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DIGITAL OPTIMAL CONTROL OF NONLINEAR UNCERTAIN SYSTEMS APPLIED TO RIGID MANIPULATORS

Abstract

Based on the recently developed numerical solution of the sampled-data (digital) LQG problem for linear time-varying systems we will treat the design and computation of *implementable digital compensators* for continuous-time nonlinear uncertain systems. A compensator is used to control the system about a so called ideal input-state response. The ideal input-state response is computed off-line through optimization and represents the desired system behavior. In this paper both the compensator design and the optimization are characterized by the fact that *the continuous-time system behavior and the digital nature of the controller are explicitly considered* in both problems. Usual controller designs *neglect the inter-sample behavior or the digital nature of the controller.* The digital controllers that result from our design procedure need only a very small number of on-line computations to be performed. As an example we compute and simulate digital controllers for a robotic manipulator.

1. Introduction

Industrial processes very often constitute continuous-time systems. The automatic control of industrial processes is generally performed by digital computers. In these cases the automatic control system is a digital control system that can be schematically represented by figure 1. The continuous-time system has a sampler at the output and a sampler and zero order hold circuit at the input.

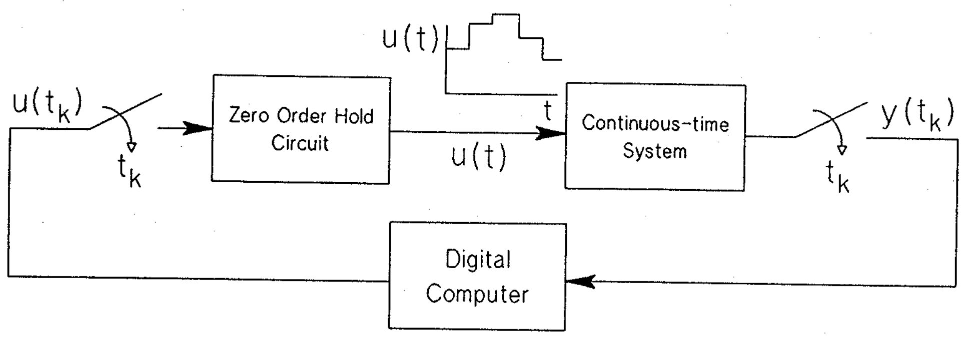

Figure 1: Digital Control System

The design of a digital controller for a continuous-time system is often referred to as a *digital control problem.* The term *digital* throughout this paper will refer to the facts that

a) We have sampled measurements since a computer cannot deal with continuous-time measurements

b) The control is of piecewise constant nature (a stair case function), since a sampler and zero order hold circuit connect the computer to the input of the system

c) We consider the continuous-time behavior of the system.

Although these seem all very straightforward considerations, much to surprise, very often at least one of these considerations is not met in the design of digital controllers for continuous-time systems. Very often the design only considers the system behavior at the sampling instants, *completely disregarding the inter-sample behavior* (Ackerman 1985, Astrom and Wittenmark 1984, Franklin and Powell 1980). So in that case consideration c) is not met. In other cases continuous-time control algorithms are designed, which then somehow have to be *approximated* by a digital controller

(Athans 1971). In these cases both consideration a) and b) are not met. In both cases there is a demand for a *"small" sampling time*, in the former case to prevent undesirable inter-sample behavior, in the latter case to properly approximate the continuous-time algorithm. This demand, for instance in the case of robot control where the computational burden on the computer is high, results in computational difficulties. Even if the sampling time is chosen to be "small" the digital controllers will only constitute *approximate solutions*.

Over the years only a few publications have appeared which consider digital control problems in the proper context just described (Levis, Schlueter and Athans 1971, Nour Eldin 1971, Halyo and Caglayan 1976, De Koning 1980,1984, Stengel 1986, Van Willigenburg and De Koning 1990 a,b). From these publications it is apparent that it hardly takes extra effort to solve digital control problems in the proper context.

Since most industrial processes are nonlinear in this paper we will deal with the design of digital controllers for continuous-time nonlinear uncertain systems where the uncertainty consists of additive white system and measurement noise. Based on the solution to the digital LQG problem (Van Willigenburg and De Koning 1990b) and a numerical procedure to compute it (Van Willigenburg 1990a) we will treat the design and computation of a digital compensator which is used to control a nonlinear system about an off-line determined so called ideal input-state response. This response defines the desired system behavior. The computation of the ideal input-state response will also be treated. The design procedure may be compared to the one presented by Athans (1971). He considered the design of continuous-time controllers where we consider the design of *implementable digital controllers.* The digital controllers that result are characterized by the fact that the number of on-line computations to be performed is very small.

As an example we will present the design and simulation of digital controllers for the IBM 7535 B 04 robot, which constitutes a highly nonlinear system. Using our approach we will demonstrate that proper results are obtained with controllers that have a sampling time of 70mS, which is generally considered too large for robot control (Craig 1986).

2. Continuous-time optimal control of nonlinear uncertain systems

Athans (1971) has excellently described the use of the solution to the continuous-time LQG problem to control continuous-time nonlinear uncertain systems about an ideal input-state response, that defines the desired system behavior. The ideal input-state response will often be referred to as the trajectory. The uncertainty is modeled by additive white gaussian system and measurement noise and the behavior is considered over a finite time interval $[t_0, t_f]$. The dynamics of the continuous-time nonlinear uncertain system are therefore given by

$$\dot{x}(t) = f(x(t),u(t),t) + \xi(t), \qquad t \in [t_0, t_f] \tag{1a}$$

$$y(t) = g(x(t),u(t),t) + \theta(t), \qquad t \in [t_0, t_f] \tag{1b}$$

where f is a nonlinear function and also g may be a nonlinear function. ξ and θ represent the additive white system and measurement noise. Based on the linearized dynamics about the trajectory, which approximately describe the dynamic behavior of small deviations from the trajectory, the solution of the continuous-time LQG problem, which constitutes a compensator, is used to control deviations from the trajectory to zero. The continuous time control system is schematically represented in figure 2.

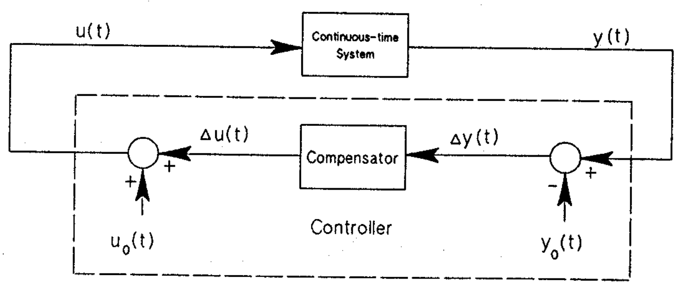

Figure 2: Continuous-time control system with a compensator based on linearized dynamics

Athans divides the controller design procedure into five parts, each part involving several steps.

Part A: *Deterministic Modeling.*

Step 1: Determination of the deterministic part of the nonlinear system equation (1a), i.e. $f(x(t),u(t),t)$.

Step 2: Determination of the deterministic part of the output equation (1b), i.e. $g(x(t),u(t),t)$.

Step 3: Based on the deterministic version of the model (1), i.e.

$$\dot{x} = f(x(t),u(t),t) \qquad [2a]$$

$$y(t) = g(x(t)u(t),t) \qquad [2b]$$

determine a so called ideal input-state-output response,

$u_0(t)$:	ideal input,	[3a]
$x_0(t)$:	ideal state response,	[3b]
$y_0(t)$:	ideal output response.	[3c]

The ideal input-state response reflects how we actually want the system to behave. $u_0(t)$ and $x_0(t)$ are related through equation (2a) and constitute the trajectory about which we want to control the system. The ideal input-state response, i.e. the trajectory, may be the outcome of a deterministic optimization problem constrained by the nonlinear dynamics (2a). The ideal output response via (2b) directly follows from the ideal state response.

Part B: *Stochastic Modeling.*

Step 4: Modeling of uncertainty in the initial state of the system (1).

Selection of the mean $\bar{x}(t_0)$.
Selection of the covariance

$$\Sigma_0 = \text{cov}[x(t_0);x(t_0)]. \quad [4]$$

Step 5: Modeling of uncertainty in the system (1).

Selection of the covariance

$$\Xi(t)\delta(t-\tau) = \text{cov}[\xi(t);\xi(\tau)] \quad [5]$$

where $\xi(t)$ is the additive white system noise in (1a).

Step 6: Modeling of measurement uncertainty.

Selection of the covariance

$$\Theta(t)\delta(t-\tau) = \text{cov}[\theta(t);\theta(\tau)] \quad [6]$$

where $\theta(t)$ is the additive white measurement noise in (1b).

Part C: *Linearization modeling.*

Step 7: Establishing of the linearized model

$$\delta\dot{x}(t) = A_0(t)\ \delta x(t) + B_0(t)\ \delta u(t), \qquad [7a]$$

$$\delta y(t) = C_0(t)\ \delta x(t), \qquad [7b]$$

about the ideal input-state response $u_0(t)$, $x_0(t)$, below referred to by o, i.e.

$$A_0(t) = \left.\frac{\partial f}{\partial x(t)}\right|_o, \quad B_0(t) = \left.\frac{\partial f}{\partial u(t)}\right|_o, \quad C_0(t) = \left.\frac{\partial g}{\partial x(t)}\right|_o, \qquad [7c]$$

which approximately describes the dynamic behavior of the perturbation variables

$$\delta u(t) = u(t) - u_0(t), \qquad [8a]$$

$$\delta x(t) = x(t) - x_0(t), \qquad [8b]$$

$$\delta y(t) = y(t) - y_0(t), \qquad [8c]$$

as long as they are small.

Step 8: With due consideration of Σ_0, $\Xi(t)$ and $\Theta(t)$ and depending on the degree of nonlinearity of the system (1) select the cost weighting matrices $Q_0(t)$, $R_0(t)$ and F_0 of the cost criterion

$$J(u) = \delta x^T(t_f) F_0 \delta x(t_f) + \int_{t_0}^{t_f} \delta x^T(t) Q_0(t) \delta x(t) +$$

$$\delta u^T(t) R_0(t) \delta u(t)\ dt. \qquad [9]$$

which is used to keep the perturbation variables, i.e. deviations from the trajectory, small.

Part D: *Control problem computation.*

Step 9: Given the matrices established in steps 7 and 8, solve the linear regulator problem [8], [9], i.e. solve backward in time the control Ricatti equation,

$$\dot{K}_0(t) = -K_0(t)A_0(t) - A_0^T(t)K_0(t) - Q_0(t) +$$
$$K_0(t)B_0(t)R_0^{-1}(t)B_0^T(t)K_0(t),\ K_0(t_f) = F_0. \qquad [10]$$

Step 10: From the solution $K_0(t)$, determine the feedback gain matrix,

$$G_0(t) = R_0^{-1}(t)B_0^T(t)K_0(t). \qquad [11]$$

Part E: *Filtering problem computation.*

Step 11: Given the matrices established in steps 4,5 and 6 solve forward in time the filter Ricatti equation,

$$\dot{\Sigma}_0(t) = A_0(t)\Sigma_0(t) + \Sigma_0(t)A_0^T(t) + \Xi(t) -$$
$$\Sigma_0(t)C_0(t)\Theta_0^{-1}(t)C_0^T(t)\Sigma_0(t),\ \Sigma_0(t_0) = \Sigma_0 \qquad [12]$$

Step 12: From the solution $\Sigma_0(t)$, determine the filter gain matrix,

$$H_0(t) = \Sigma_0(t)C_0^T(t)\Theta^{-1}(t). \qquad [13]$$

Part F: *Construction of the linearized dynamic compensator.*

Step 13: The linearized dynamic compensator is given by

$$\delta\dot{\hat{x}}(t) = [A_0(t) - B_0(t)G_0(t) - H_0(t)C_0(t)]\ \delta\hat{x}(t)$$

$$+ H_0(t)\delta y(t),\ \delta\hat{x}(t_0) = \bar{x}_0 - x_0(t_0), \qquad [14a]$$

$$\delta u(t) = -G_0(t)\ \delta\hat{x}(t) \qquad [14b]$$

where $\delta\hat{x}(t)$ is the minimum variance estimate of $\delta x(t)$, generated by the filter.

Since all matrices appearing in the compensator equation (14) can be computed off-line we observe that the number of on-line computations to be performed is very small which is a very attractive property. Athans stresses the importance of part A and B and step 8 of part C since these all involve modeling issues, where the ability of the engineer is crucial, since no recipes exist to translate the "real world" into a mathematical model. The other steps Athans calls mechanical since a variety of computational techniques to solve the Ricatti differential equation were already available then.

Concerning the implementation Athans suggest the implementation of an approximation of (14) on a digital computer. The exact implementation of the continuous-time control algorithm (14) on a digital computer is impossible since the digital nature prevents the ability to deal with continuous-time measurements as well as it prevents the generation of a control which is adjusted continuously in time.

3. Digital (sampled-data) optimal control of nonlinear uncertain systems

Since the compensator (14) cannot be implemented in a digital computer approximations have to be made, or we have to incorporate the digital nature of the computer into the control problem. Consider the general digital control system depicted in figure 1 with a sampler at the output and a sampler and zero order hold at the input of the continuous-time system. The sampling process is characterized by

a) sampling instants $t_0<t_1<t_2<...<t_N$
b) sampling periods $T_k=t_{k+1}-t_k$, $k=0,1,2,..,N-1$
c) sampling intervals $[t_{k+1},t_k)$, $k=0,1,2,..,N-1$

The tasks to be performed by the computer during the sampling interval $[t_{k+1},t_k)$ are schematically represented by figure 3. At time t_k the computer must adjust the control $u(t_k)$ and observe the output $y(t_k)$. Within $[t_{k+1},t_k)$ the next control $u(t_{k+1})$ must be computed.

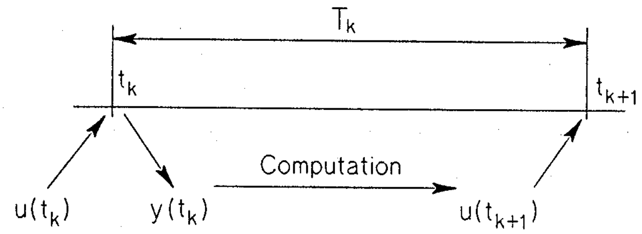

Figure 3: Task sequence of the computer

Observe that the sampling instants are not necessarily equidistant. Because of the sampler and the zero order hold at the input of the continuous-time system the continuous-time control is now of the following form

$$u(t) = u(t_k), \quad t\in[t_k,t_{k+1}), \quad k=0,1,2,..,N-1 \qquad [15]$$

The continuous-time control (15) is called a piecewise constant control and is uniquely determined by the finite sequence

$$u(t_k), \quad k=0,1,2,..,N-1 \qquad [16]$$

Because of the sampler at the output we now obtain a finite discrete-time sequence of measurements given by

$$y_k = y(t_k) \qquad k=0,1,2,..,N-1 \qquad [17]$$

Summarizing instead of continuous-time measurements we now have a finite discrete-time sequence of measurements (17) and instead of an unconstrained continuous-time control we now have a piecewise constant control (15). Finally we assume the final sampling instant to satisfy

$$t_N = t_f \qquad [18]$$

where t_f is given by (1).

The piecewise constant nature of the control restricts the choice of the ideal input $u_0(t)$ determined in step 3. When considered over the finite time interval $[t_0,t_f]$, unconstrained continuous-time controls constitute an infinite dimensional space, while piecewise constant controls constitute a finite dimensional space, since they are uniquely determined by the finite sequence (16).

Consider the ideal input-state response in step 3 to be the outcome of an optimization problem. This problem is of the following general form (Lewis 1986). Given the deterministic initial state

$$x(t_0) = x_0 \qquad [19]$$

of the deterministic system (2) minimize the integral criterion

$$J(x,u) = \Psi(x(t_f),t_f) + \int_{t_0}^{t_f} L(x,u,t)dt \qquad [20]$$

constrained by the dynamics (2) and some additional state and control constraints,

$$x(t) \in X, \quad t \in [t_0, t_f] \tag{21a}$$

$$u(t) \in U, \quad t \in [t_0, t_f] \tag{21b}$$

Note that any admissible control, i.e. a control satisfying (21b) via (2) and (19) uniquely determines the value of the integral criterion (20). So the optimization problem constitutes an optimal control problem, i.e. the problem of finding a control satisfying (21b) which minimizes (20) such that (21a) is satisfied. If the control constraints (21b) limit the controls to belong to a finite dimensional space, each control is uniquely determined by a finite sequence. An example is the piecewise constant constrained (15) on the control, each piecewise constant control being uniquely determined by the finite sequence (16). In these cases the optimal control problem may be regarded as the problem of minimizing the generally nonlinear function (20) of a finite sequence, i.e. the sequence (16) in case of a piecewise constant constrained on the control. Summarizing in these cases the optimal control problem may be regarded as the constrained minimization of a nonlinear function of a finite number of variables (Goh and Teo 1988). This problem is generally much more easy to solve than the original one. The nonlinear function value can be computed by numerical integration of equation (2a) and (20) given the control (15). In other words, to account for the piecewise constant constrained on the control *simplifies* the determination of the ideal input-state response through optimization. Very often however, the optimization is performed without considering the piecewise constant constrained on $u_0(t)$!

Obviously the fact that we now have a finite discrete-time sequence of measurements and a piecewise constant constraint on the control also affects the compensator (14). We now obtain a digital control system schematically represented by figure 4.

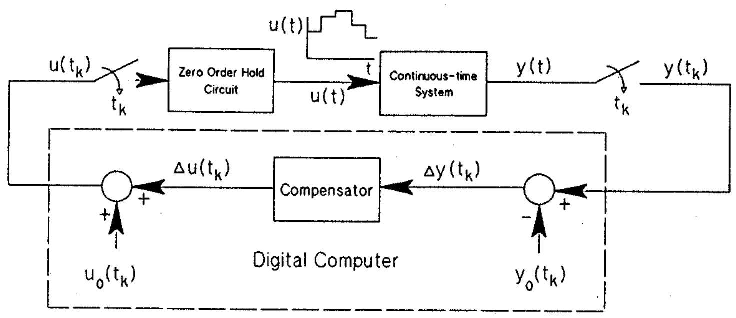

Figure 4: Digital control system with a digital compensator based on linearized dynamics

We now have to solve a different LQG problem, called the sampled-data LQG problem (Halyo and Caglayan 1976) or the digital LQG problem (Van Willigenburg and De Koning 1990b). Halyo and Caglayan only partially solved the sampled-data LQG problem since they did not present expressions for the minimum cost of the problem. Expressions for the minimum cost, explicit in the system and criterion matrices, were presented by Van Willigenburg and De Koning, who solved both the digital regulator and tracking problem completely. Both publications however were not concerned with the numerical computation of the solution. The numerical computation is not straightforward since it for instance involves the computation and integration of expressions involving the state-transition matrix of time-varying linear systems. These problems were solved by Van Willigenburg (1990a) who presented a numerical solution to the digital LQG problem.

If we incorporate the digital nature of the control system into the control problem, we have to replace the solution of the continuous-time LQG problem by the solution of the digital (sampled-data) LQG problem. Considering the design procedure presented in the previous paragraph only Part D,E and F, i.e. only the mechanical part of the design changes. So the modeling issues, where the ability of the engineer is crucial, are not at all affected! Although the modeling issues are unchanged we do have to reconsider step 3, and as a result the mechanical step 7, since step 3 is affected by the piecewise constant constraint on the control. Still using the same models and reconsidering step 3, and as a result step 7, by replacing the numerical tools to compute the solution to the continuous-time LQG problem by tools to compute the solution to the digital LQG problem (Van Willigenburg 1990a) we design a *truly implementable digital controller.* The new versions of part D,E and F of the design procedure are given below

Part D: *Control problem computation.*

Step 9: Given the matrices established in steps 7 and 8, solve the digital regulator problem associated with [8], [9], and the digital control system in figure 1, i.e. solve the discrete-time control Ricatti equation,

$$S_k=(\Phi'_k-\Gamma_k G_k)^T S_{k+1}(\Phi'_k-\Gamma_k G_k)+G_k^T R_k G_k+Q'_k, \quad S_N=F_0. \qquad [22]$$

where the index k refers to values at the sampling instant t_k. The matrices in equation (22) are given by

$$\Phi'_k = \Phi_k - \Gamma_k R_k^{-1} M_k^T, \qquad [23a]$$

$$Q'_k = Q_k - M_k R_k^{-1} M_k^T, \qquad [23b]$$

$$G_k=(\Gamma_k S_{k+1}\Gamma_k+R_k)^{-1}\Gamma_k^T S_{k+1}\Phi'_K, \qquad [23c]$$

where

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi^T(t,t_k)\; Q_0(t)\; \Phi(t,t_k)\; dt, \qquad [23d]$$

$$M_k = \int_{t_k}^{t_{k+1}} \Phi^T(t,t_k)\ Q_0(t)\ \Gamma(t,t_k)\ dt, \qquad [23e]$$

$$R_k = \int_{t_k}^{t_{k+1}} [R_0(t) + \Gamma^T(t,t_k)\ Q_0(t)\ \Gamma(t,t_k)]\ dt, \qquad [23f]$$

$$\Gamma_k = \Gamma(t_{k+1}, t_k), \qquad [23g]$$

in which

$$\Gamma(t,t_k) = \int_{t_k}^{t} \Phi(t,s)\ B_0(s)\ ds, \qquad [23h]$$

$$\Phi_k = \Phi(t_{k+1}, t_k), \qquad [23i]$$

Φ being the state transition matrix of the linearized system (7a).

Step 10: The feedback gain matrices are given by (23c) so

$$G_k = (\Gamma_k S_{k+1} \Gamma_k + R_k)^{-1} \Gamma_k^T S_{k+1} \Phi_K', \qquad k=0,1,2,..,N-1 \qquad [24]$$

Part E: *Estimator problem computation.*

Step 11: Given the matrices established in steps 4,5 and 6 solve forward in time the discrete-time predictor Ricatti equation,

$$P_{k+1} = (\Phi_k - H_k C_k) P_k (\Phi_k - H_k C_k)^T + H_k \Theta_k H_k^T + \Xi_k, \quad P_0 = \Sigma_0(t_0) \qquad [25]$$

where again the index k refers to the values at the sampling instants t_k and

$$H_k = \Phi_k P_k C_k^T (C_k P_k C_k^T + \Theta_k)^{-1} \qquad [26a]$$

and finally

$$C_k = C_0(t_k). \qquad [26b]$$

Step 12: The Kalman one step ahead predictor gain matrices are given by (26a) so

$$H_k = \Phi_k P_k C_k^T (C_k P_k C_k^T + \Theta_k)^{-1}. \qquad [27]$$

Part F: *Construction of the linearized dynamic compensator.*

Step 13: The linearized dynamic compensator is given by

$$\delta\hat{x}(t_{k+1}) = [\Phi_k - H_k C_k]\delta\hat{x}(t_k) + H_k \delta y(t_k) + \Gamma_k \delta u(t_k),$$

$$\delta\hat{x}_0 = x(t_0) - \bar{x}_0, \qquad [28a]$$

$$\delta u(t_k) = -G_k \delta\hat{x}(t_k) \qquad [28b]$$

where $\delta\hat{x}(t_k)$ is the minimum variance estimate of $\delta x(t_k)$, generated by the one step ahead predictor.

Part F from the continuous-time Kalman Filter now turns into the discrete-time Kalman one step ahead predictor (Van Willigenburg and De Koning 1990b), which is very well known. Part D from the continuous-time linear optimal regulator turns into the digital optimal regulator (Van Willigenburg and De Koning 1990b) or sampled-data optimal regulator (Halyo and Caglayan 1976). This regulator problem considers the minimization of a *continuous-time quadratic criterion* of the form (9) by means of a *piecewise constant control* given *complete state information at the sampling instants*. Since this problem is of vital importance in the context of digital control of continuous-time systems it is rather surprising, to say the least, that this regulator problem has only received minor attention. The reason is that instead the solution to the discrete-time regulator problem is used. However, this regulator problem minimizes a discrete-time quadratic criterion which only considers the continuous-time behavior of the perturbation variables (8a,b) at the sampling instants. It therefore constitutes only an *approximate solution* to the problem since it *completely disregards the inter-sample behavior*! This creates the demand for a "small" sampling time to prevent undesirable inter-sample behavior. The discrete-time criterion furthermore must be selected to generate a desired continuous-time behavior!

4. First order controllability and reconstructibility of a nonlinear continuous-time system about a trajectory

A first order approximation, i.e. the linearized model (7), is used to approximate the dynamic behavior of the perturbation variables $\delta x(t)$ and $\delta u(t)$. The error using this approximation is primarily determined by quadratic terms in $\delta x(t)$ and $\delta u(t)$ as long as these perturbation variables remain small. This justifies the use of the quadratic criterion (9) which tries to minimize quadratic terms in $\delta x(t)$ and $\delta u(t)$. Still the use of the quadratic criterion (9) does not guarantee that the perturbation variables remain small. If the linearized dynamics (7) are differentially uncontrollable over a time interval (t_1,t_2) within $[t_0,t_f]$ this implies that certain deviations $\delta x(t)$, $t\in(t_1,t_2)$ from the trajectory cannot be controlled to zero within (t_1,t_2) (Van Willigenburg 1990b). This however implies that those deviations $\delta x(t)$ are not influenced by the control during (t_1,t_2), so they cannot in general be expected to remain small. Since the controller design is based on the idea that the perturbation variables do remain small, the differential controllability of the linearized dynamics is an important property considering the successful application of the controller presented in section 2. Van Willigenburg (1990b) demonstrated that for a rigid manipulator with friction, which we will consider in this paper as an example of a nonlinear system, any trajectory is first order controllable, i.e. the linearized dynamics (7) about any trajectory are differentially controllable. In case of rigid manipulators the linearized dynamics about any trajectory are also differentially reconstructible. The property of differential reconstructibility is dual to the property of differential controllability (Van Willigenburg 1990b). Although not recognized as such by Van Willigenburg (1990b) the differential reconstructibility of the linearized dynamics about the trajectory, which may be called first order reconstructibility, presents another important property for the successful application of the controller in section 2. This can be intuitively understood since in case the linearized dynamics are differentially unreconstructible over a time interval (t_1,t_2) within $[t_0,t_f]$ certain deviations $\delta x(t)$ do not affect the measurements during (t_1,t_2), and therefore $\delta\hat{x}(t)$ may become unreliable and we cannot in general expect $\delta x(t)$ to remain small within (t_1,t_2).

In case of the digital controller of section 3, aspects of controllability and reconstructibility of the linearized dynamics about the trajectory should be reconsidered, since we have sampled measurements and a piecewise constant control. Because of this, loosely speaking, the system will always be less controllable and reconstructible, so compared to the controller of section 2, things will not improve. Concerning controllability of linear systems by means of piecewise constant controls there is the result of Furi et al. (1985), stating that complete controllability by means of continuous-time controls is equivalent to complete controllability by means of piecewise constant controls. Since we are interested in results over the finite time interval $[t_0,t_f]$ this is not the exact result we are looking for. In this paper we will not further concern ourselves with these problems, although they certainly are of interest. We will just apply the digital controller design procedure outlined in section 2 and 3 to a robotic manipulator and observe the result.

5. Digital optimal control of the IBM 7535 B 04 robotic

manipulator

In this section we present examples of the numerical computation of digital optimal controllers for the IBM 7535 B 04 robotic manipulator, designed according to the procedure described in section 2 and 3. Simulation results are also presented to demonstrate the behavior of the digital optimal robot control system.

We will use a dynamic model of the IBM 7535 B 04 robot taken from the literature (Geering et al. 1986, Van Willigenburg and Loop 1990). Before we present the examples we want to stress that our aim is not to apply these controllers in practice, our aim is merely to demonstrate how the design procedure works and that the numerically computed controllers, when applied to the nonlinear system disturbed by additive white noise, result in a proper performance. To be more specific, we will not be concerned with the careful choice of the design parameters, i.e. the covariance matrices (4), (5) and (6) and the matrices appearing in the cost criterion (9). If the aim is to apply the controller in practice the careful choice of the design parameters is essential, and that is where the ability of the engineer comes in.

Although the numerical solution to the digital LQG problem (Van Willigenburg 1990a) allows for the choice of time-varying covariance and criterion matrices we will chose both the covariance and the criterion matrices to be time-invariant to simplify the examples.

Step by step we will now follow the design steps, i.e. the parts *A*, *B* and *C* outlined in section 2.

Part A: Deterministic Modeling.

Step 1: Determination of $f(x(t),u(t),t)$.

The dynamics of a rigid N link manipulator with friction are given by (Van Willigenburg and Loop 1990)

$$\dot{x}_1 = x_2 \qquad [29a]$$

$$\dot{x}_2 = -M^{-1}(x_1)T(x_1,x_2) + M^{-1}(x_1)u. \qquad [29b]$$

where x_1 is a vector of dimension N containing the joint angles of the links, x_2 is a vector of dimension N containing the joint angular velocities, $M(x_1)$ is a inertia matrix depending on the momentary configuration of the robot and $T(x_1,x_2)$ represents centrifugal, coriolis, gravity and friction forces. Finally u is the control vector of dimension N, containing the actuation torque applied to each joint. Since $T(x_1,x_2)$ in (29b) is highly nonlinear manipulators constitute nonlinear systems, while x_1 and x_2 constitute a natural choice for the state variables. Observe that the system is linear in the control. The dynamics of the IBM 7535 B 04 robotic manipulator can be modeled using the closed form dynamics of a rigid two link manipulator (Van Willigenburg and Loop 1990) given by Asada and Slotine (1986). In terms of equation (28) we obtain

$$x_1 = (\theta_1,\theta_2)^T, \qquad [30a]$$

$$x_2 = (\dot{\theta}_1,\dot{\theta}_2)^T, \qquad [30b]$$

where θ_1 and θ_2 are the joint angles of the links,

$$M(x_1) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad [31]$$

where

$$M_{11} = m_1 l_{C1}^2 + I_1 + m_2[l_1^2 + l_{C2}^2 + 2l_1 l_{C2}\cos(\theta_2)] + I_2, \qquad [32a]$$

$$M_{12} = m_2 l_1 l_2 \cos(\theta_2) + m_2 l_{C2}^2 + I_2, \qquad [32b]$$

$$M_{21} = M_{12}, \tag{32c}$$

$$M_{22} = m_2 l_{C2}^2 + I_2, \tag{32d}$$

and

$$T(x_1, x_2) = \begin{bmatrix} -h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 + G_1 + F_1 \\ -h\dot{\theta}_1^2 + G_2 + F_2 \end{bmatrix} \tag{33}$$

where

$$h = m_2 l_1 l_{C2} sin(\theta_2), \tag{34a}$$

$$G_1 = m_1 l_{C1} g\ cos(\theta_1) + m_2 g\{l_{C2} cos(\theta_1 + \theta_2) + l_1 cos(\theta_1)\}, \tag{34b}$$

$$G_2 = m_2 l_{C2} g\ cos(\theta_1 + \theta_2), \tag{34c}$$

$$F_1 = c_1 sgn(\dot{\theta}_1) + v_1\dot{\theta}_1, \tag{34d}$$

$$F_2 = c_2 sgn(\dot{\theta}_2) + v_2\dot{\theta}_2. \tag{34e}$$

I_1 and I_2 are the moments of inertia with respect to the center of mass, m_1 and m_2 are the total masses, l_{C1} and l_{C2} the distances between the center of mass and the joint, v_1 and v_2 the viscous friction coefficients and c_1 and c_2 are the coulomb friction coefficients of the corresponding link. Finally g is the acceleration due to gravity. Equations (33b) and (33c) represent gravity forces, in case the robot moves in a vertical plane. Since the IBM 7535 B 04 robot moves in a horizontal plane they should be disregarded. The Coulomb and viscous friction forces (33d) and (33e) are neglected in case of the IBM 7535 B 04 robot. The remaining parameter values appearing in (31) and (33) for the IBM 7535 B 04 robot are as follows (Geering et al. 1986, Van Willigenburg and Loop 1990)

$l_1 = 0.4$ m $\quad l_2 = 0.25$ m $\quad l_{C2} = 0.161$ m

$m_2 = 21$ kg $\quad m_1 l_{C1}^2 + I_1 = 1.6$ m^{2}kg $\quad I_2 = 0.273$ m^{2}kg. [35]

Step 2: Determination of $g(x(t),u(t),t)$

Each joint angle of the manipulator is measured by an encoder, and often each joint angular velocity is measured by a tacho generator. We will consider both cases so we either assume the complete state is measured, i.e.

$$y(t) = C_c(t)\ x(t) \qquad [36a]$$

where

$$C_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [36b]$$

or we assume

$$y(t) = C_i(t)\ x(t), \qquad [37a]$$

where

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [37b]$$

Step 3: Determination of ideal input-state-output response, $u_0(t)$, $x_0(t)$, $y_0(t)$.

As the ideal input-state response we take a time-optimal solution computed by Van Willigenburg and Loop (1990) who presented numerical procedures to compute time-optimal solutions for rigid manipulators. The time-optimal control problem is to find the control which drives the manipulator from a fixed initial to a fixed final state in minimum time given bounds on the control variables and a deterministic dynamic model of the manipulator. We take a solution computed by a method based on control parameterization (Teo et al 1989). The control parameterization consists of the assumption that the control is piecewise constant, which is the necessary assumption to be made, in case of digital control. Furthermore the sampling times are assumed to be equidistant. Given the fixed initial state

$$x_0(t_0) = [0\ 0\ 0\ 0]^T, \qquad [38a]$$

the fixed final state

$$x_0(t_f) = [2.5\ 0\ 0\ 0]^T, \qquad [38b]$$

the bounds

$$|u_1(t)| \le 25.0 \qquad t \in [t_0, t_f], \qquad [39a]$$

$$|u_2(t)| \le 9.0 \qquad t \in [t_0, t_f] \qquad [39b]$$

on the control variables and the deterministic dynamics (30)-(35) of the IBM 7535 B 04 robot the time-optimal piecewise constant control is shown by the broken lines in figure 5b and the corresponding state-trajectory by the broken lines in figure 5a. The time-optimal solution presents the control in an open-loop fashion. To obtain a solution in feedback form, which we need in practice to overcome modeling and measurement errors and uncertainty, we need to recompute the solution on-line at every sampling instant. Since sampling times for robot manipulators are of the order of 10-100mS, even for very fast computers, this is impossible. Therefore we design a digital compensator, as described in sections 2 and 3, to control the system about the time-optimal input-state response. In this case the number of on-line computations is very small. To be able to control the system about the time-optimal input-state response, the bounds (39) must constitute *conservative bounds*, since they must allow for control corrections $\delta u(t)$. This is the price one has to pay for the fact that we incorporate uncertainty into the time-optimal control problem.

Part B: Stochastic Modeling.

As already mentioned, our aim is not to design a controller which will be used in practice, but merely to demonstrate the controller design procedure. Therefore we will only briefly, or not at all, motivate the choice of the following design parameters. In practice the choice of the design parameters is essential and that is where the ability of the engineer comes in.

Step 4: Modeling of uncertainty in the initial state of the system (1).

We chose

$$\bar{x}(t_0)=x(t_0) \qquad [40]$$

and

$$\Sigma_0 = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.09 & 0 \\ 0 & 0 & 0 & 0.09 \end{bmatrix} \qquad [41]$$

Equation (41) assumes the standard deviation in the initial joint angles to be 0.1 rad and the standard deviation in the initial joint velocities to be 0.3 rad/s. Finally it assumes the uncertainty in the initial values of the state variables to be uncorrelated.

Step 5: Modeling of uncertainty in the system (1).

We chose

$$\Xi(t) = \begin{bmatrix} .235 & 0 & 0 & 0 \\ 0 & .563 & 0 & 0 \\ 0 & 0 & 3.20 & 0 \\ 0 & 0 & 0 & 34.3 \end{bmatrix}, \quad t\in[t_0,t_f]. \qquad [42]$$

This choice is such that at each time t the standard deviation of each component of $\dot{x}(t)$ equals 10 percent of the maximum value of the corresponding component of $f(x_0(t),u_0(t),t)$ over the interval $[t_0,t_f]$. The choice furthermore assumes the noise on each component of $\dot{x}$ to be uncorrelated with the noise on the other components of $\dot{x}$.

Step 6: Modeling of measurement uncertainty.

We chose

$$\Theta(t) = \begin{bmatrix} 1e-4 & 0 & 0 & 0 \\ 0 & 1e-4 & 0 & 0 \\ 0 & 0 & 9e-2 & 0 \\ 0 & 0 & 0 & 9e-2 \end{bmatrix} \quad t\in[t_0,t_f] \qquad [43]$$

in case the output equation (36) holds and we chose

$$\Theta(t) = \begin{bmatrix} 1e-4 & 0 \\ 0 & 1e-4 \end{bmatrix} \quad t\in[t_0,t_f] \qquad [44]$$

in case the output equation (37) holds. This choice assumes the standard deviation of the joint angle measurements to be 0.01 rad. This low value reflects the fact that encoders, which measure the joint angles, give very accurate results. The standard deviation on the joint velocities, measured by tacho-generators is chosen to be 0.3 rad/s. Again the measurement noise on each output variable is assumed to be uncorrelated with the noise on the other output variables.

Part C: *Linearization modeling.*

Step 7: Establishing of the linearized model

According to (7c) we have to compute

$$A_0(t) = \left.\frac{\partial f}{\partial x(t)}\right|_0, \quad B_0(t) = \left.\frac{\partial f}{\partial u(t)}\right|_0, \quad C_0(t) = \left.\frac{\partial g}{\partial x(t)}\right|_0.$$

The numerical algorithm to compute the solution to the digital LQG problem demands the evaluation of (7c) at a finite number of time-instants, depending on the number of integration steps performed during each sampling interval (Van Willigenburg 1990a). We perform the linearization (7c) numerically and chose 10 numerical integration steps during each sampling interval.

Step 8: Selection of $Q_0(t)$, $R_0(t)$ and F_0

We chose

$$Q_0(t) = \begin{bmatrix} 10.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \qquad t\in[t_0,t_f], \qquad [45]$$

$$R_0(t) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad t\in[t_0,t_f], \qquad [46]$$

$$F_0 = \begin{bmatrix} 100.0 & 0 & 0 & 0 \\ 0 & 10.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \qquad [47]$$

When we chose Q_0, R_0 and F_0 to be diagonal matrices, as in (45)-(47) the feedback gain matrix is uniquely determined by the ratios of the diagonal elements of these matrices. Note that $F_0=10Q_0$ which reflects the fact that we want the final state to be reached closely, which corresponds to the objective of the time-optimal control problem. The other ratios were chosen experimentally using simulation results as the ones presented in the figures 5-8.

The simulation result in figure 5 is obtained with the digital robot controller designed using the above design parameter values. The simulation was performed using the *uncertain dynamics* (1) determined by (29)-(35), (40)-(42), (37), and (44). We simulated the white gaussian system and measurement noise using random number generators. To demonstrate the effect of the white gaussian system noise we included figure 9 which shows a response of the robot when we only apply the ideal input, i.e. the open loop control of figure 5b to the system. As expected, if we do not compensate for deviations, the system behavior becomes highly undesirable.

The ideal input-state response was obtained using the fifth and sixth order Runge Kutta integration algorithm IVPRK from the IMSL library (Van Willigenburg and Loop 1990) with a variable self adjustable step size. This routine can not be used to integrate an uncertain system since it repeats integration steps and compares the results to determine whether the accuracy is appropriate. Obviously the uncertainty prevents the results of repeated steps to match. We used a fourth order Runge Kutta integration routine, with a fixed step size equal to 1/10 of the sampling time. This proved to be sufficient to closely reobtain the ideal state response from the ideal input, demonstrating that the accuracy of integration is comparable to that of the routine IVPRK.

Since (1) constitutes an uncertain system figure 5 shows just one *realization* of the robot control system response. We have depicted another in figure 6. Finally figure 7 and 8 show two realizations of the response of the robot control system when the complete state is measured, i.e. where equations (37) and (44) are replaced by (36) and (43).

The simulation results are merely presented to demonstrate that the design "works" and results in a "proper" system behavior given the uncertain dynamics (1). We do not intend here to specify what "proper" is neither compare it to other results. As already mentioned several times, the success of the digital LQG compensator design, depends very much on the ability of the engineer to translate the "real world" into the mathematical model (1) and the criterion (9). What can be said is that given the linearization (7) of the model (1), which approximately describes the dynamic behavior of small deviations $\delta x(t)$ and $\delta u(t)$, and given the criterion (9) the digital LQG compensator constitutes a *truly implementable optimal solution* to the digital control problem (7), (9).

Conclusions

Since automatic control is almost exclusively performed by digital computers it is rather surprising that the proper adaptation of the continuous-time LQG problem and solution, to incorporate the digital nature of the controller, has drawn very little attention over the years. In this paper we used a recently developed result to numerically compute the solution to the digital LQG problem, which does incorporate the digital nature of the controller properly, to treat the design and computation of *truly implementable digital controllers for nonlinear uncertain systems.* The design procedure may be compared to the one presented by Athans (1971) who treated the design of continuous-time controllers for nonlinear uncertain systems. It has been demonstrated in this paper that only the "mechanical" part of this design procedure has to be adjusted, i.e. the *design parameters do not have to be adjusted!.* We simply have to replace software to compute continuous-time controllers by software to compute digital controllers. The software to compute the digital controllers has been recently developed.

Both the design of continuous-time and digital controllers presented in this paper is characterized by the fact that the controllers need only a very small number of computations to be performed on-line. Considering the control of robot manipulators, which constitute highly nonlinear systems where the computational burden on the computer is high since sampling times are often in between 10 and 40mS, this property is crucial. Again it is rather surprising that the design procedure based on the solution to the LQG problem has found very little application in this area. In this paper we computed digital controllers for an industrial robot. Through simulation we demonstrated that the demand for "small" sampling times, caused by in proper incorporation of the digital nature of the controller in the design, can be relaxed using our controller design procedure which incorporates the digital nature of the controller properly.

Concerning the applicability of the continuous-time controllers we have briefly mentioned the properties of differential controllability and reconstructibility of the linearized dynamics about the trajectory, called first order controllability and reconstructibility. In case these properties are not met questions remain how serious the proper behavior of the control system is affected. In case of digital controllers we also have to consider the effect of sampling on the controllability and reconstructibility of the linearized dynamics about the trajectory. Results are known with regard to complete controllability by means of piecewise constant controls (Furi et al. 1985), furthermore the reconstructibility of discrete-time systems is a well known property. However these properties are defined over an infinite time-interval. Our concern is with properties defined over a finite time-interval. This presents a new area of research.

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Figure 5a Trajectory and robot control system response

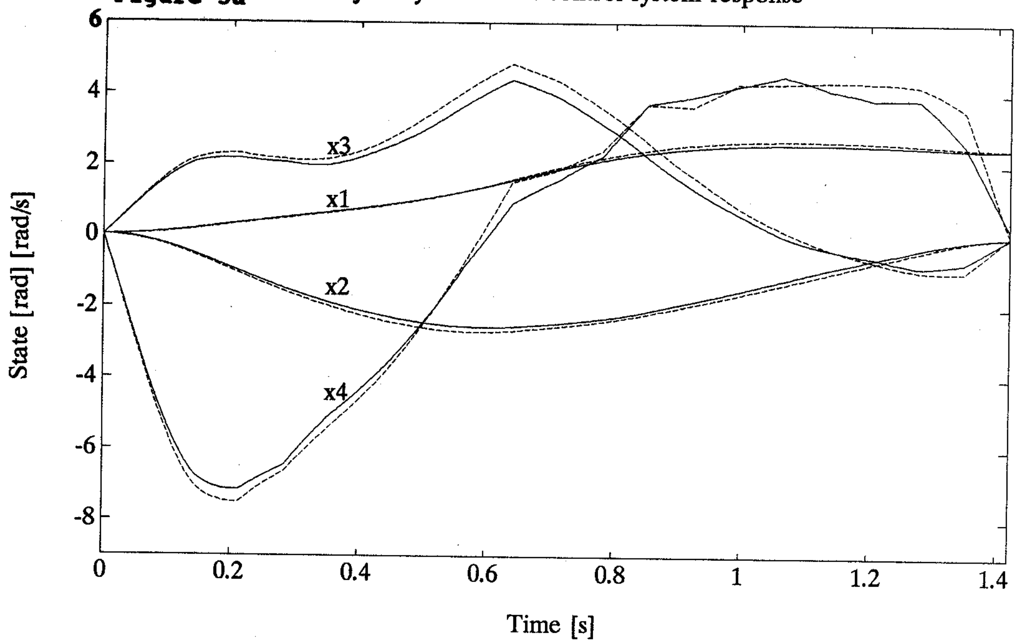

Figure 5b Ideal and actual control

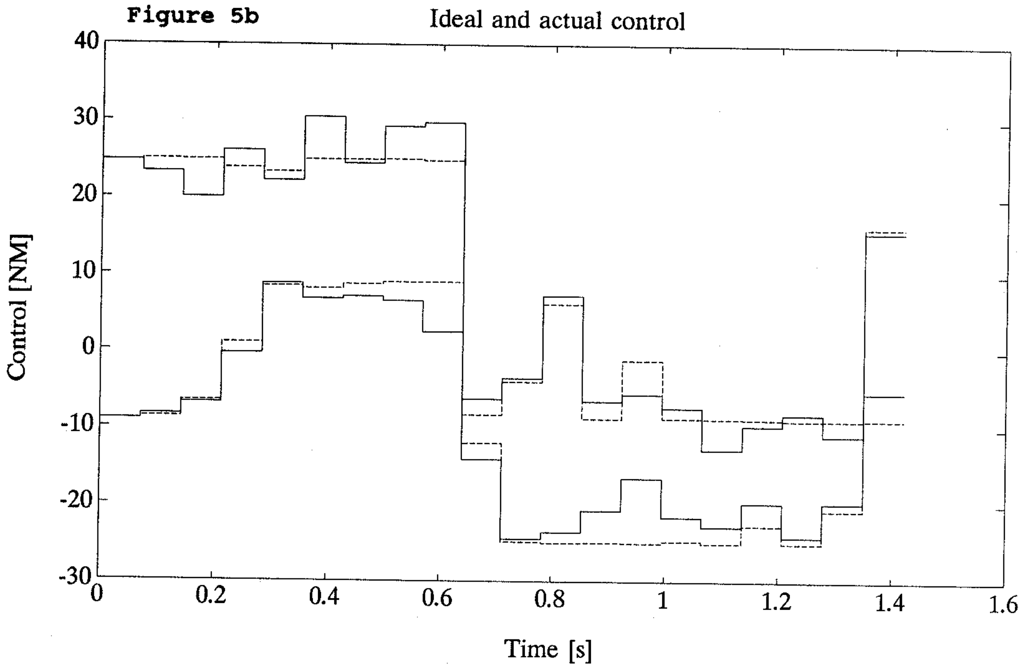

Figure 6a Trajectory and robot control system response

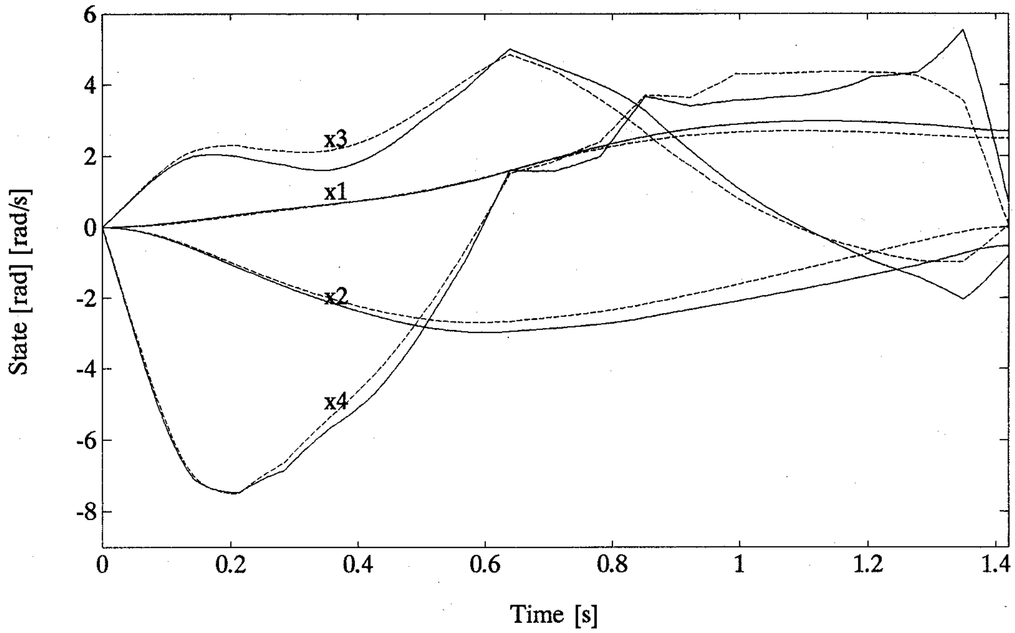

Figure 6b Ideal and actual control

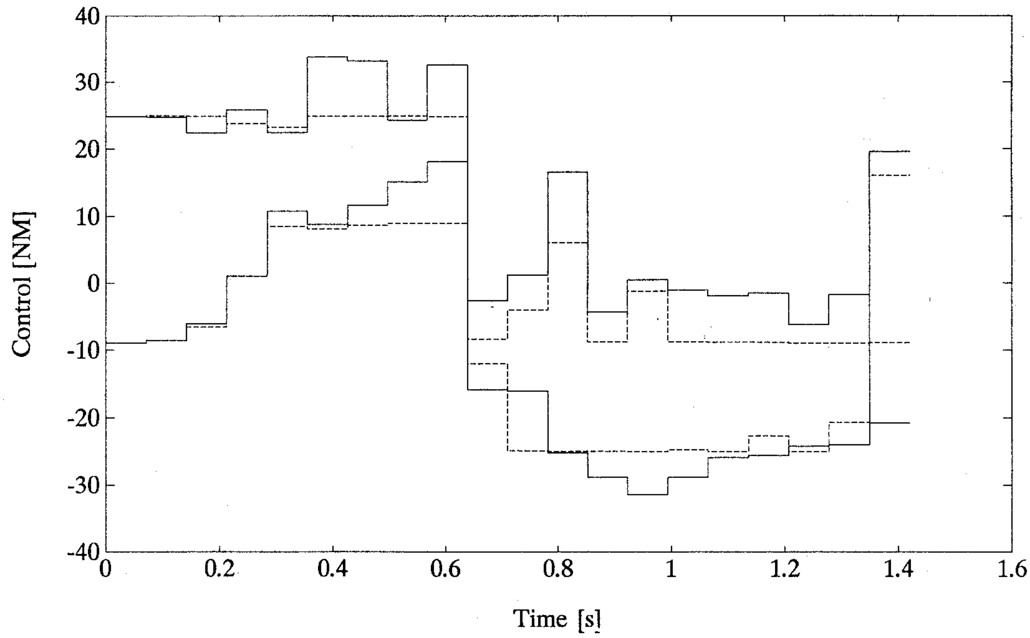

Figure 7a Trajectory and robot control system response

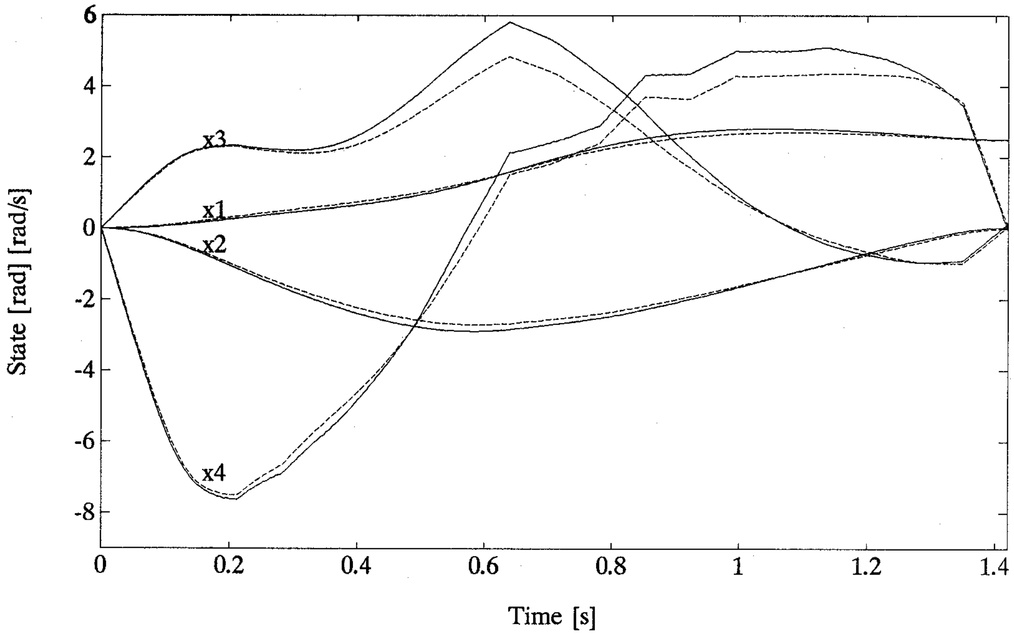

Figure 7b Ideal and actual control

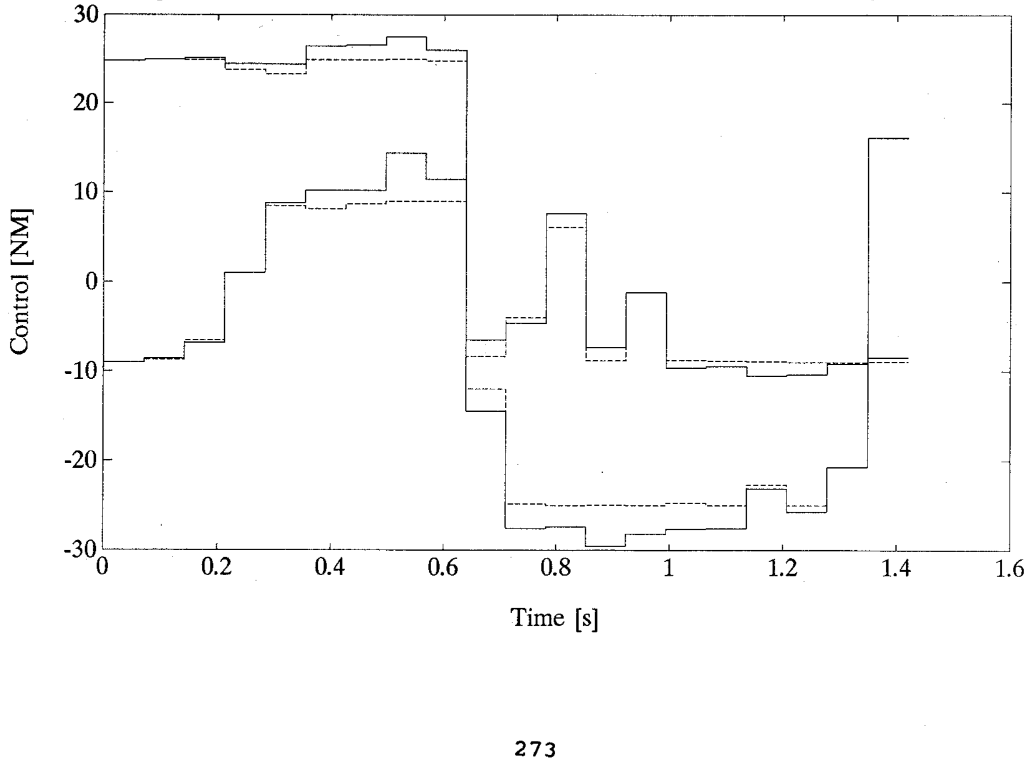

Figure 8a Trajectory and robot control system response

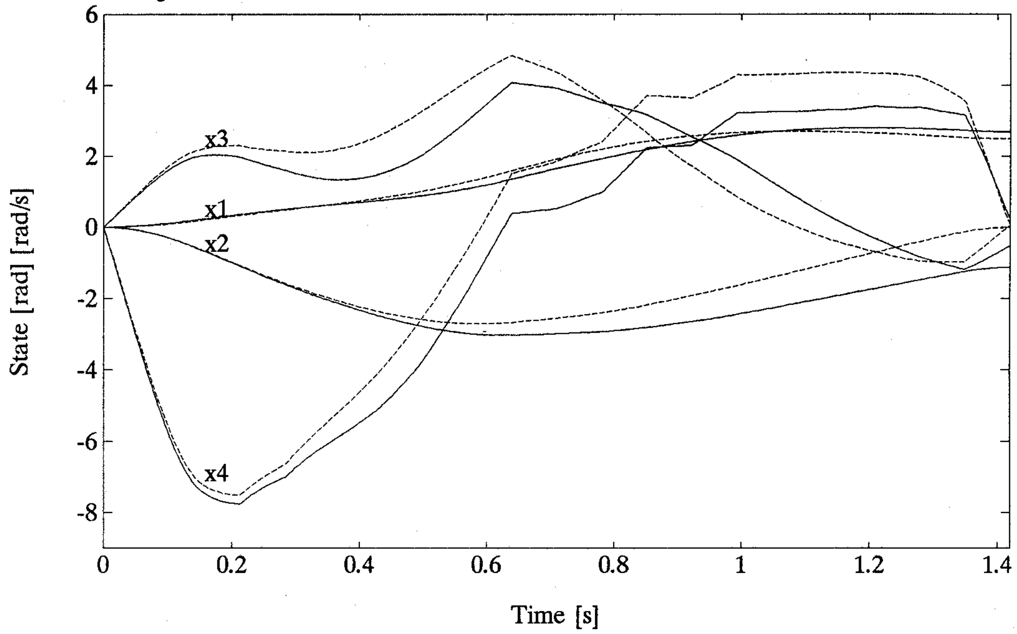

Figure 8b Ideal and actual control

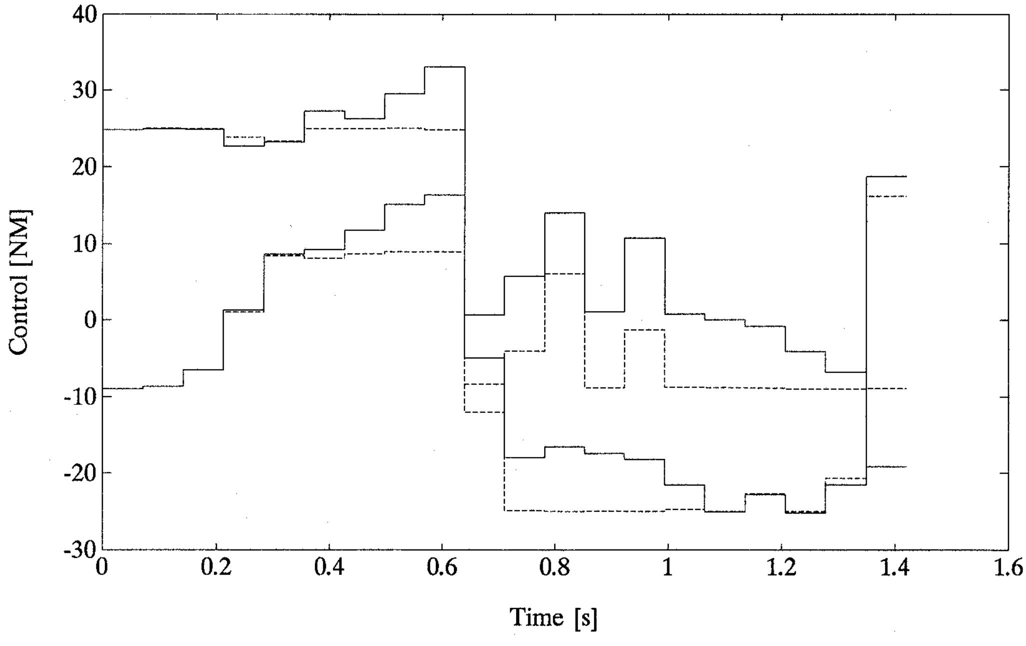

Figure 9 Trajectory and uncertain system response