n} \binom{k}{j} D^{k-j} J^j \right) \bar{d}_k \right\|_2 \tag{22}$$

$$\leqslant \sum_{k=N}^{\infty} \left(\sum_{j=0}^{n} \left\| \binom{k}{j} D^{k-j} J^j \bar{d}_k \right\|_2 \right) \tag{23}$$

$$\leqslant \sum_{k=N}^{\infty} \left(\sum_{j=0}^{n} \left\| \binom{k}{j} D^{k-j} \bar{d}_k \right\|_2 \right) \tag{24}$$

$$\leqslant \sum_{k=N}^{\infty} \left(\sum_{j=0}^{n} \binom{k}{j} \rho^{k-j} \gamma^k \|d\|_2 \right) \tag{25}$$

$$= \sum_{j=0}^{n} \gamma^j \sum_{k=N}^{\infty} \binom{k}{j} (\rho\gamma)^{k-j} \|\bar{d}\|_2 \tag{26}$$

$$= \sum_{j=0}^{n} \gamma^j \cdot \frac{1}{j!} \left(\frac{(\rho\gamma)^{N+1}}{1 - \rho\gamma} \right)^{(j)} \|\bar{d}\|_2, \tag{27}$$

where $()^{(j)}$ denotes the j-th derivative w.r.t. $\rho\gamma$, γ is the growth rate of the deterministic variables and ρ is the absolute largest entry of matrix D.

Using Leibniz's rule, we can write this sum as:

$$\sum_{j=0}^{n} \gamma^j \frac{1}{j!} \sum_{k=0}^{j} \binom{j}{k} ((\rho\gamma)^{N+1})^{(k)} \cdot \left(\frac{1}{1-\rho\gamma}\right)^{(j-k)} \|\bar{d}\|_2 \tag{28}$$

$$= \sum_{j=0}^{n} \gamma^j \frac{1}{j!} \sum_{k=0}^{j} \binom{j}{k} k! \binom{N+1}{k} (j-k)! \frac{1}{(1-\rho\gamma)^{j-k}} \|\bar{d}\|_2 \tag{29}$$

$$= \sum_{j=0}^{n} \gamma^j \sum_{k=0}^{j} \binom{N+1}{k} \frac{(\rho\gamma)^{n+1-k}}{(1-\rho\gamma)^{j-k}} \|\bar{d}\|_2 \tag{30}$$

$$= \sum_{j=0}^{n} \left(\frac{\gamma}{1-\rho\gamma}\right)^j \sum_{k=0}^{j} \binom{N+1}{k} (\rho\gamma)^{N+1-k} (1-\rho\gamma)^k \|\bar{d}\|_2 \tag{31}$$

$$= \sum_{j=0}^{n} \left(\frac{\gamma}{1-\rho\gamma}\right)^j \frac{B_{\rho\gamma}(N+1-j, j+1)}{(N-j)!j!} \cdot (N+1)! \|\bar{d}\|_2 \tag{32}$$

where $B_{\rho\gamma}(x, y)$ is the Bêta-function $\int_0^{\rho\gamma} t^{x-1}(1-t)^{y-1}dt$. So, if N^* is such that

$$\sum_{j=0}^{N} \left(\frac{\gamma}{1-\rho\gamma}\right) \frac{B_{\rho\gamma}(N^*+1-j, j+1)}{(N^*-j)!j!} (N^*+1)! < \epsilon \qquad \text{(ii)} \tag{33}$$

then N^* is a choice for the planning horizon that yields an approximation $\hat{h}_i^{N^*}$ of h_i that satisfies equation (i).

To avoid cumbersome calculation we finally note that whenever N^* satisfies the following inequality, then (ii) is satisfied:

$$\epsilon \geqslant (\rho\gamma)^{N^*+1} \cdot (N^*+1)^n \cdot \frac{1 - \left(\frac{\gamma}{\rho\gamma(1-\rho\gamma)}\right)^{n+1}}{1 - \frac{\gamma}{\rho\gamma(1-\rho\gamma)}} \qquad \text{(iii)} \tag{34}$$

References

Anderson, B.D.O. and Moore, J.B., 1989, Optimal Control: Linear Quadratic Methods, Prentice-Hall, London.

Aoki, M., 1973, Notes and comments: On sufficient conditions for optimal stabilization policies, *Review of Economic Studies* **47**, 131–138.

Aström, K.J. and Wittenmark, B., 1990, Computer Controlled Systems, Prentice-Hall, Englewood Cliffs, N.J.

Chow, G.C., 1975, Analyses and Control of Dynamic Economic Systems, John Wiley & Sons, New York.

De Koning, W.L., 1980, Equivalent discrete optimal control problem for randomly sampled digital control systems, *Int. J. Sys. Sci.,* **11**, 841–855.

Dorato, P. and Levis, A.H., 1971, Optimal Linear Regulators: The Discrete-Time Case, *IEEE Trans. Aut. Contr.,* **AC-16**, **6**, 613–620.

Engwerda, J.C., 1990a, The solution of the infinite horizon tracking problem for discrete-time systems possessing an exogenous component, *Journal of Economic Dynamics and Control,* **14**, 741–762.

Engwerda, J.C. and Van Willigenburg, L.G., 1992, LQ-control of sampled continuous-time systems, In V. Strejc (Ed.), Proceedings of the 2nd IFAC workshop on system structure and control, IFAC, Prague, Czechoslovakia, pp. 128–131.

Franklin, G.F. and Powell D., 1980, Digital Control of Dynamic Systems, Addison and Wesley, London.

Kabamba, P.J., 1987, Control of linear systems using generalized sampled-data hold functions, *I.E.E.E. Trans. Aut. Control,* **AC-32** 772–783.

Kalman, R.E., Ho, Y.C. and Narendra, K.S., 1963, Contributions to Differential Equations, vol. I, Interscience, New York.

Kwakernaak, H. and Sivan, R., 1972, Linear Optimal Control System, John Wiley and Sons, New York.

Levis, A.H., Schlueter, R.A. and Athans, M., 1971, On the behavior of optimal linear sampled data regulators, *Int. J. Contr.*, **13**, 343–361.

Lewis, F.L., 1986, Optimal Control, John Wiley & Sons, New York.

Maybeck, P.S., 1982, Stochastic Models Estimation and Control, Vol. 141-3 in the series of Mathematics in Science and Engineering, Academic Press, London.

Pindyck, R.S., 1973, Optimal planning for Economic Stabilization, North Holland, Amsterdam.

Powell, S.R., 1967, B.S. Thesis, Department of Electrical Engineering, M.I.T.

Preston, A.J. and Pagan, A.R., 1982, The Theory of Economic Policy, Cambridge University Press, New York.

Pitchford, J.D. and Turnovsky, S.J., 1977, Applications of Control Theory to Economic Analysis, North-Holland, Amsterdam.

Rosen, I.G. and Wang, C., 1992, Sampling of the linear quadratic regulator problem for distributed parameter systems, *Siam J. Control and Optimization*, **30**, **4** 942–974.

Shats, S. and Shaked, U., 1991, Discrete-time filtering of noise correlated continuous-time processes: modeling and derivation of the sampling period sensitivities, *IEEE Trans. Auto. Contr.,* **AC-36**, 115–119.

Tinbergen, J., 1952, On the Theory of Economic Policy, North Holland, Amsterdam.

Turnovsky, S.J., 1972, Optimal stabilization policies for deterministic and stochastic linear economic systems, *Review of Economic Studies,* pp. 79–95.

Urikura, S. and Nagata, A., 1987, Ripple-free deadbeat control for sampled-data systems, *IEEE Trans. Aut. Contr.,* **AC-32**, **6**, 474–482.

Van Willigenburg, L.G. and De Koning W.L., 1992, The digital optimal regulator and tracker for stochastic time-varying systems, *International Journal of Systems Science,* **23**, **12**, 2309–2322.

Van Willigenburg, L.G., 1992, Computation of the digital LQG regulator and tracker for time-varying systems, *Optimal Control Applications and Methods*, **13**, 289–299.