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Brief Paper

Optimal reduced-order compensation of time-varying discrete-time systems with deterministic and white parameters[1]

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Abstract

The finite-horizon optimal compensation problem is considered in the case of linear time-varying discrete-time systems with deterministic and white stochastic parameters and quadratic criteria. The dimensions of the compensator are *a priori* fixed and may be time varying. Also the dimensions of the system may be time varying. Strengthened discrete-time optimal projection equations (SDOPE) are developed which, within the class of minimal compensators, are equivalent to the first-order necessary optimality conditions. Based on the SDOPE and their associated boundary conditions, two numerical algorithms are presented to solve the two point boundary value problem. One is a homotopy algorithm while the second iterates the SDOPE repeatedly forward and backward in time. The latter algorithm is much more efficient and constitutes a generalization of the single iteration of the control and estimation Riccati equations, associated with the full-order problem for systems with deterministic parameters. The algorithms are illustrated with a numerical example. The case of systems with deterministic parameters will be treated as a special case of systems with white parameters. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

There are mainly two reasons why discrete-time systems with white parameters are important. Firstly, these systems arise naturally in the field of digital control where some of the parameters may be white such as the sampling period (De Koning, 1980), the controller parameters (Wingerden and De Koning, 1984), or the parameters of the plant (Wagenaar and De Koning, 1989). In all these cases it is possible to convert such a digital control system to an equivalent discrete-time system with white parameters (De Koning, 1980; Tiedeman and De Koning, 1984). Also inherent discrete-time systems, such as economic systems, may have white parameters. Secondly, the parameters of an inherent or equivalent discrete-time system may be assumed to be white for the purpose of a robust control system design. It is well known that the standard LQG design does not lead in general to a robust control system with respect to parameter deviations (Doyle, 1978). The use of white parameters to model the system uncertainty offers ways to design robust control systems (Banning and De Koning, 1995; Bernstein, 1987; Bernstein and Greeley, 1986). The advantage of a model with white parameters is that it fits naturally in the LQ design context. Therefore, this approach allows for non-conservative robust control system design with respect to structured parameter variations.

Among others time-varying discrete-time LQG problems are obtained in the case of digital LQG compensation of non-linear continuous-time systems tracking (optimal) trajectories, or as a result of aperiodic and/or asynchronous sampling (Athans, 1971; Van Willigenburg and De Koning, 1995; Van Willigenburg, 1995). The optimal full-order compensator for systems with

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deterministic parameters is a time-varying compensator (Kwakernaak, 1972). The same holds for systems with white parameters as demonstrated in this paper. Due to its time-varying nature the full-order compensator may occupy a serious amount of computer memory. From this point of view, compared to the time-invariant case, there is an even stronger need for controller reduction.

In the case of time-invariant continuous-time and discrete-time systems with deterministic parameters, necessary conditions for the solution of the optimal infinite-horizon reduced-order LQG compensation problem, have been presented in terms of four coupled matrix equations, known as the optimal projection equations (Hyland and Bernstein, 1984; Bernstein et al., 1986) since an oblique projection plays a central role. It was only after the presentation of these equations that a good insight in the reduced-order LQG problem was obtained. This e.g. allowed to trace relations with standard (full-order) LQG theory and provided a very attractive alternative to compute numerical solutions (Bernstein and Greeley, 1986; Richter, 1987; Richter and Collins, 1989; De Koning and De Waard, 1991). Recently, *strengthened* discrete-time optimal projection equations (SDOPE) have been presented in the steady-state case (Van Willigenburg and De Koning, 1997). Within the class of minimal stabilizing compensators the SDOPE were proved to be *equivalent* to first-order necessary optimality conditions as opposed to the *conventional* optimal projection equations, which were proved to be *weaker* and having solutions which do not correspond to optimal reduced-order compensators. Based on the SDOPE two numerical algorithms were proposed, a homotopy algorithm and an iterative algorithm. The latter algorithm constitutes a generalization of the algorithm that solves the control and estimation Riccati equations of full-order steady-state LQG control through iteration. Finally, these results were carried over to systems with white parameters (De Koning and Van Willigenburg, 1997) using results from De Koning (1992) which deals with full-order compensation.

While reduced-order steady-state LQG control received much attention the authors are aware of only three results that address the LQG problem in the case of time-varying systems and a finite horizon (Haddad and Tadmor, 1993; Van Willigenburg and De Koning, 1998; 1997b). Haddad and Tadmor (1993) treated the continuous-time case. They generalized the optimal projection equations to describe necessary conditions for the solution in the finite horizon time-varying case. New features of the equations are the time-varying nature and the dependence on boundary conditions that play no role in the steady-state case. However, the boundary conditions of the four coupled matrix differential equations could not be specified in terms of the parameters making up the LQG problem. This prevents the development of numerical algorithms. In the discrete-time case, Van Willigenburg and De Koning (1998) solved this problem and showed that it relates to the *change* of the dimension of a minimal compensator at the boundaries. This paper generalizes the approach for discrete-time systems with deterministic parameters to systems with white parameters.

Although Van Willigenburg and De Koning (1998) presented the SDOPE and associated boundary conditions in terms of the LQG problem parameters, no methods to solve the resulting two point boundary-value problem (TPBVP) were provided. Of course standard methods to solve the TPBVP may be used. On the other hand, it seems advantageous to exploit the relations between the SDOPE and the control and estimation Riccati equations of finite-horizon full-order LQG control, similar to De Koning and Van Willigenburg (1997) and Van Willigenburg and De Koning (1997). This is the approach adopted in this paper. Based on the SDOPE, two numerical algorithms are proposed, a homotopy algorithm and an iterative algorithm which constitutes a generalization of the single iteration of the control and estimation Riccati equations of full-order LQG control of systems with deterministic parameters. The latter algorithm is much more efficient, and using different initializations, is capable of finding multiple solutions, if they exist. The algorithms are illustrated with a numerical example. The example demonstrates the possible local optimality of an optimal reduced-order LQG compensator and the capability of the iterative algorithm to generate multiple solutions, if they exist.

Observe that the results in Van Willigenburg and De Koning (1998), and the results presented here, allow for time-varying dimensions of the discrete-time system. Time-varying dimensions of discrete-time systems arise in digital control problems, if the sampling is performed in an asynchronous manner (Van Willigenburg and De Koning, 1995; Van Willigenburg 1997).

2. The optimal reduced-order compensation problem

Consider the time-varying discrete-time system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i, \quad i = 0, 1, \ldots, N-1, \tag{1a}$$

$$y_i = C_i x_i + w_i, \quad i = 0, 1, \ldots, N, \tag{1b}$$

where $x_i \in R^{n_i}$ is the state, $u_i \in R^{m_i}$ is the control, $y_i \in R^{l_i}$ is the observation. The processes $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are sequences of independent random matrices with appropriate dimensions and time-varying statistics and $\{v_i\}$, $\{w_i\}$ are sequences of independent stochastic vectors with time-varying statistics. Observe that Φ_i may not be square. The initial condition x_0 is stochastic with mean $\bar{x}_0$ and covariance X and is independent of $\{\Phi_i, \Gamma_i, C_i, v_i, w_i\}$. Moreover Γ_i and C_i are independent and $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are independent of $v_j, w_j, i \neq j$ and uncorrelated with v_i, w_i. The processes $\{v_i\}$, $\{w_i\}$ are zero-mean with covariance's $V_i \geq 0$ and $W_i > 0$ and cross covariance V_i'. As a controller the following time-varying dynamic compensator is chosen:

$$\hat{x}_{i+1} = F_i\hat{x}_i + K_i y_i, \quad i = 0, 1, \ldots, N-1, \tag{2a}$$

$$u_i = -L_i\hat{x}_i, \quad i = 0, 1, \ldots, N, \tag{2b}$$

where $\hat{x}_i \in R^{n_i^c}$ is the compensator state. The dimension n_i^c of the compensator state may be time-varying and $n_i^c \leq n_i$, $i = 0, 1, \ldots, N$. $F_i \in R^{n_{i+1}^c \times n_i^c}$, $K_i \in R^{n_{i+1}^c \times l_i}$ and $L_i \in R^{m_i \times n_i^c}$ are real matrices. Note that F_i may not be square. The initial condition $\hat{x}_0 \in R^{n_0^c}$ is deterministic. Compensator (2) is denoted by $(\hat{x}_0, F^N, K^N, L^N)$ where

$$F^N = \{F_i, i = 0, 1, \ldots, N-1\},$$

$$K^N = \{K_i, i = 0, 1, \ldots, N-1\},$$

$$L^N = \{L_i, i = 0, 1, \ldots, N-1\}.$$

Since the input–output behavior of the compensator is independent of the internal realization only *minimal* compensators need to be considered. Minimal compensators require less storage and computation and furthermore prevent the use of the Moore-Penrose Pseudo inverse, which complicates the optimal projection approach (Van Willigenburg and De Koning, 1998). Therefore, from now on, only minimal compensators will be considered. Finite-horizon minimal reduced-order compensators for the system (1), in general, have time-varying dimensions which satisfy (Van Willigenburg and De Koning, 1998),

$$\bar{x}_0 \neq 0 \Rightarrow n_0^c = 1, \qquad \bar{x}_0 = 0 \Rightarrow n_0^c = 0, \qquad n_N^c = 0, \tag{3a}$$

$$n_i^c - m_i \leq n_{i+1}^c \leq n_i^c + \overset{l}{\cancel{m}}_i, \quad i = 0, 1, \ldots, N-1. \tag{3b}$$

Eq. (3a) states that, at the boundaries, the dimension of the compensator state drops to zero and one. Eq. (3b) states that the *change* of the dimension of the compensator state, from each discrete time instant to the next, is bounded from above and below. Given the restriction to minimal compensators, the designer must specify prescribed compensator dimensions which satisfy Eq. (3) and $n_i^c \leq n_i$. If the prescribed compensator dimensions do not satisfy Eq. (3) and $n_i^c \leq n_i$, they can be reduced to dimensions which do satisfy Eq. (3) and $n_i^c \leq n_i$, without loss of performance (Van Willigenburg and De Koning, 1998). The algorithms, presented in Section 4, have the property that, even if the designer does not specify prescribed dimensions which satisfy Eq. (3), a minimal optimal reduced-order compensator is obtained with dimensions which do satisfy Eq. (3).

Problem formulation

Given the system (1) the optimal reduced-order compensation problem is to find a minimal compensator (2), with prescribed dimensions n_i^c which satisfy Eq. (3) and $n_i^c \leq n_i$, that minimizes the criterion

$$J_N(\hat{x}_0, F^N, K^N, L^N) = E\left\{x_N^{\mathrm{T}} Z x_N + \sum_{i=0}^{N-1} (x_i^{\mathrm{T}} Q_i x_i + 2x_i^{\mathrm{T}} Q_i' u_i + u_i^{\mathrm{T}} R_i u_i)\right\}, \tag{4a}$$

$$Q_i \geq 0,\ R_i > 0,\ Q_i - Q_i' R_i^{-1} Q_i'^{\mathrm{T}} \geq 0,\ Z \geq 0, \tag{4b}$$

and to find the minimum value of J_N.

3. The strengthened discrete-time optimal projection equations (SDOPE)

As in Bernstein et al. (1986), Bernstein and Haddad (1987), Bernstein and Hyland (1988), using the Lagrange multiplier technique first-order necessary optimality conditions can be obtained for the solution of the optimal reduced-order compensation problem. As in De Koning and Van Willigenburg (1997), Van Willigenburg and De Koning (1998) the SDOPE and their associated boundary conditions are obtained through rearrangement of these first-order necessary optimality conditions.

Let $A^{\#}$ denote the group inverse of the matrix $A \in R^{p \times p}$. This inverse is unique and given by (Rao and Mitra, 1971),

$$A^{\#} = A(A^3)^+ A, \tag{5}$$

where $+$ denotes the Moore–Penrose inverse. To state the main theorem the following lemma is needed.

Lemma 1 (Bernstein and Hyland, 1988). *Suppose $\hat{P}$, $\hat{S} \in R^{n \times n}$ are symmetric nonnegative definite and $rank(\hat{P}\hat{S}) = n^c$. Then there exist $G, H \in R^{n^c \times n}$ and $M \in R^{n^c \times n^c}$ such that*

$$\hat{P}\hat{S} = G^{\mathrm{T}} M H, \tag{6a}$$

$$\tau^2 = \tau = \hat{P}\hat{S}(\hat{P}\hat{S})^{\#}, \tag{6b}$$

where τ is defined by

$$\tau = G^{\mathrm{T}} H. \tag{6c}$$

From (6b) τ is an oblique projection (idempotent matrix) uniquely determined by $\hat{P}$ and $\hat{S}$. G, M, H are unique up to a change of basis in R^{n^c}. The triple (G, M, H) is called a projective factorization of $\hat{P}\hat{S}$. Furthermore,

$$(\hat{P}\hat{S})^{\#} = G^{\mathrm{T}} M^{-1} H, \tag{6d}$$

$$H G^{\mathrm{T}} = I_{n^c}, \tag{6e}$$

$$rank(G) = rank(M) = rank(H) = rank(\tau) = n^c. \tag{6f}$$

$\hat{P}\hat{S}$ in Lemma 1 is diagonalizable and has n_c non-zero eigenvalues which are positive (Bernstein and Hyland, 1988). Hence G, M, H and τ can be computed from an eigenvalue decomposition of $\hat{P}\hat{S}$ as follows:

$$\hat{P}\hat{S} = U_{\hat{P}\hat{S}}\Lambda_{\hat{P}\hat{S}}U_{\hat{P}\hat{S}}^{-1}, \quad \Lambda_{\hat{P}\hat{S}} = \begin{bmatrix} \Lambda^{r}_{\hat{P}\hat{S}} & 0 \\ 0 & 0 \end{bmatrix} \tag{7a}$$

$$G = [A^{\mathrm{T}} \quad 0]U_{\hat{P}\hat{S}}^{\mathrm{T}}, \tag{7b}$$

$$M = A^{-1}\Lambda^{r}_{\hat{P}\hat{S}}A, \tag{7c}$$

$$H = [A^{-1} \quad 0]U_{\hat{P}\hat{S}}^{-1}, \tag{7d}$$

$$\tau = U_{\hat{P}\hat{S}}\begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix}U_{\hat{P}\hat{S}}^{-1}, \tag{7e}$$

where the columns of $U_{\hat{P}\hat{S}}$ are eigenvectors of $\hat{P}\hat{S}$ and the elements of the diagonal matrix $\Lambda_{\hat{P}\hat{S}}$ are the eigenvalues of $\hat{P}\hat{S}$ and the n_c non-zero diagonal elements of $\Lambda_{\hat{P}\hat{S}}$ appear first. $A \in R^{n_c \times n_c}$ in Eqs. (7b)–(7d) is an arbitrary non-singular matrix. This reflects the uniqueness of G, M, H up to a change of basis in R^{n_c}.

Let an overbar denote expectation and introduce the following notation:

$$\tilde{\Phi}_i = \Phi_i - \bar{\Phi}_i, \qquad \tilde{\Gamma}_i = \Gamma_i - \bar{\Gamma}_i, \qquad \tilde{C}_i = C_i - \bar{C}_i \tag{8}$$

and also

$$K_{P_i,\hat{P}_i} = (\overline{\Phi_i P_i C_i^{\mathrm{T}}} + \overline{\tilde{\Phi}_i \hat{P}_i \tilde{C}_i^{\mathrm{T}}} + V_i') \times (\overline{C_i P_i C_i^{\mathrm{T}}} + \overline{\tilde{C}_i \hat{P}_i \tilde{C}_i^{\mathrm{T}}} + W_i)^{-1}, \quad i = 0, 1, \ldots, N-1, \tag{9a}$$

$$L_{S_{i+1},\hat{S}_{i+1}} = (\overline{\Gamma_i^{\mathrm{T}} S_{i+1}\Gamma_i} + \overline{\tilde{\Gamma}_i^{\mathrm{T}} \hat{S}_{i+1}\tilde{\Gamma}_i} + R_i)^{-1} \times (\overline{\Gamma_i^{\mathrm{T}} S_{i+1}\Phi_i} + \overline{\tilde{\Gamma}_i^{\mathrm{T}} \hat{S}_{i+1}\tilde{\Phi}_i} + Q_i'^{\mathrm{T}}) \quad i = 0, 1, \ldots, N-1, \tag{9b}$$

$$\Psi_i^1 = (\bar{\Phi}_i - \bar{\Gamma}_i L_{S_{i+1},\hat{S}_{i+1}})\hat{P}_i(\bar{\Phi}_i - \bar{\Gamma}_i L_{S_{i+1},\hat{S}_{i+1}})^{\mathrm{T}} + K_{P_i,\hat{P}_i}(\overline{C_i P_i C_i^{\mathrm{T}}} + \overline{\tilde{C}_i \hat{P}_i \tilde{C}_i^{\mathrm{T}}} + W_i)K^{\mathrm{T}}_{P_i,\hat{P}_i}, \quad i = 0, 1, 2, \ldots, N-1, \tag{9c}$$

$$\Psi_{i+1}^2 = (\bar{\Phi}_i - K_{P_i,\hat{P}_i}\bar{C}_i)^{\mathrm{T}}\hat{S}_{i+1}(\bar{\Phi}_i - K_{P_i,\hat{P}_i}\bar{C}_i) + L^{\mathrm{T}}_{S_{i+1},\hat{S}_{i+1}}(\overline{\Gamma_i^{\mathrm{T}} S_{i+1}\Gamma_i} + \overline{\tilde{\Gamma}_i^{\mathrm{T}} \hat{S}_{i+1}\tilde{\Gamma}_i} + R_i)L_{S_{i+1},\hat{S}_{i+1}}, \quad i = 0, 1, 2, \ldots, N-1, \tag{9d}$$

$$\tau_{\perp i} = I_{n_i} - \tau_i, \quad i = 0, 1, \ldots, N. \tag{9e}$$

Now the main theorem can be stated.

Theorem 1. *The compensator* $(\hat{x}_0, F^N, K^N, L^N)$ *satisfies the first-order necessary optimality conditions for optimal reduced-order compensation and is minimal if and only if there exist nonnegative symmetric* $n_i \times n_i$ *matrices* $P_i, \hat{P}_i$, $i = 0, 1, 2, \ldots, N$ *and* $S_i, \hat{S}_i$, $i = 0, 1, 2, \ldots, N$, *such that for some projective factorizations* (G_i, M_i, H_i) *of* $\hat{P}_i\hat{S}_i$, $i = 0, 1, \ldots, N$.

$$F_i = H_{i+1}[\bar{\Phi}_i - K_{P_i,\hat{P}_i}\bar{C}_i - \bar{\Gamma}_i L_{S_{i+1},\hat{S}_{i+1}}]G_i^{\mathrm{T}} \in R^{n^c_{i+1} \times n^c_i}, \quad i = 0, 1, \ldots, N-1, \tag{10a}$$

$$K_i = H_{i+1}K_{P_i,\hat{P}_i} \in R^{n^c_{i+1} \times l_i}, \quad i = 0, 1, \ldots, N-1, \tag{10b}$$

$$L_i = L_{S_{i+1},\hat{S}_{i+1}}G_i^{\mathrm{T}} \in R^{m_i \times n^c_i}, \quad i = 0, 1, \ldots, N-1, \tag{10c}$$

$$\hat{x}_0 = H_0\bar{x}_0 \in R^{n^c_0}, \tag{10d}$$

and such that P_i, S_i, $\hat{P}_i$, $\hat{S}_i$, τ_i, $i = 0, 1, \ldots, N$, *satisfy*

$$P_{i+1} = \overline{\Phi_i P_i \Phi_i^{\mathrm{T}}} - K_{P_i,\hat{P}_i}(\overline{C_i P_i C_i^{\mathrm{T}}} + \overline{\tilde{C}_i \hat{P}_i \tilde{C}_i^{\mathrm{T}}} + W_i)K^{\mathrm{T}}_{P_i,\hat{P}_i} + \overline{\tilde{\Phi}_i \hat{P}_i \tilde{\Phi}_i^{\mathrm{T}}} - \overline{\tilde{\Phi}_i \hat{P}_i L^{\mathrm{T}}_{S_{i+1},\hat{S}_{i+1}}\tilde{\Gamma}_i^{\mathrm{T}}} - \overline{\tilde{\Gamma}_i L_{S_{i+1},\hat{S}_{i+1}}\hat{P}_i\tilde{\Phi}_i^{\mathrm{T}}} + \overline{\tilde{\Gamma}_i L_{S_{i+1},\hat{S}_{i+1}}\hat{P}_i L^{\mathrm{T}}_{S_{i+1},\hat{S}_{i+1}}\tilde{\Gamma}_i^{\mathrm{T}}} + V_i + \tau_{\perp i+1}\Psi_i^1\tau^{\mathrm{T}}_{\perp i+1}, \quad i = 0, 1, \ldots, N-1, \quad P_0 = X, \tag{10e}$$

$$S_i = \overline{\Phi_i^{\mathrm{T}} S_{i+1}\Phi_i} - L^{\mathrm{T}}_{S_{i+1},\hat{S}_{i+1}} \times (\overline{\Gamma_i^{\mathrm{T}} S_{i+1}\Gamma_i} + \overline{\tilde{\Gamma}_i^{\mathrm{T}} \hat{S}_{i+1}\tilde{\Gamma}_i} + R_i)L_{S_{i+1},\hat{S}_{i+1}} + \overline{\tilde{\Phi}_i^{\mathrm{T}}\hat{S}_{i+1}\tilde{\Phi}_i} - \overline{\tilde{\Phi}_i^{\mathrm{T}}\hat{S}_{i+1}K_{P_i,\hat{P}_i}\tilde{C}_i} - \overline{\tilde{C}_i^{\mathrm{T}}K^{\mathrm{T}}_{P_i,\hat{P}_i}\hat{S}_{i+1}\tilde{\Phi}_i} + \overline{\tilde{C}_i^{\mathrm{T}}K^{\mathrm{T}}_{P_i,\hat{P}_i}\hat{S}_{i+1}K_{P_i,\hat{P}_i}\tilde{C}_i} + Q_i + \tau^{\mathrm{T}}_{\perp i}\Psi^2_{i+1}\tau_{\perp i}, \quad i = 0, 1, \ldots, N-1, \quad S_N = Z, \tag{10f}$$

$$\hat{P}_{i+1} = \tfrac{1}{2}(\tau_{i+1}\Psi_i^1 + \Psi_i^1\tau_{i+1}^{\mathrm{T}}), \quad i = 0, 1, \ldots, N-1, \quad \hat{P}_0 = \bar{x}_0\bar{x}_0^{\mathrm{T}}, \tag{10g}$$

$$\hat{S}_i = \tfrac{1}{2}(\tau_i^{\mathrm{T}}\Psi^2_{i+1} + \Psi^2_{i+1}\tau_i), \quad i = 0, 1, \ldots, N-1, \quad \hat{S}_N = 0, \tag{10h}$$

$$rank(\hat{P}_i) = rank(\hat{S}_i) = rank(\hat{P}_i\hat{S}_i) = n_i^c, \quad i = 0, 1, \ldots, N, \tag{10i}$$

$$\tau_i = \hat{P}_i\hat{S}_i(\hat{P}_i\hat{S}_i)^{\#}, \quad i = 0, 1, \ldots, N. \tag{10j}$$

The costs of the compensator $(\hat{x}_0, F^N, K^N, L^N)$ *are given by*

$$J_N = J_{N_1} = J_{N_2}, \tag{11a}$$

$$J_{N_1} = tr[Z(P_N + \hat{P}_N)] + \sum_{i=0}^{N-1} tr[Q_i P_i + (Q_i + L^{\mathrm{T}}_{S_{i+1},\hat{S}_{i+1}}R_i L_{S_{i+1},\hat{S}_{i+1}} - 2Q_i' L_{S_{i+1},\hat{S}_{i+1}})\hat{P}_i], \tag{11b}$$

$$J_{N_2} = tr[X(S_0 + \hat{S}_0) + \bar{x}_0\bar{x}_0^{\mathrm{T}}S_0] + \sum_{i=0}^{N-1} tr[V_i S_{i+1} + (V_i + K_{P_i,\hat{P}_i}W_i K^{\mathrm{T}}_{P_i,\hat{P}_i} - 2V_i' K^{\mathrm{T}}_{P_i,\hat{P}_i})\hat{S}_{i+1}]. \tag{11c}$$

Proof. The proof follows from the proofs of comparable theorems in Bernstein and Hyland (1988), De Koning and Van Willigenburg (1997) and Van Willigenburg and De Koning (1998). □

Eqs. (10e)–(10j) constitute the *strengthened* discrete-time optimal projection equations. The difference with the *conventional* discrete-time optimal projection equations relates to the following equalities which must hold if the first-order necessary conditions are to be satisfied;

$$\hat{P}_{i+1} = \tau_{i+1}\Psi_i^1 = \Psi_i^1\tau_{i+1}^{\mathrm{T}} = \tau_{i+1}\Psi_i^1\tau_{i+1}^{\mathrm{T}}, \tag{12a}$$

$$\hat{S}_i = \tau_i^{\mathrm{T}}\Psi_{i+1}^2 = \Psi_{i+1}^2\tau_i = \tau_i^{\mathrm{T}}\Psi_{i+1}^2\tau_i. \tag{12b}$$

Now Eqs. (10g) and (10h) ensure that Eqs. (12a) and (12b) are satisfied (Van Willigenburg and De Koning 1998). The final expressions in Eqs. (12a) and (12b), instead of Eqs. (10g) and (10h), determine the conventional discrete-time optimal projection equations (De Koning and De Waard, 1991; Haddad and Moser, 1994). They however do not ensure the second and third equality in Eqs. (12a) and (12b) to hold (Van Willigenburg and De Koning, 1998).

From Eqs. (3a) and (10g) observe that $\hat{P}_N$, $\hat{S}_N$ have dimension zero, so formally they vanish. As a result F_{N-1}, K_{N-1} vanish, which reflects the fact that they do not influence the performance because $\hat{x}_N$ does not influence the performance. Eqs. (10e)–(10j) constitute a two-point boundary value problem (TPBVP). Eq. (10) can be viewed as a generalization of the *strengthened* discrete-time optimal projection equations in the steady-state case (Van Willigenburg and De Koning, 1997) since these are obtained after removal of the time indices and boundary conditions. If the system is deterministic,

$$\begin{aligned} &\Phi_i = \bar{\Phi}_i, \qquad \tilde{\Phi}_i = 0, \qquad \Gamma_i = \bar{\Gamma}_i, \qquad \tilde{\Gamma}_i = 0, \\ &C_i = \bar{C}_i, \qquad \tilde{C}_i = 0. \end{aligned} \tag{12c}$$

From (12c) the result for systems with deterministic parameters is obtained if in Eqs. (9) and (10) the overbars are deleted everywhere and also all terms involving $\tilde{\Phi}_i$, $\tilde{\Gamma}_i$, or $\tilde{C}_i$. As opposed to the steady-state case, the conventional finite-horizon full-order LQG compensator for discrete-time systems with constant dimensions and deterministic parameters, in general, is *not* a special case of Theorem 1. This is because the conventional full-order LQG compensator, in general, is not minimal because its dimensions do not satisfy Eq. (3) (Van Willigenburg and De Koning, 1998).

4. Numerical algorithms

Homotopy degree theory turned out to be the key element in proving the uniqueness of the solution to the *full-order* infinite-horizon time-invariant LQG compensation problem for systems with white parameters (De Koning, 1992). Using the uniqueness of the solution a highly efficient numerical algorithm to compute the optimal full-order compensator, based on iteration of the full-order version of the SDOPE, was derived in De Koning (1992). The algorithm was proved to converge.

In an attempt to prove the uniqueness of the solution to the reduced-order compensation problem the same approach has been applied to steady-state reduced-order LQG control of time-invariant systems with deterministic parameters (De Koning and De Waard, 1991; Richter, 1987; Richter and Collins, 1989). Convex analysis of static output feedback problems (Geromel et al., 1996) and numerical examples in De Koning and Van Willigenburg (1997), Van Willigenburg and De Koning (1997) strongly indicate that these attempts, so far, have failed. Generally speaking it seems that in the case of controller and also model *reduction*, no general uniqueness results can be obtained if quadratic criteria are involved.

Although uniqueness cannot be guaranteed a homotopy and iterative algorithm, based on the SDOPE, can still be derived in the steady-state reduced-order case for both systems with deterministic and white stochastic parameters. The algorithms however, can no longer be *guaranteed* to converge and generate the global minimum. Despite this result the algorithms turn out to work very well in practice (De Koning and Van Willigenburg, 1997; Van Willigenburg and De Koning, 1997). Here, in a similar manner, a homotopy and iterative algorithm will be derived for the finite-horizon time-varying case. The iterative algorithm is much more efficient, and, using different initializations, is capable of generating multiple solutions, if they exist.

Eqs. (10e)–(10g) are *coupled* forward and backward recursions. The coupling is due to (a) the stochastic nature of the system parameters and (b) order reduction. A homotopy constitutes a parameterized family of problems where the parameter varies from zero to one. For the parameter value 0 an "easy" problem with a known solution is obtained. In our case this is the full-order problem for systems with deterministic parameters, for which Eqs. (10e) and (10f) are *not* coupled. For the parameter value 1 the original problem is obtained. By slowly changing the parameter from 0 to 1 the easy problem is deformed into the original problem. In our case by slowly changing the parameter from 0 to 1 both the coupling due to the stochastic nature of the system parameters and the coupling due to order reduction are gradually introduced. Following the solution path as the parameter changes from zero to one, from the solution of the easy problem a solution of the original problem is obtained. If the number of solutions along the solution path remains constant the uniqueness of the solution to the "easy" problem implies the uniqueness of the solution to the original problem. Homotopy degree theory states under which conditions the number of solutions remains constant (Loyd, 1978; Richter and De Carlo, 1983).

To construct the homotopy, let S^{n_i,n_i} denote the space of real $n_i \times n_i$ symmetric matrices. Define

$$X_1^N = \{X_i^1,\ i = 0, 1, \ldots, N\}, \quad X_i^1 \in S^{n_i,n_i}, \tag{13a}$$

$$X_2^N = \{X_i^2,\ i = 0, 1, \ldots, N\}, \quad X_i^2 \in S^{n_i,n_i}, \tag{13b}$$

$$X_3^N = \{X_i^3,\ i = 0, 1, \ldots, N\}, \quad X_i^3 \in S^{n_i,n_i}, \tag{13c}$$

$$X_4^N = \{X_i^4,\ i = 0, 1, \ldots, N\}, \quad X_i^4 \in S^{n_i,n_i}, \tag{13d}$$

and

$$Y_1^N = \{Y_i^1,\ i = 0, 1, \ldots, N\}, \quad Y_i^1 \in S^{n_i,n_i}, \tag{14a}$$

$$Y_2^N = \{Y_i^2,\ i = 0, 1, \ldots, N\}, \quad Y_i^2 \in S^{n_i,n_i}, \tag{14b}$$

$$Y_3^N = \{Y_i^3,\ i = 0, 1, \ldots, N\}, \quad Y_i^3 \in S^{n_i,n_i}, \tag{14c}$$

$$Y_4^N = \{Y_i^4,\ i = 0, 1, \ldots, N\}, \quad Y_i^4 \in S^{n_i,n_i}. \tag{14d}$$

To gradually introduce the coupling due to order reduction define

$$\tau_i^\beta = U_{X_i^3X_i^4}\begin{bmatrix} I_{n_i^{c'}} & 0 \\ 0 & (1-\beta)I_{n_i - n_i^{c'}} \end{bmatrix} U_{X_i^3X_i^4}^{-1},$$

$$\tau_{\perp i}^\beta = I_{n_i} - \tau_i^\beta, \quad \beta \in [0, 1], \tag{15a}$$

$$n_i^{c'} = \min(n_i^c, rank(X_i^3X_i^4)) \tag{15b}$$

where the columns of $U_{X_i^3X_i^4}$, as in Eq. (7), are eigenvectors of $X_i^3X_i^4$ obtained from an eigenvalue decomposition of $X_i^3X_i^4$ arranged such that the *largest positive eigenvalues* appear first on the diagonal of the diagonal matrix $\Lambda_{X_i^3X_i^4}$. Eq. (15b) ensures that τ_i^β is still properly computed, i.e. according to Eq. (10j), if, for some reason, $rank(X_i^3X_i^4) < n_i^c$. Therefore, even if the designer does not specify prescribed compensator dimensions which satisfy Eq. (3), an optimal reduced-order compensator with dimensions which do satisfy Eq. (3) is still obtained. These and other consequences of Eq. (15b) are thoroughly discussed in Van Willigenburg and De Koning (1998). To gradually introduce the coupling due to the stochastic nature of the system parameters, as in De Koning (1992), define

$$\Phi_i^\gamma = \bar{\Phi}_i + \gamma\tilde{\Phi}_i, \qquad \Gamma_i^\gamma = \bar{\Gamma}_i + \gamma\tilde{\Gamma}_i, \qquad C_i^\gamma = \bar{C}_i + \gamma\tilde{C}_i,$$

$$\gamma \in [0, 1]. \tag{16}$$

Note that γ may be conceived as an uncertainty measure. If $\gamma = 0$ the system has deterministic parameters. The homotopy parameter will be denoted by α and three different parameterizations will be considered,

$$\beta = \gamma = \alpha, \qquad \alpha \in [0, 1], \tag{17a}$$

$$\beta = \alpha,\ \gamma = 0, \qquad \alpha \in [0, 1], \tag{17b}$$

$$\beta = 0,\ \gamma = \alpha, \qquad \alpha \in [0, 1]. \tag{17c}$$

Observe from Eqs.(15), (16) that α in (17a) gradually introduces *both* the coupling due to order reduction and the coupling due to the stochastic nature of the system parameters. In Eq. (17b) only the coupling due to order reduction is gradually introduced. Finally, in Eq. (17c), only the coupling due to the stochastic nature of the system parameters is gradually introduced. Therefore, Eq. (17a) amounts to the original problem, Eq. (17b) to the reduced-order problem for systems with deterministic parameters and Eq. (17c) to the full-order problem for systems with white stochastic parameters. Consider the following parameterized nonlinear transformation, based on the SDOPE:

$$(Y_1^N, Y_2^N, Y_3^N, Y_4^N) = \Re_\alpha(X_1^N, X_2^N, X_3^N, X_4^N), \quad \alpha \in [0,1]$$

defined by,

$$\begin{aligned} Y_{i+1}^1 &= \overline{\Phi_i^\gamma Y_i^1 \Phi_i^{\gamma^{\mathrm{T}}}} - K_{Y_i^1,Y_i^3}(\overline{C_i^\gamma Y_i^1 C_i^{\gamma^{\mathrm{T}}}} + \gamma^2\overline{\tilde{C}_i Y_i^3 \tilde{C}_i^{\mathrm{T}}} + W_i)K_{Y_i^1,Y_{ii}^3}^{\mathrm{T}} \\ &\quad + \gamma^2\overline{\tilde{\Phi}_i Y_i^3 \tilde{\Phi}_i^{\mathrm{T}}} - \gamma^2\overline{\tilde{\Phi}_i Y_i^3 L_{Y_{i+1}^2,Y_{i+1}^4}^{\mathrm{T}} \tilde{\Gamma}_i^{\mathrm{T}}} \\ &\quad - \gamma^2\overline{\tilde{\Gamma}_i L_{Y_{i+1}^2,Y_{i+1}^4} Y_i^3 \tilde{\Phi}_i^{\mathrm{T}}} \\ &\quad + \gamma^2\overline{\tilde{\Gamma}_i L_{Y_{i+1}^2,Y_{i+1}^4} Y_i^3 L_{Y_{i+1}^2,Y_{i+1}^4}^{\mathrm{T}} \tilde{\Gamma}_i^{\mathrm{T}}} + V_i \\ &\quad + \tau_{\perp i+1}^\beta \Psi_i^1 \tau_{\perp i+1}^{\beta^{\mathrm{T}}}, \\ &\quad i = 0, 1, \ldots, N-1, \quad Y_0' = X, \end{aligned} \tag{18a}$$

$$\begin{aligned} Y_i^2 &= \overline{\Phi_i^{\gamma\mathrm{T}} Y_{i+1}^2 \Phi_i^\gamma} - L_{Y_{i+1}^2,Y_{i+1}^4}^{\mathrm{T}} \\ &\quad \times(\overline{\Gamma_i^{\gamma^{\mathrm{T}}} Y_{i+1}^2 \Gamma_i^\gamma} + \gamma^2\overline{\tilde{\Gamma}_i^{\mathrm{T}} Y_{i+1}^4 \tilde{\Gamma}_i} + R_i)L_{Y_{i+1}^2,Y_{i+1}^4} \\ &\quad + \gamma^2\overline{\tilde{\Phi}_i^{\mathrm{T}} Y_{i+1}^4 \tilde{\Phi}_i} - \gamma^2\overline{\tilde{\Phi}_i^{\mathrm{T}} Y_{i+1}^4 K_{X_i^1,X_i^3} \tilde{C}_i} \\ &\quad - \gamma^2\overline{\tilde{C}_i^{\mathrm{T}} K_{X_i^1,X_i^3}^{\mathrm{T}} Y_{i+1}^4 \tilde{\Phi}_i} \\ &\quad + \gamma^2\overline{\tilde{C}_i^{\mathrm{T}} K_{X_i^1,X_i^3}^{\mathrm{T}} Y_{i+1}^4 \tilde{C}_i} + Q_i + \tau_{\perp i}^{\beta^{\mathrm{T}}} \Psi_{i+1}^2 \tau_{\perp i}^{\beta^{\mathrm{T}}}, \\ &\quad i = 0, 1, \ldots, N-1, \quad Y_N^2 = Z, \end{aligned} \tag{18b}$$

$$\begin{aligned} Y_{i+1}^3 &= \tfrac{1}{2}(\tau_{i+1}^\beta \Psi_i^1 + \Psi_i^1 \tau_{i+1}^{\beta^{\mathrm{T}}}), \quad i = 0, 1, \ldots, N-1, \\ Y_0^3 &= \bar{x}_0\bar{x}_0^{\mathrm{T}}, \end{aligned} \tag{18c}$$

$$\begin{aligned} Y_i^4 &= \tfrac{1}{2}(\tau_i^{\beta^{\mathrm{T}}} \Psi_{i+1}^2 + \Psi_{i+1}^2 \tau_i^\beta), \quad i = 0, 1, \ldots, N-1, \\ Y_N^4 &= 0, \end{aligned} \tag{18d}$$

where

$$\Psi_i^1 = (\bar{\Phi}_i - \bar{\Gamma}_i L_{X_{i+1}^2, X_{i+1}^4}) Y_i^3 (\bar{\Phi}_i - \bar{\Gamma}_i L_{X_{i+1}^2, X_{i+1}^4})^{\mathrm{T}} + K_{Y_i^1, Y_i^3} (\overline{C_i^\gamma Y_i^1 C_i^{\gamma^{\mathrm{T}}}} + \gamma^2 \overline{\tilde{C}_i Y_i^3 \tilde{C}_i^{\mathrm{T}}} + W_i) K_{Y_i^1, Y_i^3}^{\mathrm{T}},$$

$$i = 0, 1, 2, \ldots, N-1, \tag{19a}$$

$$\Psi_{i+1}^2 = (\bar{\Phi}_i - K_{X_i^1, X_i^3} \bar{C}_i)^{\mathrm{T}} Y_{i+1}^4 (\bar{\Phi}_i - K_{X_i^1, X_i^3} \bar{C}_i) + L_{Y_{i+1}^2, Y_{i+1}^4}^{\mathrm{T}} (\overline{\Gamma_i^{\gamma^{\mathrm{T}}} Y_{i+1}^2 \Gamma_i^\gamma} + \gamma^2 \overline{\tilde{\Gamma}_i^{\mathrm{T}} Y_{i+1}^4 \tilde{\Gamma}_i} + R_i) L_{Y_{i+1}^2, Y_{i+1}^4},$$

$$i = 0, 1, 2, \ldots, N-1. \tag{19b}$$

Call $(X_1^N, X_2^N, X_3^N, X_4^N)$ nonnegative if $X_i^1,\ X_i^2,\ X_i^3,\ X_i^4 \geq 0,\ i = 0, 1, 2, \ldots, N$. Denote the parameterized equation $(Y_1^N, Y_2^N, Y_3^N, Y_4^N) = \mathfrak{R}_\alpha(Y_1^N, Y_2^N, Y_3^N, Y_4^N)$ by $H(Y^\alpha, \alpha) = 0$, where Y^α denotes a nonnegative solution of $(X_1^N, X_2^N, X_3^N, X_4^N) = \mathfrak{R}_\alpha(X_1^N, X_2^N, X_3^N, X_4^N)$. The function $H(Y^\alpha, \alpha)$ is called a homotopy. From Eqs. (15), (16) and (17a) for $\alpha = 1$ $(X_1^N, X_2^N, X_3^N, X_4^N) = \mathfrak{R}_\alpha(X_1^N, X_2^N, X_3^N, X_4^N)$ is equivalent to Eqs. (10e)–(10j) if $n_i^{c'} = n_i^c$ and $X_i^1 = P_i$, $X_i^2 = S_i$, $X_i^3 = \hat{P}_i$ and $X_i^4 = \hat{S}_i$. From Eqs. (15), (16) and (17a) for $\alpha = 0$ $(X_1^N, X_2^N, X_3^N, X_4^N) = \mathfrak{R}_\alpha(X_1^N, X_2^N, X_3^N, X_4^N)$ are the equations associated with the "easy" problem, i.e. the full-order problem for systems with deterministic parameters. In that case, Eqs. (18a) and (18b) are uncoupled and have a unique solution (Kwakernaak, 1972). Therefore Y^0 is unique.

Observe from Eqs. (15), (18) and (19) that to compute $\mathfrak{R}_\alpha$, first Eqs. (18b) and (18d) have to be iterated backward in time and then Eqs. (18a) and (18c) forward in time. Therefore, *repeated* application of $\mathfrak{R}_\alpha$, denoted by $(X_1^{N^k}, X_2^{N^k}, X_3^{N^k}, X_4^{N^k}) = \mathfrak{R}_\alpha^k(X_1^{N^0}, X_2^{N^0}, X_3^{N^0}, X_4^{N^0})$, where k is the number of repetitions, is equivalent to repeated forward and backward iteration of the SDOPE, using partly values obtained in previous iterations, because the forward and backward recursions (9) and (10) are *coupled* through $K_{P_i, \hat{P}_i}$, $L_{S_{i+1}, \hat{S}_{i+1}}$ and τ_i. In the full-order deterministic parameter case, i.e. if $\alpha = 0$, Eqs. (18a) and (18b) constitute the *uncoupled* estimation and control Riccati equations. Then, $(X_1^{N^k}, X_2^{N^k}, X_3^{N^k}, X_4^{N^k}) = (X_1^{N^2}, X_2^{N^2}, X_3^{N^2}, X_4^{N^2})$ $k \geq 2$, because the solution is obtained after one backward and one forward iteration of Eqs. (18b) and (18a). The second iteration is needed to fix Eqs. (18c) and (18d). So in this case convergence is obtained after two iterations. Similar to the homotopy $H(Y^\alpha, \alpha)$ the solution method based on repeated application of $\mathfrak{R}_\alpha$, until convergence is obtained, may be viewed as a *generalization* of the single iteration of the control and estimation Riccati equations of full-order LQG control for systems with deterministic parameters. Summarizing, repeated application of $\mathfrak{R}_\alpha$ results in the following homotopy algorithm, where Θ_{n_i} and I_{n_i} denote the $n_i \times n_i$ zero and identity matrix respectively,

Algorithm 1
Initialization:

$$X_i^1 = \Theta_{n_i}, \quad X_i^2 = \Theta_{n_i}, \quad X_i^3 = I_{n_i}, \quad X_i^4 = I_{ni},$$
$$i = 0, 1, \ldots, N.$$

$\alpha = 0$, $\Delta a = 1/N$, $N \geq 1$ and integer.
compute $Y^\alpha = \lim_{k \to \infty} \mathfrak{R}_\alpha^k(X_1^N, X_2^N, X_3^N, X_4^N)$ through iteration

Loop:

$\alpha := \alpha + \Delta\alpha$
determine, through iteration, whether
$Y^\alpha = \lim_{k \to \infty} \mathfrak{R}_\alpha^k(Y^{\alpha - \Delta\alpha})$ exists
stop when $\alpha = 1$.

Because in the reduced-order case the compensation problem may not have a unique solution, and so the number of solutions along the homotopy path may *change* the convergence of Algorithm 1 cannot be guaranteed. Also if the algorithm converges it cannot be guaranteed to converge to the global minimum. Similar arguments apply to the next algorithm which repeatedly iterates the SDOPE forward and backward in time and which is more efficient than algorithm 1, because it does not involve the homotopy parameter.

Algorithm 2
Initialization:

$$X_i^1 = \Theta_{n_i}, X_i^2 = \Theta_{n_i}, X_i^3 = \Lambda_{n_i}^1, X_i^4 = \Lambda_{n_i}^2,$$
$$i = 0, 1, \ldots, N.$$

with $\Lambda_{n_i}^1, \Lambda_{n_i}^2 \geq 0$, symmetric, random, and with rank n_i^c.

Computation:
Determine, through iteration, whether $Y^1 = \lim_{k \to \infty} \mathfrak{R}_1^k (X_1^N, X_2^N, X_3^N, X_4^N)$ exists.

Theorem 2. *If Algorithm 1 or 2 converges to $Y^1 \geq 0$ it generates a minimal compensator (F^N, K^N, L^N), given by Eqs. (9) and (10) when P_i, S_i, $\hat{P}_i$, $\hat{S}_i$ are replaced by Y_i^1, Y_i^2, Y_i^3, Y_i^4, respectively. This compensator is a local or global minimum of the optimal reduced-order compensation problem with prescribed compensator dimensions equal to rank$(Y_i^3, Y_i^4) = n_i^{c'} \leq n_i^c$.*

Proof. From the definition of $\mathfrak{R}_\alpha$, if the algorithms converge to $Y^1 \geq 0$, the corresponding minimal compensator is given by Eqs. (9) and (10) when P_i, S_i, $\hat{P}_i$, $\hat{S}_i$, are replaced by $Y_i^1, Y_i^2, Y_i^3, Y_i^4$, respectively. From Theorem 1 this compensator satisfies the first-order necessary optimality conditions when the prescribed compensator dimensions equal *rank*$(Y_i^3 Y_i^4) = n_i^{c'} \leq n_i^c$. Because both Algorithms 1 and 2 are *generalizations* of the algorithm that solves the two Riccati equations of full-order LQG control for systems with deterministic parameters, they converge to local (global) minima, *not* to local (global) maxima, which also satisfy the first-order necessary optimality conditions. □

The numerical example in Section 5 and the numerical examples and arguments in De Koning and Van Willigenburg (1997), Van Willigenburg and De Koning (1997) seem to indicate that Algorithms 1 and 2 have the following two important properties. Firstly $n_i^{c'} = n_i^c$ unless a minimal (locally) optimal compensator with dimensions n_i^c does not exist. Secondly, if the algorithms converge, then $Y^1 \geq 0$.

5. Numerical issues and example

In the transformation $\mathfrak{R}_\alpha$ terms like $\overline{\Phi^\gamma A \Gamma^{\gamma^{\mathrm{T}}}}$ and $\overline{\Phi^{\gamma^{\mathrm{T}}} A C^\gamma}$ for some matrix A occur. They may be written as $st^{-1}[(\overline{\Phi^\gamma \otimes \Gamma^\gamma}) st(A)]$ and $st^{-1}[(\overline{C^\gamma \otimes \Phi^\gamma})^{\mathrm{T}} st(A)]$, respectively, where st denotes the stack operator and $\otimes$ the Kronecker product (Bellman, 1970). Furthermore $\overline{\Phi^\gamma \otimes \Gamma^\gamma} = \bar{\Phi} \otimes \bar{\Gamma} + \gamma^2 \overline{\tilde{\Phi} \otimes \tilde{\Gamma}}$. In view of this it is convenient to specify the needed statistics of the parameters by $\bar{\Phi}_i$, $\bar{\Gamma}_i$, $\bar{C}_i$, $\overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i}$, $\overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i}$, $\overline{\tilde{\Gamma}_i \otimes \tilde{\Gamma}_i}$, $\overline{\tilde{\Phi}_i \otimes \tilde{C}_i}$ and $\overline{\tilde{C}_i \otimes \tilde{C}_i}$. In Eq. (15b), $rank(X_i^3\, X_i^4)$ is computed as the number of eigenvalues of $X_i^3 X_i^4$ with a magnitude larger than 10^{-6} times the largest. Furthermore, in Eq. (10), $G_i = [A^{\mathrm{T}}\ 0]\, U_{X_i^3 X_i^4}$ and $H_i = [A^{-1}\ 0]\, U^{-1}_{X_i^3 X_i^4}$ where $A \in R^{n_i^c \times n_i^c}$ is an arbitrary invertible matrix. In the algorithms A is the identity matrix. Since $U_{X_i^3 X_i^4}$ may be ill-conditioned, instead $U_{X_i^3 X_i^4 + 10^{-12} I_{n_i}}$ is computed. The iteration $(X_1^{N^k}, X_2^{N^k}, X_3^{N^k}, X_4^{N^k}) = \mathfrak{R}_\alpha(X_1^{N^0}, X_2^{N^0}, X_3^{N^0}, X_4^{N^0})$ is numerically stable in general. In critical situations the symmetry of X_i^1 and X_i^2 must be enforced further by performing, $X_i^1 = \frac{1}{2}(X_i^1 + X_i^{1^{\mathrm{T}}})$, $X_i^2 = \frac{1}{2}(X_i^2 + X_i^{2^{\mathrm{T}}})$ at the end of each iteration. In the case of numerically difficult examples, where the magnitude of the state of the closed loop system becomes very large, the *numerical* stability is greatly enhanced if we apply the following computation at the end of each iteration, $X_i^{1^k} = (1-a) X_i^{1^k} + a X_i^{1^{k-1}}$ and similarly for X_i^2, X_i^3, X_i^4 where k is the iteration index and $a \in [0, 1]$ the coefficient which determines this numerical damping. In the following example, the computations mentioned above were applied with $a = 0.25$. Based on arguments in De Koning (1992), convergence is assumed when the relative difference between successive values of $trace(S_0 + P_N)$ falls repeatedly below a certain tolerance, in our case 10^{-8}. The random nonnegative symmetric matrices with prescribed rank, by which Algorithm 2 is initialized, are obtained as follows. Using a random number generator a random real square matrix A with the appropriate dimensions is generated. From a singular value decomposition $A = U \Lambda V^{\mathrm{T}}$ is obtained. The smallest singular values on the diagonal of the diagonal matrix Λ are set to zero such that the resulting matrix Λ' has the prescribed rank. Then A', i.e. the random nonnegative symmetric matrix with prescribed rank, is calculated as $A' = U \Lambda' U^{\mathrm{T}}$.

The following example exhibits most key-features of the problem formulation, i.e. the system is time-varying, Φ_i, Γ_i and Φ_i, C_i are dependent, cross products Q_i' are present in the criterion, the additive system and measurement noise have non-zero cross-covariance matrices V_i', and the prescribed dimension of the compensator state varies over time. To limit space only the outcome of the two expressions for the minimum costs (11b) and (11c), i.e. J_{N_1}, J_{N_2} are presented, since they require the computation of all the other matrices which appear in the SDOPE.

Example. Consider the finite-horizon discrete-time LQG compensation problem (1), (2) and (4) with

$$\bar{\Phi}_i = (1 + 0.2\sin(i)) \begin{bmatrix} -0.9653 & 0.7942 \\ -0.7942 & -0.9653 \end{bmatrix},$$

$$\bar{\Gamma}_i = \begin{bmatrix} 0.4492 \\ 0.1784 \end{bmatrix}, \qquad \bar{C}_i = [0.6171 \quad 0.3187],$$

$$\overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i} = \lambda(\bar{\Phi}_i \otimes \bar{\Phi}_i), \quad \overline{\tilde{\Gamma}_i \otimes \tilde{\Gamma}_i} = \lambda(\bar{\Gamma}_i \otimes \bar{\Gamma}_i),$$

$$\overline{\tilde{C}_i \otimes \tilde{C}_i} = \lambda(\bar{C}_i \otimes \bar{C}_i),$$

$$\overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i} = \lambda(\bar{\Phi}_i \otimes \bar{\Gamma}_i), \quad \cancel{\overline{\tilde{\Gamma}_i \otimes \tilde{\Phi}_i} = \lambda(\bar{\Gamma}_i \otimes \bar{\Phi}_i)},$$

$$\overline{\tilde{\Phi}_i \otimes \tilde{C}_i} = \lambda(\bar{\Phi}_i \otimes \bar{C}_i), \quad \cancel{\overline{\tilde{C}_i \otimes \tilde{\Phi}_i} = \lambda(\bar{C}_i \otimes \bar{\Phi}_i)},$$

$$V_i = diag(0.7327 \quad 0.8612),$$

$$V_i' = [-0.0677 \quad -0.05360]^{\mathrm{T}}, \quad W_i = 0.9334,$$

$$Q_i = diag(0.0437 \quad 0.1108), \quad Q_i' = [-0.0859 \quad -0.0107]^{\mathrm{T}}$$

$$R_i = 0.3311,$$

$$n_i^c = 1, \quad i = 0, 1, 2, 3, 6, 7, 8,$$

$$n_i^c = 2, \quad i = 4, 5, \; N = 9,$$

$$\bar{x}_0 = [1 \quad 1]^{\mathrm{T}}, \quad X = diag(0.1 \quad 0.1), \quad Z = diag(0.1 \quad 0.1).$$

The spectral radius of $\bar{\Phi}_i$ equals $(1 + 0.2\sin(i)) \times 1.25$ so the system is mean-square-unstable (De Koning, 1992). The parameter λ may be viewed as a system parameter uncertainty measure (De Koning, 1992). If $\lambda = 0$ then the system has deterministic parameters and as λ increases the parameter uncertainty increases. For three different values of λ the solutions found by Algorithms 1 and 2 are presented in Table 1. In each case Algorithm 2 finds two different solutions. Of these two solutions, in each case, Algorithm 1 finds the best. Unfortunately, Algorithm 1 cannot be guaranteed to find the global minimum (Van Willigenburg and De Koning, 1997). The final column in Table 1 gives the minimum costs obtained with the optimal full-order compensator.

Table 1
Solutions obtained from Algorithms 1 and 2 for different values of the system parameter uncertainty measure λ with a convergence tolerance of 10^{-8}

	Algorithm 1: $\Delta\alpha = 0.1$		Algorithm 2	Full-order
$\lambda = 0$	$J_{N_1} = 33.5487$ $J_{N_2} = 33.5487$	$J_{N_1} = 33.5487$ $J_{N_2} = 33.5487$	$J_{N_1} = 33.7895$ $J_{N_2} = 33.7895$	$J_{N_1} = 29.3773$ $J_{N_2} = 29.3773$
$\lambda = 0.01$	$J_{N_1} = 35.2902$ $J_{N_2} = 35.2902$	$J_{N_1} = 35.2902$ $J_{N_2} = 35.2902$	$J_{N_1} = 36.1523$ $J_{N_2} = 36.1523$	$J_{N_1} = 30.8781$ $J_{N_2} = 30.8781$
$\lambda = 0.1$	$J_{N_1} = 55.0898$ $J_{N_2} = 55.0898$	$J_{N_1} = 55.0898$ $J_{N_2} = 55.0898$	$J_{N_1} = 55.5067$ $J_{N_2} = 55.5067$	$J_{N_1} = 47.8533$ $J_{N_2} = 47.8533$

Table 2
Performance of the algorithms with a convergence tolerance of 10^{-6} when $\lambda = 0.1$

Number of iterations of Algorithm 2 for 10 random Initializations: average = 58									
44	51	49	64	63	60	62	66	72	49
Number of iterations of algorithm 1 with $\Delta\alpha = 0.1$: 301									

If one specifies $n_i^c = n_i$, $i = 0, 1, \ldots, N-1$, Algorithms 1 and 2, due to Eq. (15b), generate a *minimal realization* of the optimal full-order compensator with dimensions which satisfy Eq. (3). Table 2 gives an impression of the performance of the algorithms on the example. Using Matlab 4.2c2 on a pentium 90 MHz PC each iteration takes approximately 0.26 s. With a convergence tolerance of 10^{-6} J_{N_1} and J_{N_2} and are equal up to the third decimal.

6. Conclusions

To numerically solve the optimal finite-horizon reduced-order LQG compensation problem for systems with white parameters strengthened discrete-time optimal projection equations (SDOPE) were presented together with two numerical algorithms to solve the associated two point boundary value problem. As in the infinite-horizon time-invariant case the algorithms exploit the resemblance between the SDOPE and the two Riccati equations of full-order LQG control for systems with deterministic parameters. The algorithms were illustrated with a numerical example.

Despite the strengthening of the optimal projection equations, unfortunately, in the case of reduced-order compensation, it seems that in general multiple solutions may exist. This phenomenon prevents the use of homotopy degree theory to prove the convergence of the algorithms. Despite this result, in the infinite-horizon time-invariant case the algorithms turn out to work very well in practice (De Koning and Van Willigenburg, 1997; Van Willigenburg and De Koning, 1997). Although not reported in this paper similar numerical experiences have been obtained with the algorithms in this paper. In the case of full-order compensation of systems with white parameters the situation is much more favorable. In that case homotopy degree theory can be used to prove the uniqueness of the solution and the convergence of the algorithms in the infinite-horizon time-invariant case (De Koning, 1992). A similar result is expected in the finite-horizon time-varying case. However, in the latter case there is the technical difficulty that the optimal full-order compensator, in general, is not minimal (Van Willigenburg and De Koning, 1998). Therefore, this subject is left for future research.

With respect to the possible local optimality of optimal reduced-order compensators the following practical approach is suggested. Apply Algorithm 2, which is initialized randomly, several times and pick the best solution. Of course, one can never be sure that better solutions do not exist. However, compared with the performance of the optimal full-order compensator, the loss of performance may serve as a criterion for acceptance of a (locally) optimal reduced-order compensator.

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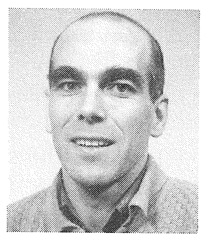

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