# Compensatability and Optimal Compensation of Systems with White Parameters

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*Abstract*—**The optimal compensation problem is considered in the case of linear discrete-time systems with stationary white parameters and quadratic criteria. A generalization of the notion of mean square stabilizability, namely mean square compensatability, is introduced. It is shown that suitable mean square compensatability and detectability conditions are sufficient, and necessary in general, for the existence of a unique optimal mean square stabilizing compensator. Tests are given to determine if a system is mean square compensatable or not. It is indicated how to calculate numerically the tests and the optimal mean square stabilizing compensator. The results are illustrated with examples.**

## I. Introduction

In this paper linear discrete-time systems with white parameters are considered. There are mainly two reasons why discrete-time systems with white parameters are important. Firstly, these systems arise naturally in the field of digital control systems where some of the parameters may be white such as the sampling period [1], the controller parameters due to the finite word length of the computer [2], or the parameters of the plant [3]. In all these cases it is possible to convert such a digital control system to an equivalent discrete-time system with white parameters [4], [5]. Also inherent discrete-time systems, such as economic systems, may have white parameters. Secondly, the parameters of an equivalent or inherent discrete-time system may be *assumed* to be white for the purpose of a robust control system design. It is well known that the standard LQG design does not lead in general to a robust control system with respect to parameter deviations [6]. A possible approach for robust control system design is by modeling the uncertainty in the parameters as white stochastic fluctuations [7], [8]. The advantage of a model with white parameters is that it fits naturally in the LQ design context. Therefore, this approach seems promising for nonconservative robust control system design with respect to structured parameter variations.

Here we will study the optimal dynamic output feedback, called optimal compensation, of linear discrete-time systems with stationary white parameters and where the criteria are quadratic. In the case of deterministic parameters the optimal compensation problem leads to separate control and estimation problems. In the stochastic parameter case this is no longer true [9]. Control and estimation has to be done simultaneously by the compensator. The question arises as to the conditions for which an optimal compensator exists and the control system is stable in a mean square sense.

The optimal compensator problem in the case of white parameters has been studied in [10]–[12] for continuous-time systems, and in [13], [14] for discrete-time systems. They derive necessary conditions for the existence of an optimal compensator in various cases. Sufficient conditions for the existence of an optimal mean square stabilizing compensator are derived in [10]. However, the results are restricted to a very special class of systems. In [15] sufficient conditions are given for mean square stability of the compensated system.

In this paper we introduce a generalization of the notion of mean square stabilizability [16], called mean square compensatability, for linear discrete-time systems with white parameters. A system is called mean square compensatable if there exists a mean square stabilizing compensator. The relation of mean square compensatability with the existing notions of mean square stabilizability and detectability [16], [17] is investigated. It is shown that suitable conditions of mean square compensatability and mean square detectability are sufficient, and necessary in general, for the existence of a unique optimal mean square stabilizing compensator. The above mentioned conditions coincide with the usual stabilizability and detectability conditions if the parameters are deterministic, i.e., there is no uncertainty in the parameters. Moreover, we give two tests, explicit in the system matrices, for systems with white parameters to be mean square compensatable or not. One of these tests is based on the maximal mean square stability of the closed-loop system achievable through a compensator, which can be conceived as a measure of mean square compensatability rather than merely a test. Also we indicate how to actually calculate numerically the compensatability tests given a system, and the optimal stabilizing compensator, if it exists, given a system and a criterion. It should be noted that a sufficient and necessary test for mean square detectability is already given in [17]. The results are illustrated with some examples.

## II. Compensatability

For easy reference we shall first repeat some results from [16], [17] concerning mean square stability, stabilizability, and detectability.

Consider the system

$$x_{i+1} = \Phi_i x_i, \qquad i = 0, 1, \cdots, \tag{1}$$

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where $x_i \in R^n$ is the state and $\Phi_i$ is a real matrix of appropriate dimensions. The process $\{\Phi_i\}$ is a sequence of independent random matrices with constant statistics and the initial condition $x_0$ is deterministic. System (1) is characterized by $(\Phi_i)$. Let *ms* denote mean square and let an overbar denote expectation.

*Definition 1:* $(\Phi_i)$ is called ms-stable if $\overline{\|x_i\|^2} \to 0$ as $i \to \infty$ for all $x_0$. □

Let $S^n$ denote the linear space of real symmetric $n \times n$ matrices and define the linear transformation $A$: $S^n \to S^n$ by

$$AX = \overline{\Phi^T X \Phi}, \qquad X \in S^n \tag{2}$$

where index $i$ is deleted without ambiguity because $AX$ is independent of $i$. Let $\rho$ denote spectral radius and $I$ the $n \times n$ identity matrix. Then $\overline{\|x_i\|^2} = \overline{x_i^T x_i} = x_0^T A^i I x_0$ and $\|A^i I\|^{1/i} = \|A^i\|^{1/i} \to \rho(A)$ as $i \to \infty$. Thus, $\rho(A)$ is a measure of ms-stability of $(\Phi_i)$. In particular, $(\Phi_i)$ ms-stable $\Leftrightarrow \rho(A) < 1$. If $\Phi_i$ is deterministic and constant then ms-stability is identical to stability in the usual sense.

Consider the open-loop system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \qquad i = 0, 1, \cdots, \tag{3}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ the control, and $\Phi_i$, $\Gamma_i$ are real matrices of appropriate dimensions. The processes $\{\Phi_i\}$ and $\{\Gamma_i\}$ are sequences of independent random matrices with constant statistics and the initial condition $x_0$ is deterministic. System (3) is characterized by the pair $(\Phi_i, \Gamma_i)$.

Consider the static state feedback controller

$$u_i = -Lx_i \tag{4}$$

where $L$ is a real matrix of appropriate dimensions. Then from (3) we have the closed-loop system

$$x_{i+1} = (\Phi_i - \Gamma_i L) x_i. \tag{5}$$

*Definition 2:* $(\Phi_i, \Gamma_i)$ is called ms-stabilizable if there exists an $L$ such that $(\Phi_i - \Gamma_i L)$ is ms-stable. □

We have $(\Phi_i)$ ms-stable $\Rightarrow (\Phi_i, \Gamma_i)$ ms-stabilizable. If $\Phi_i = \Phi$ and $\Gamma_i = \Gamma$ where $\Phi$, $\Gamma$ are deterministic and constant then ms-stabilizability is identical to stabilizability in the usual sense. It is well known that $\Gamma$ invertible $\Rightarrow$ $(\Phi, \Gamma)$ stabilizable. However, $\Gamma$ invertible $\not\Rightarrow (\Phi_i, \Gamma)$ ms-stabilizable. For instance, take the scalar case $\Phi_i = \phi_i$, $\Gamma = \gamma$, $L = l$, and $\gamma \neq 0$. Then $\overline{x_{i+1}^2} = \overline{(\phi_i - \gamma l)^2 x_i^2} = [(\bar{\phi} - \gamma l)^2 + \overline{\tilde{\phi}_i^2}]\overline{x_i^2}$, where $\tilde{\phi}_i = \phi_i - \bar{\phi}$. The expression between brackets can never be made smaller than $\overline{\tilde{\phi}_i^2}$.

Consider the system

$$x_{i+1} = \Phi_i x_i, \tag{6a}$$

$$y_i = C_i x_i, \qquad i = 0, 1, \cdots, \tag{6b}$$

where $X_i \in R^n$ is the state, $y_i \in R^l$ the observation, and $\Phi_i$, $C_i$ are real matrices of appropriate dimensions. The processes $\{\Phi_i\{$ and $\{C_i\}$ are sequences of independent random matrices with constant statistics and the initial condition $x_0$ is deterministic. System (6) is characterized by the pair $(\Phi_i, C_i)$.

*Definition 3:* $(\Phi_i, C_i)$ is called ms-detectable if $\overline{\|y_i\|^2} = 0$, $i = 0, 1, \cdots$, implies that $\overline{\|x_i\|^2} \to 0$ as $i \to \infty$. □

Using the transformation $A$ defined by (2), we have $(\Phi_i, C_i)$ ms-detectable $\Leftrightarrow$ $(x_0^T A^i \overline{C^T C} x_0 = 0$, $i = 0, 1, \cdots \Rightarrow x_0^T A^i I x_0 \to 0$ as $i \to \infty)$. Also we have $(\Phi_i)$ ms-stable $\Rightarrow (\Phi_i, C_i)$ ms-detectable. If $\Phi_i = \Phi$ and $C_i = C$ where $\Phi, C$ are deterministic and constant, then ms-detectability is identical to detectability in the usual sense. Furthermore, $C$ invertible $\Rightarrow (\Phi_i, C)$ ms-detectable.

Suitable conditions of ms-stabilizability and ms-detectability are sufficient, and necessary in general, to solve the optimal state feedback control problem in the white parameter case. In order to solve the optimal compensation problem, the condition of ms-stabilizability appears to be too weak, contrary to the deterministic parameter case. If the parameters are stochastic the operations of control and estimation are not independent of each other. This interaction should be expressed in a generalized stabilizability condition. Therefore, we introduce the notion of ms-compensatability. In this connection it is interesting to note the following. If $\Phi_i = \Phi$ and $\Gamma_i = \Gamma$ then $(\Phi, \Gamma)$ stabilizable $\Leftrightarrow (\Phi^T, \Gamma^T)$ detectable. This duality of stabilizability and detectability does not hold in the stochastic parameter case. In fact, ms-stabilizability is a stronger property than ms-detectability in the sense that $(\Phi_i, \Gamma_i)$ ms-stabilizable $\Rightarrow (\not\Leftarrow)(\Phi_i^T, \Gamma_i^T)$ ms-detectable. This expresses the fundamental fact that the presence of uncertainty in the system in the form of stochastic parameters makes stabilizing more difficult, while detecting may even be easier.

Consider the system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \tag{7a}$$

$$y_i = C_i x_i, \qquad i = 0, 1, \cdots, \tag{7b}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ the control, $y_i \in R^l$ the observation, and $\Phi_i$, $\Gamma_i$, $C_i$ are real matrices of appropriate dimensions. The processes $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are sequences of independent random matrices with constant statistics and the initial condition $x_0$ is deterministic. System (7) is characterized by the triple $(\Phi_i, \Gamma_i, C_i)$. Consider the dynamic output feedback compensator

$$\hat{x}_{i+1} = F\hat{x}_i + Ky_i, \tag{8a}$$

$$u_i = -L\hat{x}_i, \qquad i = 0, \cdots, \tag{8b}$$

where $\hat{x}_i \in R^n$ is the compensator state and $F$, $K$, $L$ are real matrices of appropriate dimensions. The initial condition $\hat{x}_0$ is deterministic. Compensator (8) is characterized by the triple $(F, K, L)$. Now from (7) we have the closed-loop system

$$\begin{bmatrix} x_{i+1} \\ \hat{x}_{i+1} \end{bmatrix} = \begin{bmatrix} \Phi_i & -\Gamma_i L \\ KC_i & F \end{bmatrix} \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}. \tag{9}$$

Define $x_i'$ and $\Phi_i'$ by

$$x_i' = \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, \qquad \Phi_i' = \begin{bmatrix} \Phi_i & -\Gamma_i L \\ KC_i & F \end{bmatrix},$$

then (9) becomes

$$x'_{i+1} = \Phi'_1 x'_1, \qquad i = 0, 1, \cdots, \tag{10}$$

where $\{\Phi'_i\}$ is a sequence of independent random matrices with constant statistics. The initial condition $x'_0$ is deterministic.

*Definition 4:* $(\Phi_i, \Gamma_i, C_i)$ is called ms-compensatable if there exists an $F$, $K$, and $L$ such that $(\Phi'_i)$ is ms-stable. □

A number of properties concerning ms-compensatability are now stated.

*Theorem 1:*

a) $(\Phi_i)$ ms-stable $\Rightarrow (\Phi_i, \Gamma_i, C_i)$ ms-compensatable.
b) $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Leftrightarrow (\Phi_i^T, C_i^T, \Gamma_i^T)$ ms-compensatable.
c) $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Rightarrow (\Phi_i, \Gamma_i)$ and $(\Phi_i^T, C_i^T)$ both ms-stabilizable.
d) If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$, $C_i = C$, then $(\Phi, \Gamma, C)$ ms-compensatable $\Leftrightarrow (\Phi, \Gamma)$ and $(\Phi^T, C^T)$ both stabilizable in the usual sense.

*Proof:* Part a) is clear by choosing $F = 0$, $K = 0$, $L = 0$. Then $(\Phi'_i)$ is ms-stable. Part b) follows from the structure of $\Phi_i'^T$ and the fact that $(\Phi'_i)$ ms-stable $\Leftrightarrow (\Phi_i'^T)$ ms-stable. Part c) will be proven in Section III of this paper. Finally part d).

Choose $F = \Phi - \Gamma L - KC$ and denoting the $n \times n$ zero matrix by $\Theta$, then

$$\begin{bmatrix} I & \Theta \\ I & -I \end{bmatrix} \begin{bmatrix} \Phi & -\Gamma L \\ KC & F \end{bmatrix} \begin{bmatrix} I & \Theta \\ I & -I \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & \Gamma L \\ \Theta & \Phi - KC \end{bmatrix}$$

which proves this part. □

Note that if $\Gamma_i = \Gamma$, $C_i = C$, then $\Gamma, C$ invertible $\not\Rightarrow (\Phi_i, \Gamma, C)$ ms-compensatable.

In contrast with ms-stabilizability we will not need a new notion for ms-detectability. However, it will appear convenient to introduce the following definition concerning detectability of a triple instead of a pair of random matrices.

*Definition 5:* $(\Phi_i, \Gamma_i, C_i)$ is called ms-detectable if $(\Phi_i, C_i)$ and $(\Phi_i^T, \Gamma_i^T)$ are both ms-detectable. □

We have the following properties concerning ms-detectability.

*Theorem 2:*

a) $(\Phi_i)$ ms-stable $\Rightarrow (\Phi_i, C_i, \Gamma_i)$ ms-detectable.
b) $(\Phi_i, \Gamma_i, C_i)$ ms-detectable $\Leftrightarrow (\Phi_i^T, C_i^T, \Gamma_i^T)$ ms-detectable.
c) If $C_i = C$, $\Gamma_i = \Gamma$ then $C, \Gamma$ invertible $\Rightarrow (\Phi_i, C, \Gamma)$ ms-detectable.
d) If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$, $C_i = C$ then $(\Phi, \Gamma, C)$ ms-detectable $\Leftrightarrow (\Phi, C)$ and $(\Phi^T, \Gamma^T)$ both detectable in the usual sense.

*Proof:* Follows immediately from Definition 5. □

If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$, and $C_i = C$ then, from Theorem 1 and 2, $(\Phi, \Gamma, C)$ ms-compensatable $\Leftrightarrow (\Phi, \Gamma, C)$ ms-detectable. This duality of compensatability and detectability does not hold in the stochastic parameter case. From Theorem 1 and Definition 5 it follows that ms-compensatability is a stronger property than ms-detectability in the sense that $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Rightarrow (\not\Leftarrow)(\Phi_i, \Gamma_i, C_i)$ ms-detectable.

In the next section we will show that under suitable compensatability and detectability conditions there exists a unique optimal mean square stabilizing compensator.

## III. Optimal Compensation

Consider the system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i, \qquad i = 0, 1, \cdots, \tag{11a}$$

$$y_i = C_i x_i + w_i, \qquad i = 0, 1, \cdots, \tag{11b}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ the control, $y_i \in R^l$ the observation, $v_i \in R^n$ the system noise, $w_i \in R^l$ the observation noise, and $\Phi_i$, $\Gamma_i$, $C_i$ are real matrices of appropriate dimensions. The processes $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are sequences of independent random matrices and $\{v_i\}$, $\{w_i\}$ are mutually independent sequences of independent stochastic vectors with constant statistics. Initial condition $x_0$ is stochastic with mean $\bar{x}_0$ and covariance $P_0$, and is independent of $\{\Phi_i, \Gamma_i, C_i, v_i, w_i\}$. Moreover, $\Phi_i$, $\Gamma_i$, and $C_i$ are independent of $v_j$, $w_j$, $i \neq j$ and uncorrelated with $v_i$, $w_i$. The processes $\{v_i\}$ and $\{w_i\}$ are zero-mean with covariances $V$ and $W$, with $V \geq 0$ and $W \geq 0$.

We choose as controller the compensator

$$\hat{x}_{i+1} = F\hat{x}_i + Ky_i, \qquad i = 0, 1, \cdots, \tag{12a}$$

$$u_i = -L\hat{x}_i, \qquad i = 0, 1, \cdots, \tag{12b}$$

where $\hat{x}_i \in R^n$ is the compensator state and $F$, $K$, $L$ are real matrices of appropriate dimensions. The initial condition $\hat{x}_0$ is deterministic. A compensator is called ms-stabilizing if $\overline{\|x_i\|^2}$ and $\overline{\|\hat{x}_i\|^2}$ converge as $i \to \infty$ to values which do not depend on $x_0$ and $\hat{x}_0$. The *optimal compensation problem* is to find the optimal ms-stabilizing compensator $(F^*, K^*, L^*)$ which minimizes the criterion

$$\sigma_\infty(F, K, L) = \lim_{N \to \infty} \frac{1}{N} E\left\{ \sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i) \right\} \tag{13}$$

where $Q$ and $R$ are real symmetric matrices of appropriate dimensions with $Q \geq 0$ and $R \geq 0$, and to find the minimum value $\sigma_\infty^* = \sigma_\infty(F^*, K^*, L^*)$.

The closed-loop system may be described by

$$\begin{bmatrix} x_{i+1} \\ \hat{x}_{i+1} \end{bmatrix} = \begin{bmatrix} \Phi_i & -\Gamma_i L \\ KC_i & F \end{bmatrix} \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix} + \begin{bmatrix} v_i \\ Kw_i \end{bmatrix}. \tag{14}$$

Define

$$x'_i = \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, v'_i = \begin{bmatrix} v_i \\ Kw_i \end{bmatrix}, \Phi'_i = \begin{bmatrix} \Phi_i & -\Gamma_i L \\ KC_i & F \end{bmatrix},$$

$$V' = \begin{bmatrix} V & \Theta \\ \Theta & KWK^T \end{bmatrix}$$

then (14) becomes

$$x'_{i+1} = \Phi'_i x'_i + v'_i, \qquad i = 0, 1, \cdots, \tag{15}$$

where $\{\Phi'_i\}$ is a sequence of independent random matrices and $\{v'_i\}$ is a sequence of independent stochastic vectors.

Initial condition $x_0'$ is independent of $\{\Phi_i', v_i'\}$. Moreover, $\Phi_i'$ is independent of $v_j'$, $i \neq j$, and uncorrelated with $v_i'$. The process $\{v_i'\}$ is zero-mean with covariance $V'$. Let $P_i'$ denote $\overline{x_i' x_i'^T}$, then from (15)

$$P_{i+1}' = \overline{\Phi' P_i' \Phi'^T} + V'. \tag{16}$$

Suppose $(\Phi_i')$ is ms-stable, then [18] the compensator $(F, K, L)$ is ms-stabilizing and $P' = \lim_{i\to\infty} P_i'$ exists, $P' \geq 0$, and $P'$ is the unique solution of the equation

$$P' = \overline{\Phi' P' \Phi'^T} + V', \qquad P' \in S^{2n}. \tag{17}$$

Furthermore, criterion (13) exists and may be written as

$$\sigma_\infty(F, K, L) = \mathrm{tr}\,(Q'P') \tag{18}$$

where $Q'$ is defined by

$$Q' = \begin{bmatrix} Q & \Theta \\ \Theta & L^T R L \end{bmatrix}.$$

Therefore, we restrict our attention to the following set of admissible compensators:

$$C_{adm} = \{(F, K, L) | (\Phi_i') \text{ is ms-stable}\}.$$

Since the value of $\sigma_\infty(F, K, L)$ is independent of the internal realization of $(F, K, L)$, we may further restrict our attention to the set of minimal compensators

$$C_{adm}^m = \{(F, K, L) \in C_{adm} | (F, K) \text{ reachable}, (F, L) \text{ observable}\}.$$

The optimal compensation problem may be restated as to find the optimal compensator $(F^*, K^*, L^*) \in C_{adm}^m$ which minimizes (18) subject to (17), and to find the minimum criterion value $\sigma_\infty^* = \sigma_\infty(F^*, K^*, L^*)$.

Define the linear transformation $A'$: $S^{2n} \to S^{2n}$ by

$$A'X = \overline{\Phi'^T X \Phi'}, \qquad X \in S^{2n} \tag{19}$$

then $(\Phi_i')$ ms-stable $\Leftrightarrow \rho(A') < 1$. Because the eigenvalues of $A'$ depend continuously on $(F, K, L)$, the set $C_{adm}^m$ is open. Therefore, we may apply the matrix minimum principle [19] to find necessary conditions for the solution of the optimal compensation problem. To that end, define the Hamiltonian $H$ by

$$H(F, K, L, P', S') = \mathrm{tr}\left[Q'P' + \left(\overline{\Phi' P' \Phi'^T} + V' - P'\right)S'\right] \tag{20}$$

where $S' \in S^{2n}$ is the Lagrange multiplier. Then the necessary optimality conditions are

$$\frac{\partial H}{\partial F} = \frac{\partial}{\partial F}\mathrm{tr}\left(\overline{\Phi' P' \Phi'^T S'}\right) = 0, \tag{21a}$$

$$\frac{\partial H}{\partial K} = \frac{\partial}{\partial K}\mathrm{tr}\left(\overline{\Phi' P' \Phi'^T S'} + V'S'\right) = 0, \tag{21b}$$

$$\frac{\partial H}{\partial L} = \frac{\partial}{\partial L}\mathrm{tr}\left(\overline{\Phi' P' \Phi'^T S'} + Q'P'\right) = 0, \tag{21c}$$

$$\frac{\partial H}{\partial P'} = \overline{\Phi'^T S' \Phi'} + Q' - S' = 0, \tag{22a}$$

$$\frac{\partial H}{\partial S'} = \overline{\Phi' P' \Phi'^T} + V' - P' = 0 \tag{22b}$$

where $S' \geq 0$ and $P' \geq 0$. Partition $S'$ and $P'$ as

$$S' = \begin{bmatrix} S_1 & S_{12} \\ S_{12}^T & S_2 \end{bmatrix}, \qquad P' = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$$

according to the partitioning of $\Phi_i'$ and define $S = S_1 - S_2$, $\hat{S} = S_2$, $P = P_1 - P_2$, and $\hat{P} = P_2$. Note that $P = \lim_{i\to\infty} \overline{\tilde{x}_i \tilde{x}_i^T}$, where $\tilde{x}_i = x_i - \hat{x}_i$, and $\hat{P} = \lim_{i\to\infty} \overline{\hat{x}_i \hat{x}_i^T}$. Define also $\tilde{\Phi}_i = \Phi_i - \overline{\Phi}$, $\tilde{\Gamma}_i = \Gamma_i - \overline{\Gamma}$ and $\tilde{C}_i = C_i - \overline{C}$. Then in [13] it is shown that (21a) may be transformed to

$$F = \overline{\Phi} - \overline{\Gamma} L - K\overline{C} \tag{23a}$$

and, assuming that $\Gamma_i$ and $C_i$ are independent, (21b), (21c) may be transformed to

$$K = \left(\overline{\Phi} P \overline{C}^T + \overline{\tilde{\Phi}\hat{P}\tilde{C}^T}\right)\left(\overline{CPC^T} + W + \overline{\tilde{C}\hat{P}\tilde{C}^T}\right)^+, \tag{23b}$$

$$L = \left(\overline{\Gamma^T S \Gamma} + R + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Gamma}}\right)^+ \left(\overline{\Gamma}^T S \overline{\Phi} + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Phi}}\right) \tag{23c}$$

where $+$ denotes the Moore–Penrose pseudo-inverse and where $S \geq 0$, $S_{12} = S_{12}^T = -\hat{S}$, $P \geq 0$ and $P_{12} = P_{12}^T = \hat{P}$. Moreover, $\hat{S} > 0$ and $\hat{P} > 0$ due to the minimality assumption in $C_{adm}^m$ [20]. The fact $P_{12} = \hat{P}$ implies that $\overline{\tilde{x}_i \hat{x}_i^T} \to 0$ as $i \to \infty$. We may substitute $F$, $K$, and $L$ from (23) into (22) which gives six coupled nonlinear $n \times n$ matrix equations in $S_1$, $S_2$, $S_{12}$, $P_1$, $P_2$, and $P_{12}$. After finding a possible solution of these equations we may calculate $F$, $K$, and $L$ from (23). However, the existence of $F$, $K$, and $L$ does not guarantee that the compensator $(F, K, L)$ is a ms-stabilizing one and if so, that it is the optimal one. Before investigating this issue we make some remarks.

Define the linear transformation $B'$: $S^{2n} \to S^{2n}$ by

$$B'X = \overline{\Phi' X \Phi'^T}, \qquad X \in S^{2n}. \tag{24}$$

Using $A'$ and $B'$ (22) may be written as

$$S' = A'S' + Q', \tag{25a}$$

$$P' = B'P' + V'. \tag{25b}$$

Equations (23) and (25) form together the necessary optimality conditions.

Using $F = \overline{\Phi} - \overline{\Gamma} L - K\overline{C}$, (25) may be transformed to [13]

$$S = \overline{(\Phi - \Gamma L)^T S (\Phi - \Gamma L)} + Q + \overline{(\tilde{\Phi} - K\tilde{C})^T \hat{S} (\tilde{\Phi} - K\tilde{C})} + L^T\left(R + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Gamma}}\right)L, \tag{26a}$$

$$\hat{S} = (\overline{\Phi} - K\overline{C})^T \hat{S} (\overline{\Phi} - K\overline{C}) + L^T\left(\overline{\Gamma^T S \Gamma} + R + \overline{\tilde{\Gamma}^T \hat{S} \tilde{\Gamma}}\right)L, \tag{26b}$$

$$P = \overline{(\Phi - KC)P(\Phi - KC)^T}$$
$$+ V + \overline{(\tilde{\Phi} - \tilde{\Gamma}L)\hat{P}(\tilde{\Phi} - \tilde{\Gamma}L)^T}$$
$$+ K\left(W + \overline{\tilde{C}\hat{P}\tilde{C}^T}\right)K^T, \tag{26c}$$
$$\hat{P} = (\overline{\Phi} - \overline{\Gamma}L)\hat{P}(\overline{\Phi} - \overline{\Gamma}L)^T$$
$$+ K\left(\overline{CPC^T} + W + \overline{\tilde{C}\hat{P}\tilde{C}^T}\right)K^T. \tag{26d}$$

Note that (23), (25) and (23), (26) are equivalent. They will be used accordingly, as it suits us.

Also using $F = \overline{\Phi} - \overline{\Gamma}L - K\overline{C}$, compensator (12) may be written as

$$\hat{x}_{i+1} = \overline{\Phi}\hat{x}_i + \overline{\Gamma}u_i + K(y_i - \overline{C}\hat{x}_i), \tag{27a}$$
$$u_i = -L\hat{x}_i. \tag{27b}$$

The compensator state $\hat{x}_i$ given by (27a) is precisely the *optimal* linear estimator of $x_i$ given the control (27b) and the observations $y_0, \cdots, y_{i-1}$, and $P$ given by (26c) is the estimation error covariance, as $i \to \infty$ [18].

Using the partition of $Q'$ and $P'$, the criterion value (18) for the optimal $(F, K, L)$ is given by

$$\sigma_\infty^* = \operatorname{tr}(Q'P') = \operatorname{tr}\left[QP + (Q + L^{*T}RL^*)\hat{P}\right] \tag{28a}$$

which from (22a) may also be written as

$$\sigma_\infty^* = \operatorname{tr}(V'S') = \operatorname{tr}\left[VS + (V + K^*WK^{*T})\hat{S}\right]. \tag{28b}$$

If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$, and $C_i = C$, where $\Phi$, $\Gamma$, $C$ are deterministic and constant, then (23), (26) reduces to the well-known *uncoupled* control and estimation algebraic Riccati equations, respectively, (23c), (26a) and (23b), (26b), and the superfluous equations (26b) and (26d).

Necessary conditions for the existence of an optimal ms-stabilizing compensator are stated above. What we want are sufficient conditions which are necessary in general. In order to state the main result in this direction, we need the following two lemmas.

*Lemma 1:* Either $R > 0$, $W > 0$, $(\Phi_i, V^{\frac{1}{2}}, Q^{\frac{1}{2}})$ ms-detectable, or $Q > 0$, $V > 0 \Rightarrow (\Phi_i', V'^{\frac{1}{2}}, Q'^{\frac{1}{2}})$ ms-detectable.

*Proof:* We will prove i) either $R > 0$, $(\Phi_i, Q^{\frac{1}{2}})$ ms-detectable, or $Q > 0 \Rightarrow (\Phi_i', Q'^{\frac{1}{2}})$ ms-detectable; and ii) either $W > 0$, $(\Phi_i^T, V^{\frac{1}{2}})$ ms-detectable, or $V > 0 \Rightarrow (\Phi_i', V'^{\frac{1}{2}})$ ms-detectable. Only the first implication will be proven. The second one goes analogously. Note that $\Phi_i'$, $A'$, and $Q'$ depend on $F$, $K$, and $L$. If $F = 0$, $K = 0$, and $L = 0$ we will indicate this with a lower index 0. Referring to system (7) we have $x_0'^T A_0'^i Q_0' x_0' = x_0^T A^i Q x_0$, and $x_0'^T A_0'^i I' x_0' = x_0^T A^i I x_0$, where $I'$ is the $2n \times 2n$ identity matrix. Thus, using the first result below Definition 3, $(\Phi_i, Q^{\frac{1}{2}})$ ms-detectable $\Rightarrow (\Phi_{0,i}', Q_0'^{\frac{1}{2}})$ ms-detectable. Furthermore, $x_0'^T A'^i Q' x_0' = E\{x_i^T Q x_i + u_i^T R u_i\}$, $u_i = -L\hat{x}_i$. Suppose $x_0'^T A'^i Q' x_0' = 0$, then from $E\{u_i^T R u_i\} = 0$ and $R > 0$ we have that $u_i = 0$ almost surely [21]. So $F$, $K$, and $L$ may have any value, including zero. Now let $(\Phi_{0,i}', Q_0'^{\frac{1}{2}})$ be ms-detectable. Then $x_0'^T A'^i Q' x_0' = 0$, $\forall i \Rightarrow x_0'^T A_0'^i Q_0' x_0' = 0$, $\forall i \Rightarrow x_0'^T A_0'^i I' x_0' \to 0$, as $i \to \infty \Rightarrow x_0'^T A'^i I' x_0' \to 0$ as $i \to \infty$. Thus, $(\Phi_i', Q'^{\frac{1}{2}})$ is ms-detectable. Now suppose $Q > 0$. Then there exists $\alpha > 0$ such that $\alpha Q' \geq I'$. Hence, $x_0'^T A'^i I' x_0' \leq \alpha x_0' A'^i Q' x_0'$. Therefore, $x_0'^T A'^i Q' x_0' = 0$, $\forall i \Rightarrow x_0'^T A'^i I' x_0' = 0$, $\forall i$. So $(\Phi_i', Q'^{\frac{1}{2}})$ is ms-detectable. □

Note that from Definitions 5 and 3, $(\Phi_i' V'^{\frac{1}{2}}, Q'^{\frac{1}{2}})$ ms-detectable means that the fictitious systems $x_{i+1}' = \Phi_i' x_i'$, $y_i' = Q'^{\frac{1}{2}} x_i'$, and $x_{i+1}' = \Phi_i'^T x_i'$, $y_i' = V'^{\frac{1}{2}} x_i'$ are both ms-detectable.

Define $\Phi_i^\alpha$, $\Gamma_i^\alpha$, $C_i^\alpha$, and $\Phi_i'^\alpha$ by

$$\Phi_i^\alpha = \overline{\Phi} + \alpha\tilde{\Phi}_i, \tag{29a}$$
$$\Gamma_i^\alpha = \overline{\Gamma} + \alpha\tilde{\Gamma}_i, \tag{29b}$$
$$C_i^\alpha = \overline{C} + \alpha\tilde{C}_i \tag{29c}$$
$$\Phi_i'^\alpha = \begin{bmatrix} \Phi_i^\alpha & -\Gamma_i^\alpha L \\ KC_i^\alpha & F \end{bmatrix}. \tag{29d}$$

Note that $\Phi_i^0 = \overline{\Phi}$, $\Gamma_i^0 = \overline{\Gamma}$, $C_i^0 = \overline{C}$, $\Phi_i'^0 = \overline{\Phi'}$ and $\Phi_i^1 = \Phi_i$, $\Gamma_i^1 = \tilde{\Gamma}_i$, $C_i^1 = \tilde{C}_i$, $\Phi_i'^1 = \Phi_i'$.

*Lemma 2:* $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Rightarrow (\Phi_i^\alpha, \Gamma_i^\alpha, C_i^\alpha)$ ms-compensatable, $\alpha \in [0, 1]$, and $(\Phi_i, \Gamma_i, C_i)$ ms-detectable $\Rightarrow (\Phi_i^\alpha, \Gamma_i^\alpha, C_i^\alpha)$ ms-detectable, $\alpha \in [0, 1]$.

*Proof:* Consider closed-loop system (9) where $x_i'$, $\Phi_i$, $\Gamma_i$, $C_i$, and $\Phi_i'$ are replaced by, respectively, $x_i'^\alpha$, $\Phi_i^\alpha$, $\Gamma_i^\alpha$, $C_i^\alpha$, and $\Phi_i'^\alpha$. Using the definition of $\Phi_i'^\alpha$ it is easy to see that $\Phi_i'^\alpha = \overline{\Phi'} + \alpha\tilde{\Phi}_i'$. Thus $\overline{\| x_i'^\alpha \|^2} = \|\overline{\Phi'}^i x_0'\|^2 + \alpha^{2i}\overline{\|\tilde{\Phi}_{i-1}' \cdots \Phi_0' x_0'\|^2}$, hence $\overline{\| x_i'^\alpha \|^2}$ is an increasing function of $\alpha$. So if $(\Phi_i'^\alpha)$ is ms-stable for some $\alpha$, then it is also ms-stable for a smaller $\alpha$, which proves the first part. Now consider system (6) where $x_i$, $y_i$, $\Phi_i$, and $C_i$ are replaced by, respectively, $x_i^\alpha$, $y_i^\alpha$, $\Phi_i^\alpha$, and $C_i^\alpha$. Then ms-detectability of $(\Phi_i^\alpha, C_i^\alpha)$ follows from $\overline{\| y_i^\alpha \|^2} = 0 \Rightarrow \overline{\| y_i \|^2} = 0$ and $\overline{\| x_i^\alpha \|^2} \leq \overline{\| x_i \|^2}$. A similar argument leads to ms-detectability of $(\Phi_i^{\alpha T}, \Gamma_i^{\alpha T})$. □

We are now in a position to state the solution of the optimal compensation problem. From now on we assume that $\Phi_i$, $\Gamma_i$, and $C_i$ are mutually independent. Independence of $\Gamma_i$ and $C_i$ is needed in order to use (23b), (23c). Independence of $\Phi_i$ from $\Gamma_i$ and $C_i$ is to simplify the presentation. Independence of $\Phi_i$, $\Gamma_i$, and $C_i$ is not a severe assumption in the context of robust control, i.e., where white parameters are used to *model* the uncertainty in the parameters. Now define the nonlinear transformation $C_{K,L}$: $S^n \times S^n \times S^n \times S^n \to S^n \times S^n \times S^n \times S^n$ by

$$C_{K,L}X = \Big(\overline{\Phi^T X_1 \Phi} - L^T\left(\overline{\Gamma^T X_1 \Gamma} + R + \overline{\tilde{\Gamma}^T X_2 \tilde{\Gamma}}\right)L + Q$$
$$+ \overline{\tilde{\Phi}^T X_2 \tilde{\Phi}} + \overline{\tilde{C}^T K^T X_2 K \tilde{C}},$$
$$(\overline{\Phi} - K\overline{C})^T X_2 (\overline{\Phi} - K\overline{C})$$
$$+ L^T\left(\overline{\Gamma^T X_1 \Gamma} + R + \overline{\tilde{\Gamma}^T X_2 \tilde{\Gamma}}\right)L,$$
$$\overline{\Phi X_3 \Phi^T} - K\left(\overline{C X_3 C^T} + W + \overline{\tilde{C} X_4 \tilde{C}^T}\right)K^T$$
$$+ V + \overline{\tilde{\Phi} X_4 \tilde{\Phi}^T} + \overline{\tilde{\Gamma} L X_4 L^T \tilde{\Gamma}^T},$$
$$(\overline{\Phi} - \overline{\Gamma}L) X_4 (\overline{\Phi} - \overline{\Gamma}L)^T + K\Big(\overline{C X_3 C^T}$$
$$+ W + \overline{\tilde{C} X_4 \tilde{C}^T}\Big)K^T\Big) \tag{30}$$

where $X = (X_1, X_2, X_3, X_4)$, $X_1, X_2, X_3, X_4 \in S^n$. Also define $F_X$, $K_X$, and $L_X$ by

$$F_X = \overline{\Phi} - \overline{\Gamma} L_X - K_X \overline{C}, \tag{31a}$$

$$K_X = \overline{\Phi} X_3 \overline{C}^T \left( \overline{C X_3 C^T} + W + \overline{\tilde{C} X_4 \tilde{C}^T} \right)^+, \tag{31b}$$

$$L_X = \left( \overline{\Gamma^T X_1 \Gamma} + R + \overline{\tilde{\Gamma}^T X_2 \tilde{\Gamma}} \right)^+ \overline{\Gamma}^T X_1 \overline{\Phi} \tag{31c}$$

and the nonlinear transformation $C: S^n \times S^n \times S^n \times S^n \to S^n \times S^n \times S^n \times S^n$ by

$$CX = C_{K_X, L_X} X. \tag{32}$$

Note that $(S, \hat{S}, P, \hat{P}) = C(S, \hat{S}, P, \hat{P})$ is equivalent to (23), (26), where (26a), (26c) are written slightly different for convenience later on. Now consider $(X_{1i}, X_{2i}, X_{3i}, X_{4i}) = C^i(\Theta, \Theta, \Theta, \Theta)$, $i = 0, 1, \cdots$. If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$, and $C_i = C$, where $\Phi, \Gamma, C$ are deterministic and constant, then $X_{1i}$ and $X_{3i}$ for $i = 0, 1, \cdots$ are the iterations of the well-known *uncoupled* control and estimation Riccati equations with initial value $\Theta$. It is well known that $\{X_{1i}\}$ and $\{X_{3i}\}$ are monotonic in the sense that $X_{1i} \leq X_{1j}$ and $X_{3i} \leq X_{3j}$ if $i < j$. This property may be used to prove convergence of $\{X_{1i}\}$ and $\{X_{3i}\}$, which gives us an easy way to calculate a solution of the algebraic Riccati equations. However, in the stochastic parameter case $\{X_{1i}\}$ and $\{X_{3i}\}$ are *not* monotonic due to the coupling between the corresponding equations. Fortunately, it is still possible to prove convergence, using the homotopic continuation method. This method is in short: first solve an easy "similar" problem, then continuously deform this problem into the original problem and follow the path of solutions as the easy problem is deformed into the original problem. Topological degree theory tells us under what conditions the number of solutions along the path keeps constant. For more information we refer to [22], [23]. Call $(X_1, X_2, \cdots)$ nonnegative definite if $X_1, X_2, \cdots \geq 0$.

*Theorem 3:* Assume that $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable and assume that either $R > 0$, $W > 0$, $(\Phi_i, V^{\frac{1}{2}}, Q^{\frac{1}{2}})$ ms-detectable, or $Q > 0$, $V > 0$. Then $Y = \lim_{i \to \infty} C^i(\Theta, \Theta, \Theta, \Theta)$ exists, $Y$ is the unique nonnegative definite solution of the equation $X = CX$, $(F^*, K', L^*) = (F_Y, K_Y, L_Y)$ and

$$\begin{aligned} \sigma_\infty^* &= \operatorname{tr}\left[ QY_3 + \left( Q + L_Y^T R L_Y \right) Y_4 \right] \\ &= \operatorname{tr}'\left[ VY_1 + \left( V + K_Y W K_Y^T \right) Y_2 \right] \end{aligned}$$

where $Y = (Y_1, Y_2, Y_3, Y_4)$, $Y_1, Y_2, Y_3, Y_4 \in S^n$.

*Proof:* Because $(\Phi_i, \Gamma_i, C_i)$ is ms-compensatable, there exists a compensator $(F, K, L)$ such that $(\Phi_i')$ is ms-stable. Thus, the set $C_{adm}^m$ is not empty. Hence, the necessary optimality conditions (23), (25) has a nonnegative definite solution $(S', P')$. Now suppose $(S', P')$ is such a solution. Then we may conceive $(S', P')$ as a solution of (25) for certain fixed $F$, $K$, $L$. Also from Lemma 1 we have that $(\Phi_i', V'^{\frac{1}{2}}, Q'^{\frac{1}{2}})$ is ms-detectable. Now we have a solution of (25a) and $(\Phi_i', Q'^{\frac{1}{2}})$ is ms-detectable. Then from [17] it follows that $(\Phi_i')$ is ms-stable. We may also have used that $(S', P')$ is a solution of (25b) and that $(\Phi_i'^T, V'^{\frac{1}{2}})$ is ms-detectable. This leads also, from [18], to ms-stability of $(\Phi_i')$. Therefore, under the conditions of the theorem all nonnegative definite solutions of (23), (25) correspond to compensators $(F, K, L) \in C_{adm}^m$. That leaves us to prove that (23), (25), or equivalently $X = CX$, has only one nonnegative definite solution $Y$ and that $Y = \lim_{i \to \infty} C^i(\Theta, \Theta, \Theta, \Theta)$. Replace in (23), (25), (30), and (31) $\Phi_i$, $\Gamma_i$, $C_i$, $\Phi_i'$ by, respectively, $\Phi_i^\alpha$, $\Gamma_i^\alpha$, $C_i^\alpha$, and $\Phi_i'^\alpha$ defined by (29). Replace the transformation $C$ in (32) by $C^\alpha$. Denote the parametrized equation $Y^\alpha = C^\alpha Y^\alpha$ by $H(Y^\alpha, \alpha) = 0$, where $Y^\alpha$ denotes the nonnegative definite solution of $X = C^\alpha X$ with parameter $\alpha$. Now for $\alpha = 1$ we have the original stochastic parameter case, and for $\alpha = 0$ the deterministic parameter case, i.e., the optimal compensation problem for the system $(\overline{\Phi}, \overline{\Gamma}, \overline{C})$. The function $H$ is called a homotopy and we may follow the solution path $Y^\alpha$ if $\alpha$ goes from 0 to 1. Now $(\Phi_i, \Gamma_i, C_i)$ is ms-compensatable and $(\Phi_i', V'^{\frac{1}{2}}, Q'^{\frac{1}{2}})$ is ms-detectable, thus from Lemma 2 $(\Phi_i^\alpha, \Gamma_i^\alpha, C_i^\alpha)$ is ms-compensatable and $(\Phi_i'^\alpha, V'^{\frac{1}{2}}, Q'^{\frac{1}{2}})$ is ms-detectable for $\alpha \in [0, 1]$. Hence, using topological degree theory, the number of ms-stabilizing solutions $Y^\alpha$ is constant along the solution path if $\alpha$ goes from 0 to 1. For the precise conditions we refer to [22], [23]. It is well known that $Y^0$ is unique, thus $Y^1$ is also unique. Moreover, it is well known that $Y^0 = \lim_{i \to \infty} C^{0i}(\Theta, \Theta, \Theta, \Theta)$. Then, using similar arguments as above, also $Y^1 = \lim_{i \to \infty} C^{1i}(\Theta, \Theta, \Theta, \Theta)$. □

From Theorem 4 it will be clear that the conditions of Theorem 3 are not only sufficient but also necessary in general. If $\Phi_i = \Phi$, $\Gamma_i = \Gamma$, and $C_i = C$, where $\Phi, \Gamma, C$ are deterministic and constant, then Theorem 3 gives the well known solution of the usual LQ optimal control problem with infinite horizon and long-term average criterion. Note that for ms-stability of $(\Phi_i')$ we need only ms-detectability of $(\Phi_i', Q'^{\frac{1}{2}})$ or $(\Phi_i'^T, V'^{\frac{1}{2}})$. Thus, according to the proof of Lemma 1, it is needed that i) either $R > 0$, $(\Phi_i, Q^{\frac{1}{2}})$ ms-detectable, or $Q > 0$; or ii) either $W > 0$, $(\Phi_i^T, V^{\frac{1}{2}})$ ms-detectable, or $V > 0$.

Finally, in this section we prove part c) of Theorem 1 in Section II.

*Proof of Theorem 1c):* $(\Phi_i, \Gamma_i, C_i)$ is ms-compensatable, thus (26) has a nonnegative definite solution for some $K$ and $L$ and for any $Q \geq 0$, $V \geq 0$. We may write (26a) and (26c) as

$$S = \overline{(\Phi - \Gamma L)^T S (\Phi - \Gamma L)} + Q + \Delta Q, \qquad \Delta Q \geq 0,$$

$$P = \overline{(\Phi - KC) P (\Phi - KC)^T} + V + \Delta V, \qquad \Delta V \geq 0.$$

Choose $Q > 0$, $V > 0$ then $Q + \Delta Q > 0$, $V + \Delta V > 0$ and thus $(\Phi_i, (Q + \Delta Q)^{\frac{1}{2}})$ and $(\Phi_i^T, (V + \Delta V)^{\frac{1}{2}})$ are both ms-detectable. Hence, $(\Phi_i - \Gamma_i L)$ and $(\Phi_i^T - C_i^T K^T)$ are both ms-stable [17], [18], thus $(\Phi_i, \Gamma_i)$ and $(\Phi_i^T, C_i^T)$ are both ms-stabilizable. □

Note that the proof of Theorem 1c) is constructive in the sense that ms-stability of $(\Phi_i')$ for $F = \overline{\Phi} - \overline{\Gamma} L - K \overline{C}$ and certain $K$ and $L$ implies that $(\Phi_i - \Gamma_i L)$ and $(\Phi_i - KC_i)$ are both ms-stable. Also note that in the proof we do not use any assumption concerning mutual independence of $\Phi_i$, $\Gamma_i$, and $C_i$.

## IV. Compensatability Tests

First we may state the following result concerning ms-compensatability and convergence of $C^i(\Theta, \Theta, \Theta, \Theta)$ as $i \to \infty$.

*Theorem 4:* Assume that either $R > 0$, $W > 0$, $(\Phi_i, V^{\frac{1}{2}}, Q^{\frac{1}{2}})$ ms-detectable, or $Q > 0$, $V > 0$. Then $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Leftrightarrow C^i(\Theta, \Theta, \Theta, \Theta)$ converges as $i \to \infty$.

*Proof:* By Theorem 3, $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Rightarrow C^i(\Theta, \Theta, \Theta, \Theta)$ converges as $i \to \infty$. The assumptions in this theorem are not needed here. Now suppose $Y = \lim_{i \to \infty} C^i(\Theta, \Theta, \Theta, \Theta)$ exists. Because $C^{i+1}(\Theta, \Theta, \Theta, \Theta) = CC^i(\Theta, \Theta, \Theta, \Theta)$ one has, taking the limits, $Y = CY$. Also $Y \geq 0$ by definition. Hence, (25) has a nonnegative definite solution for certain fixed $F$, $K$, $L$. Also from Lemma 1 we have that $(\Phi_i', V'^{\frac{1}{2}}, Q'^{\frac{1}{2}})$ is ms-detectable. Thus, using the same arguments as in Theorem 3, $(\Phi_i')$ is ms-stable, and therefore $(\Phi_i, \Gamma_i, C_i)$ is ms-compensatable. □

From Theorem 4 we have the following sufficient and necessary test, explicit in the system matrices, for systems with white stochastic parameters to be ms-compensatable.

*Compensatability Test 1:* Choose $Q = V = I$ and $R = W = 0$. Then $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Leftrightarrow C^i(\Theta, \Theta, \Theta, \Theta)$ converges as $i \to \infty$. □

The above test determines if $(\Phi_i')$ can be made ms-stable by some $F$, $K$, $L$, or equivalently if $\rho(A')$ can be made smaller than 1, by some $F$, $K$, $L$. An interesting problem is to determine the minimal value of $\rho(A')$, or equivalently the maximal ms-stability of $(\Phi_i')$, achievable through $F$, $K$, $L$. That would give us a *measure* of ms-compensatability rather than merely a ms-compensatability test. To investigate this issue consider the system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \tag{33a}$$

$$y_i = C_i x_i, \qquad i = 0, 1, \cdots, \tag{33b}$$

which is the same as system (11) except that $V = 0$ and $W = 0$, and the compensator

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_i y_i, \tag{34a}$$

$$u_i = -L_i \hat{x}_i, \qquad i = 0, 1, \cdots, \tag{34b}$$

which is the same as compensator (8) except that it is time varying now. The closed-loop system is given by

$$x'_{i+1} = \Phi_i' x_i', \qquad i = 0, 1, \cdots, \tag{35}$$

which is the same as (10) except that $\Phi_i'$ is now defined by

$$\Phi_i' = \begin{bmatrix} \Phi_i & -\Gamma_i L_i \\ K_i C_i & F_i \end{bmatrix}$$

where the statistics are not constant. Furthermore, the linear transformation $A_i'$: $S^{2n} \to S^{2n}$ is defined by

$$A_i' X = \overline{\Phi_i'^T X \Phi_i'}, \qquad X \in S^{2n} \tag{36}$$

which is the same as (19) except that $A_i'$ depends on $i$. Finally, define $\tilde{\rho}(A')$ by

$$\tilde{\rho}(A') = \min_{F, K, L} \rho(A') \tag{37}$$

where $A'$ is defined by (19). Now suppose $x_0$ is chosen such that all modes of $(\Phi_i)$ are excited at $i = 0$ and we choose a sequence of compensators $(F_0, K_0, L_0), \cdots, (F_{N-1}, K_{N-1}, L_{N-1})$ such that the value of $\overline{\| x_N \|^2}$ is as small as possible. Then we have from [16] $\lim_{N \to \infty} (\overline{\| x_N \|^2})^{1/N} = \tilde{\rho}(A')$. Therefore, in order to determine $\tilde{\rho}(A')$ we consider the problem of minimizing $\overline{\| x_N \|^2}$, which leads automatically to time varying compensators. From (35) we have

$$P'_{i+1} = \overline{\Phi' P_i' \Phi'^T}. \tag{38}$$

Let $I''$ denote $\operatorname{diag}(I, \Theta)$. Choose $P_0' = \overline{x_0' x_0'^T} = I''$ which represents the fact that all modes of $(\Phi_i)$ are excited at $i = 0$. Define $F^N = \{F_0, \cdots, F_{N-1}\}$, $K^N = \{K_0, \cdots, K_{N-1}\}$, and $L^N = \{L_0, \cdots, L_{N-1}\}$ and the criterion

$$J_N(F^N, K^N, L^N) = \overline{\| x_N \|^2} = \operatorname{tr}(P_{1,N}) = \operatorname{tr}(I'' P_N'). \tag{39}$$

All compensators $(F_i, K_i, L_i)$ are admissible, ms-stabilizing, or not because $N < \infty$. Now minimizing $\overline{\| x_N \|^2}$ is the problem of finding $F^N$, $K^N$, $L^N$ which minimizes (39) subject to (38). Let $J_N^*$ denote the minimal criterion value, then $\lim_{N \to \infty} (J_N^*)^{1/N} = \tilde{\rho}(A')$. To solve this problem we may apply again the matrix minimum principle [19]. Define the Hamiltonian $H_i$ by

$$H_i(F_i, K_i, L_i, P_i, S'_{i+1}) = \operatorname{tr}\left[\left(\overline{\Phi_i' P_i' \Phi_i'^T} - P_i'\right) S'_{i+1}\right], \qquad i = 0, \cdots, N-1 \tag{40}$$

where $S_1', \cdots, S_N' \in S^{2n}$ are the Lagrange multipliers. Then the necessary optimality conditions are

$$\frac{\partial H_i}{\partial F_i} = \frac{\partial}{\partial F_i} \operatorname{tr}\left(\overline{\Phi' P_i' \Phi'^T} S'_{i+1}\right) = 0, \tag{41a}$$

$$\frac{\partial H_i}{\partial K_i} = \frac{\partial}{\partial K_i} \operatorname{tr}\left(\overline{\Phi' P_i' \Phi'^T} S'_{i+1}\right) = 0, \tag{41b}$$

$$\frac{\partial H_i}{\partial L_i} = \frac{\partial}{\partial L_i} \operatorname{tr}\left(\overline{\Phi' P_i' \Phi'^T} S'_{i+1}\right) = 0, \tag{41c}$$

$$\frac{\partial H_i}{\partial P_i'} = \overline{\Phi'^T S_{i+1} \Phi'} - S'_{i+1} = S_i' - S'_{i+1},$$

$$S_N' = \frac{\partial \operatorname{tr}(I'' P_N')}{\partial P_N'} = I'', \tag{42a}$$

$$\frac{\partial H_i}{\partial S'_{i+1}} = \overline{\Phi' P_i' \Phi'^T} - P_i' = P'_{i+1} - P_i', \; P_0' = I'', \qquad i = 0, \cdots, N-1 \tag{42b}$$

where $S'_{i+1}$, $P_i' \geq 0$, $i = 0, \cdots, N-1$. Partition $S_i'$, $P_i'$ and define $S_i$, $\hat{S}_i$, $P_i$, $\hat{P}_i$ as in Section III, where the index $i$ is added. Also use the decompositions of $\Phi_i$, $\Gamma_i$, $C_i$ and define the linear transformation $B_i'$: $S^{2n} \to S^{2n}$ by

$$B_i' X = \overline{\Phi_i' X \Phi_i'^T}, \qquad X \in S^{2n} \tag{43}$$

which is the same as (24) except that $B_i' X$ depends on $i$. Then in essentially the same way as in Section IV we may transform (41), (42) to

$$F_i = \overline{\Phi} - \overline{\Gamma} L_i - K_i \overline{C}, \tag{44a}$$

$$K_i = \overline{\Phi} P_i \overline{C}^T \left( \overline{C P_i C^T} + \overline{\tilde{C} \hat{P}_i \tilde{C}^T} \right)^+, \tag{44b}$$

$$L_i = \left( \overline{\Gamma^T S_{i+1} \Gamma} + \overline{\tilde{\Gamma}^T \hat{S}_{i+1} \tilde{\Gamma}} \right)^+ \overline{\Gamma}^T S_{i+1} \overline{\Phi}, \tag{44c}$$

$$S_i' = A_i' S_{i+1}', \qquad S_N' = I'', \tag{45a}$$

$$P_{i+1}' = B_i' P_i', \qquad P_0' = I'', \qquad i = 0, \cdots, N-1. \tag{45b}$$

Equations (45a) and (45b) are coupled via (41) and together they form a two-point boundary-value problem which is in general very hard to solve. However, it will appear that these equations can still be used to determine $\tilde{\rho}(A')$. First observe that (44), (45) have the same structure as (23), (25) where $Q = 0$, $R = 0$, $V = 0$, and $W = 0$. Hence, we may transform (45) to

$$\begin{aligned} S_i = {} & \overline{(\Phi - \Gamma L_i)^T S_{i+1} (\Phi - \Gamma L_i)} \\ & + \overline{(\tilde{\Phi} - K_i \tilde{C})^T \hat{S}_{i+1} (\tilde{\Phi} - K_i \tilde{C})} \\ & + L_i^T \overline{\tilde{\Gamma}^T \hat{S}_{i+1} \tilde{\Gamma}} L_i, \end{aligned} \tag{46a}$$

$$\begin{aligned} \hat{S}_i = {} & (\overline{\Phi} - K_i \overline{C})^T \hat{S}_{i+1} (\overline{\Phi} - K_i \overline{C}) \\ & + L_i^T \left( \overline{\Gamma^T S_{i+1} \Gamma} + \overline{\tilde{\Gamma}^T \hat{S}_{i+1} \tilde{\Gamma}} \right) L_i, \end{aligned} \tag{46b}$$

$$\begin{aligned} P_{i+1} = {} & \overline{(\Phi - K_i C) P_i (\Phi - K_i C)^T} \\ & + \overline{(\tilde{\Phi} - \tilde{\Gamma} L_i) \hat{P}_i (\tilde{\Phi} - \tilde{\Gamma} L_i)^T} \\ & + K_i \overline{\tilde{C} \hat{P}_i \tilde{C}^T} K_i^T, \end{aligned} \tag{46c}$$

$$\begin{aligned} \hat{P}_{i+1} = {} & (\overline{\Phi} - \overline{\Gamma} L_i) \hat{P}_i (\overline{\Phi} - \overline{\Gamma} L_i)^T \\ & + K_i \left( \overline{C P_i C^T} + \overline{\tilde{C} \hat{P}_i \tilde{C}^T} \right) K_i^T \end{aligned} \tag{46d}$$

for $i = 0, \cdots, N-1$ and where $S_N = I$, $\hat{S}_N = \Theta$, $P_0 = I$, $\hat{P}_0 = \Theta$. Of course, we have $S_{i+1}, \hat{S}_{i+1}, P_i, \hat{P}_i \geq 0$, $i = 0, \cdots, N-1$. Now define the nonlinear transformations $D_{K,L}$, $D$: $S^n \times S^n \times S^n \times S^n \to S^n \times S^n \times S^n \times S^n$ which are exactly the same as, respectively, $C_{K,L}$ and $C$ except that $Q = 0$, $R = 0$, $V = 0$, and $W = 0$. Then $(S_i, \hat{S}_i, P_{i+1}, \hat{P}_{i+1}) = D(S_{i+1}, \hat{S}_{i+1}, P_i, \hat{P}_i)$ is equivalent to (44), (46).

*Theorem 5:* Suppose $(Y_{1i}, Y_{2i}, Y_{3i}, Y_{4i}) = D^i(I, \Theta, I, \Theta)$. Then $\tilde{\rho}(A') = \tilde{\rho}(B') = \lim_{i \to \infty} [\operatorname{tr}(Y_{1i} + Y_{3i})]^{1/i}$.

*Proof:* First note that from (36) and (45) we may write

$$J_N^* = x_0'^T S_0' x_0' = \operatorname{tr}(I'' S_0') = \operatorname{tr}(S_{1,0}).$$

Thus, using $S_{1,0} = S_0 + \hat{S}_0$ and $P_{1,N} = P_N + \hat{P}_N$, we have

$$\begin{aligned} J_N^* &= \operatorname{tr}(S_0 + \hat{S}_0) = \operatorname{tr}(P_N + \hat{P}_N) \\ &= \frac{1}{2} \operatorname{tr}(S_0 + \hat{S}_0 + P_N + \hat{P}_N). \end{aligned}$$

It is easy to show that

$$\begin{aligned} \omega &= \lim_{N \to \infty} \left[ \frac{1}{2} \operatorname{tr}(S_0 + \hat{S}_0 + P_N + \hat{P}_N) \right]^{1/N} \\ &= \lim_{N \to \infty} [\operatorname{tr}(S_0 + P_n)]^{1/N}. \end{aligned}$$

Now suppose for a moment that the initial time is not 0 but $M$; then

$$\omega = \lim_{\substack{N \to \infty \\ M \to -\infty}} [\operatorname{tr}(S_0 + P_N)]^{1/N-M} = \lim_{i \to \infty} [\operatorname{tr}(Y_{1i} + Y_{3i})]^{1/i}.$$

Hence

$$\tilde{\rho}(A') = \lim_{N \to \infty} (J_N^*)^{1/N} = \lim_{i \to \infty} [\operatorname{tr}(Y_{1i} + Y_{3i})]^{1/N}.$$

Finally because $\rho(A') = \rho(B')$ we have also $\tilde{\rho}(A') = \tilde{\rho}(B')$. □

From Theorem 5 the following compensatability test is immediate.

*Compensatability Test 2:* Suppose $(Y_{1i}, Y_{2i}, Y_{3i}, Y_{4i}) = D^i(I, \Theta, I, \Theta)$. Then $(\Phi_i, \Gamma_i, C_i)$ ms-compensatable $\Leftrightarrow$ $\lim_{i \to \infty} [\operatorname{tr}(Y_{1i} + Y_{3i})]^{1/i} < 1$. □

Also test 2 is sufficient and necessary, and explicit in the system matrices, and holds for systems with white stochastic parameters.

## V. Numerical Considerations and Examples

The ms-stability of $(\Phi_i')$ may be checked by calculation of $\rho(A')$. Let $\otimes$ denote Kronecker product [24]. It is easy to show that the eigenvalues of $A'$ and of $\overline{\Phi' \otimes \Phi'}$ are the same. Hence, $\rho(A') = \rho(\overline{\Phi' \otimes \Phi'})$ which is easy to calculate with standard software. Note that for deterministic and constant $\Phi'$ we have $\rho(\Phi' \otimes \Phi') = \rho^2(\Phi')$.

The ms-compensatability of $(\Phi_i, \Gamma_i, C_i)$ may be checked by compensatability test 1 or 2. Therefore, we need to iterate $C^i(\Theta, \Theta, \Theta, \Theta)$ or $D^i(I, \Theta, I, \Theta)$ until convergence is reached. In $CX$ or $DX$, terms like $\overline{\Phi^T X \Phi}$ arise for some matrix $X$ which may equally be written as $st^{-1}[(\overline{\Phi \otimes \Phi})^T st(X)]$, where $st$ denotes the stack operator, and using Kronecker product rules [24]. So $\overline{\Phi \otimes \Phi}$ needs only to be calculated once, while the $st$ and $st^{-1}$ operations involve only the renumbering of computer memory locations. It is remarked that concerning test 2 often $\operatorname{tr}(Y_{1,i+1} + Y_{3,i+1}) / \operatorname{tr}(Y_{1i} + Y_{3i})$ converges faster to $\tilde{\rho}$ than $[\operatorname{tr}(Y_{1i} + Y_{3i})]^{1/i}$.

For checking the ms-detectability of $(\Phi_i, V^{\frac{1}{2}}, Q^{\frac{1}{2}})$ one is referred to De Koning [9]. Note that from Definition 5 we have to check that $(\Phi_i, Q^{\frac{1}{2}})$ and $(\Phi_i^T, V^{\frac{1}{2}})$ are both ms-detectable, and also that $A^i X = st^{-1}[(\overline{\Phi \otimes \Phi})^{Ti} st(X)]$.

The optimal stabilizing compensator, if it exists, may now be calculated from Theorem 3 and using the remarks above, given a system and a criterion. In view of the calculations it is convenient to specify the needed statistics of the parameters by $\overline{\tilde{\Phi}\otimes\tilde{\Phi}}$, $\overline{\tilde{\Gamma}\otimes\tilde{\Gamma}}$ and $\overline{\tilde{C}\otimes\tilde{C}}$. Furthermore, we have that $\overline{\Phi\otimes\Phi} = \bar{\Phi}\otimes\bar{\Phi} + \overline{\tilde{\Phi}\otimes\tilde{\Phi}}$, and similarly for $\Gamma_i$ and $C_i$. Now we can make the calculations straightforward.

*Example 1:* Consider system $(\Phi_i, \Gamma_i, C_i)$ which is specified by

$$\bar{\Phi} = \begin{bmatrix} 0.7092 & 0.3017 \\ 0.1814 & 0.9525 \end{bmatrix}, \qquad \bar{\Gamma} = \begin{bmatrix} 0.7001 \\ 0.1593 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0.3088 & 0.5735 \end{bmatrix},$$

$$\overline{\tilde{\Phi}\otimes\tilde{\Phi}} = \beta(\bar{\Phi}\otimes\bar{\Phi}), \qquad \overline{\tilde{\Gamma}\otimes\tilde{\Gamma}} = \beta(\bar{\Gamma}\otimes\bar{\Gamma}),$$

$$\overline{\tilde{C}\otimes\tilde{C}} = \beta(\bar{C}\otimes\bar{C})$$

where $\beta \geq 0$. Now $\rho(\bar{\Phi}\otimes\bar{\Phi}) = \rho(\bar{\Phi})^2 = 1.2$ and $\rho(\overline{\Phi\otimes\Phi}) = (1+\beta)\rho(\bar{\Phi}\otimes\bar{\Phi}) = (1+\beta)1.2$. Thus, $(\bar{\Phi})$ is not stable in the usual sense and $(\Phi_i)$ is not ms-stable. From Theorem 5 we may calculate $\tilde{\rho}(A') = \tilde{\rho}(\overline{\Phi\otimes\Phi'})$ for different values of $\beta$ which is done in Table I.

For $\beta = 0$ we have the deterministic case. Then $\tilde{\rho}(A') = 0$ because $(\bar{\Phi}, \bar{\Gamma})$ is reachable and $(\bar{\Phi}, \bar{C})$ is observable. The radius $\tilde{\rho}(A')$ is an increasing function of $\beta$. For $\beta = 0.2$ system $(\Phi_i, \Gamma_i, C_i)$ is still ms-compensatable, for $\beta = 0.3$, not any more.

*Example 2:* Consider system (11) and criterion (13) where $\Phi_i$, $\Gamma_i$, and $C_i$ are specified as in Example 1 and $V$, $W$, $Q$, and $R$ by

$$V = \begin{bmatrix} 0.5627 & 0 \\ 0 & 0.7357 \end{bmatrix}, \qquad W = \begin{bmatrix} 0.2588 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.7350 & 0 \\ 0 & 0.9820 \end{bmatrix}, \qquad R = \begin{bmatrix} 0.6644 \end{bmatrix}.$$

Choose $\beta = 0.1$, then from Example 1 we know that $(\Phi_i, \Gamma_i, C_i)$ is ms-compensatable. It also holds that $R > 0$, $W > 0$, and $(\Phi_i, V^{\frac{1}{2}}, Q^{\frac{1}{2}})$ ms-detectable. We may also use the fact that $Q > 0$ and $V > 0$. From Theorem 3 we may calculate the optimal ms-stabilizing compensator $(F, K, L)$ specified by

$$F = \begin{bmatrix} 0.0731 & -0.8496 \\ -0.2095 & 0.2334 \end{bmatrix}, \qquad K = \begin{bmatrix} 0.6457 \\ 0.9439 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.6238 & 1.1154 \end{bmatrix}.$$

It is interesting to compare the spectral radius $\rho(A')$ of the optimal compensated system with $\tilde{\rho}(A')$, and to calculate the criterion value $\sigma_\infty^*$. That is done in Table II for different values of $\beta$.

The ms-stability of the system decreases as $\beta$ increases, while the criterion value increases. For $\beta = 0.3$ the system is not ms-stable and the criterion value is infinite.

Finally, in this section we remark that all the calculations are done with the software package PC-MATLAB, version 3.2 [25] on an Olivetti M280 PC. The calculation of $\tilde{\rho}(A')$ in Example 1 and $(F, K, L)$ in Example 2 for one value of $\beta$ took, respectively, 9 s and 13 s. Suppose $n = 8$, $m = 1$, and $l = 1$, then these two calculation times are, respectively, 31 s and 51 s.

TABLE I
MINIMAL SPECTRAL RADIUS

| $\beta$ | $\rho(\overline{\Phi\otimes\Phi})$ | $\tilde{\rho}(A')$ |
|---|---|---|
| 0 | 1.20 | 0 |
| 0.05 | 1.26 | 0.47 |
| 0.10 | 1.32 | 0.67 |
| 0.20 | 1.44 | 0.95 |
| 0.30 | 1.56 | 1.17 |

TABLE II
SPECTRAL RADIUS OPTIMAL SYSTEM

| $\beta$ | $\tilde{\rho}(A')$ | $\rho(A')$ | $\sigma_\infty^*$ |
|---|---|---|---|
| 0 | 0 | 0.56 | 5.3 |
| 0.05 | 0.47 | 0.65 | 6.8 |
| 0.10 | 0.67 | 0.75 | 9.7 |
| 0.20 | 0.95 | 0.95 | 53.5 |
| 0.30 | 1.17 | 1.17 | $\infty$ |

## VI. CONCLUSIONS

In this paper the problem of optimal compensation has been considered in the case of linear discrete-time systems with stationary white parameters and quadratic criteria. A generalization of the notion of ms-stabilizability, called ms-compensatability has been introduced. It has been shown that suitable conditions of ms-compensatability and ms-detectability are sufficient, and necessary in general, for the existence of a unique optimal ms-stabilizing compensator. Two tests have been given to determine if a system is ms-compensatable or not. One of the tests is based on a measure of ms-compensatability. It has been indicated how the tests and the optimal ms-stabilizing compensator may be calculated numerically. Finally, the results have been illustrated with some examples.

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