

Lecture notes Modelling Dynamic Systems Part 1 (2016)

Connects directly to the lecture notes of Systems & Control Theory Part 2 that concerns Optimal Control.

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1 People in control

1.1 Introduction

Mankind is dominating planet earth. Whether you like it or not, that's how it is. And we are dominating because we have gained control over many things one important thing being agriculture. Control is all around us and most of our lives we are busy controlling! We are driving cars, we are raising children, we are educating, we are learning to gain control, we are managing other people and others are managing us, we are trying to control economies, we are negotiating...

After reading this chapter the student should:

- Have improved their (intuitive) notion of control, measurement and modelling.
- Be able to draw the block diagram of a control system.
- Know the meaning of control system design, system identification, optimal input design and adaptive control.
- Know the difference between scientific modelling (first principles modelling), black box modelling and grey box modelling. Know the meaning of model validation.
- Realize the importance of control in life.

1.2 Questions

Why are people most successful in control? To answer this question, we should ask ourselves: "What is needed to be successful in control?". To successfully drive a car what is needed? People drive cars pretty well, but the control of economies often is not very successful. Why is that?

To drive a car you should 'know how it works'. If you mix up the gas, the brakes and the steering wheel ... Now most of us know all this when they drive a car for the first time ... So more seems to be needed ... Knowing perfectly how the car works, there is yet another problem: "How does the traffic work?" Is this a different problem?

Next let us focus on the control of economies. At some stages we have been quite successful at this (Keynes in the thirties). However at several occasions we faced an economic crisis! Do we know how economies work?

What actually does it mean when we say : "We know how it works?". And if we know how it works, do we know how to control? Or do we need something else as well?

1.3 Answers

Let us start with "knowing how it works". Roughly what does that mean? In the case of driving a car it means that we have developed a model, usually in our brain, of what happens with the car if we push the gas, turn the steering wheel, or push the brake. To keep the example simple let us forget about the traffic rules and pretend we are alone on the road. Then is our model of the car sufficient to control it? Having this model how precisely are you going to steer? At this stage you may certainly ask: "What are you driving at?"

To know how to control the car you must know its destination. Knowing its destination we still don't know exactly how to control the car. When for instance should we arrive at our destination, when and where do we depart? Even if we know all this, there are infinitely many ways to control the car! Roughly speaking knowing the model of the car is not enough for control. We also need to know control objectives. Knowing these there may still be several ways to control the system. To make a final decision we need either additional control objectives or a rule to decide which of the possible controls to select.

Now suppose the destination is 10 km away from where we depart and we know all of the above. Then we know precisely how to control the car. Remember we are also alone on the road. Now try to answer the following question: "Can you drive the car in this manner safely to its destination with your eyes closed?"

The answer is: "no". And the reasons are twofold. (1) the model of the car cannot be correct 100% and (2) also the control cannot be realised with an accuracy of 100%. So, to drive the car safely to its destination we need to keep our eyes open. Keeping our eyes open we are actually making measurements of where we really are. These measurements must also be used to control the car properly.

1.4 A control system

Now let us generalize our example. The car we call the system that has to be controlled. Then our model of the car becomes the model of the system or the systems model. The driver of the car we call the controller of the car. This controller uses the systems model as well as measurements (car position measurements) to control (steer, gas, brake) the car. The system (the car) together with its controller (the driver) we call the control system. Figure 1-1 represents the diagram of a control system.

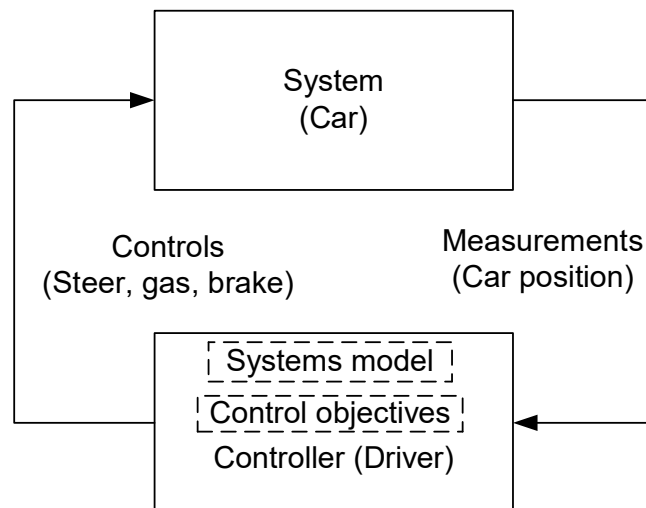


Figure 1-1 Diagram of a control system.

1.5 The Biobased Chemistry and Technology group: courses and research

In this first and introductory course taught by the Biobased Chemistry and Technology group entitled Modelling Dynamic Systems (BCT-20306) we focus on building mathematical models using scientific knowledge (first principles) and measurements (measurement data). However given the importance of control in real life, we very much recommend studying the control system design courses:

- Systems and Control Theory (BCT-31306)
- Control Engineering (BCT-21306)
- Parameter Estimation and Model Structure Identification (BCT-31806)

They rely very much on the contents of this course.

Control is a major topic in life, although very often, we are unaware of this. Control system design is one of the research topics of the Biobased Chemistry and Technology group (BCT) at Wageningen University. Two courses on this subject are taught by us: Systems and Control Theory (BCT-31306) and Control Engineering (BCT-21306). The first course is concerned with optimal control system design (researched by Gerard van Willigenburg) while the second course is closer to current industrial practice (researched by Ton van Boxtel and Rachel van Ooteghem). The systems for which we design controllers range from bioreactors, wastewater treatment plants, dryers, greenhouses and crops to robots and autonomous vehicles.

In science, control is not necessarily the ultimate objective. The scientific objective is often "to know and understand how it works". In this case the systems model is the important object. In Figure 1-1 observe that measurements are used to aid the selection of the control because the model is not accurate 100%. The measurements supply information about what

is really happening (the real position of the car) which is different from what the model says because the model is not accurate 100%. Therefore, the measurements can also be used to improve the model. If the measurements entering the controller are also used to improve the model the controller is called an adaptive controller, because the model is adapted, see Figure 1-2 (adaptive controllers are researched by Gerard van Willigenburg). When adapting the model, we have to keep in mind that also measurements are not exact 100%.

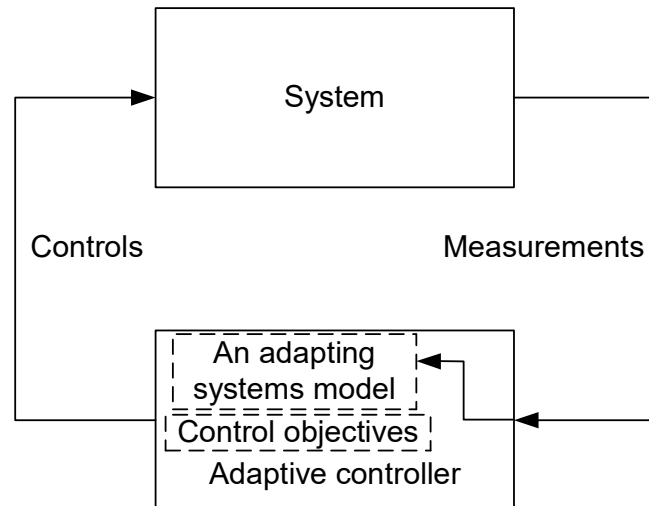


Figure 1-2 An adaptive control system.

If modelling instead of control is the objective, then improving the model by means of employing the measurements is called system identification, see Figure 1-3 (researched by Karel Keesman). A course on this topic is called Parameter Estimation and Model Structure Identification (BCT-31806).

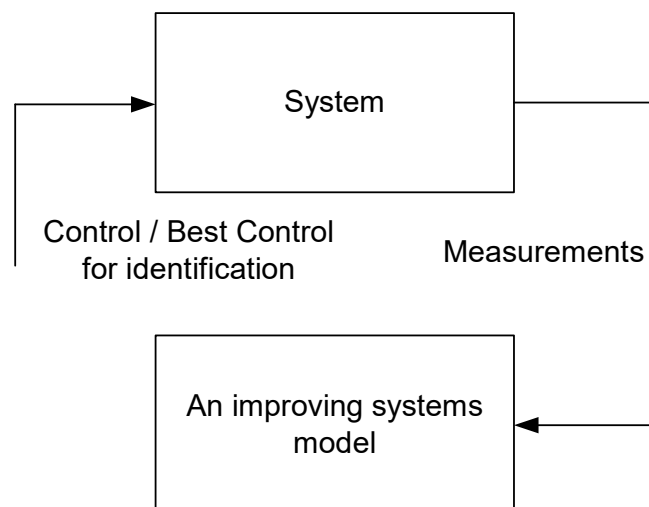


Figure 1-3 System identification / optimal input design.

Usually, the information content of the measurements depends on the control of the system. For example, if we control the car without turning the steering wheel the measurements will contain no information about the influence of steering on the car movement. So an interesting question that arises is: "Which control maximizes the information content of the measurements?" This question is raised and answered by what is called optimal input design, see Figure 1-3 (researched by Karel Keesman).

Having built a mathematical model another important question is: "How good is the model?". Measurements may be used to verify or validate the model by observing how well they correspond with the outcome of the model. This is called model validation which is part of system identification. It is well known that to validate a model properly measurements different from the ones used to build the model have to be used.

The model of the car resides in our brain and is probably not mathematical. However, mathematics is the language of science and computing while scientific knowledge, like chemical, physical and mechanical laws, form the basis for building models. Therefore, control system design requires a mathematical model of the system and a mathematical description of the control objectives. The chemical, physical and mechanical laws are sometimes referred to as first principles and the associated modelling as first principles modelling. If we build a mathematical model of the car we exploit several known laws from mechanics. Another type of mathematical modelling, called black box modelling ignores totally all scientific knowledge and builds the model entirely from measurements. So called grey box modelling combines first principles and black box modelling.

1.6 Summary

- Control is an important part of our lives although we usually do not realise this.
- Controllers require a model of the system as well as measurements.
- Control system design requires a mathematical model of the system and a mathematical description of the control objectives.
- A controller is called adaptive if the measurements are not just used to decide on the control but are also used to improve the model. System identification is not concerned with control but only with improving the model using measurement data. Optimal input design is concerned with finding the best control for system identification.
- Mathematical models are built from scientific knowledge (laws) and/or measurement data. Scientific modelling (first principles modelling) is highly based on scientific knowledge (first principles like physical, chemical or mechanical laws). Black box models

ignore totally all scientific knowledge and are entirely build from measurement data. Grey box models are in between these two. Building a mathematical model is the subject of system identification. Model validation is an important part of system identification.

2 Mathematical preliminaries: Taylor series expansions, linearization, complex numbers and matrices

2.1 Introduction

Building mathematical models, using them to compute system behaviour, and interpreting the results, are the main issues of this course. The mathematics needed to perform these tasks is summarized in this section. The four mathematical subjects mentioned in the title are the most important ones needed for this course. They will be reviewed in this chapter through several examples.

After reading this chapter the student:

- Is able to reproduce and use Taylor series expansions.
- Is able to understand and make computations with complex numbers.
- Is able to understand and make calculations with matrices.

2.2 Did you know...?

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, x \in \mathbb{R}^1 \quad (2.1)$$

$$j^2 = -1 \quad (2.2)$$

$$e^{j\phi} = \cos(\phi) + j \sin(\phi) \quad (2.3)$$

Equation (2.1) is a special case of what is called a Taylor series expansion. The formula is actually used to calculate e^x in calculators and computers because the series

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ converges for all values of x . Can you see why?

Equation (2.2) defines the so-called imaginary number j . Alternatively, we may define,

$$j = \sqrt{-1} \quad (2.4)$$

The imaginary number j appears in equation (2.3). The number on the right-hand side of the equality sign is a so-called complex number which has a real part namely $\cos(\phi)$, a real number, and an imaginary part $\sin(\phi)$, because $\sin(\phi)$ is multiplied with the imaginary

number j . To prove that equation (2.3) holds we need to know about Taylor series expansions. Equation (2.3) is important to understand the properties of complex numbers. Complex numbers and Taylor series expansions themselves turn out to be very important to analyse mathematical systems models.

2.3 Taylor series expansions

The following equation represents Taylor series expansions of the infinitely differentiable function $f(x)$,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots \quad (2.5)$$

The equation may be interpreted as follows. Knowing the value of the function and all its derivatives for one value $x = x_0$ we can compute the function values for arbitrary values x .

This is a rather remarkable property of infinitely differentiable functions $f(x)$. We will not prove that equation (2.5) holds. However, for a simple example you can check it. Take $f(x) = x^2$. So $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$. Also take $x_0 = 1$, $x = 3$. Then equation (2.5) reads,

$$3^2 = 1^2 + \frac{2}{1!}2 + \frac{2}{2!}2^2 + 0 \quad (2.6)$$

For any value of x_0 and x equation (2.5) must be satisfied. Verify this by taking some other values. Equation (2.5) is satisfied for any infinitely differentiable function. Try another second order polynomial function.

Equation (2.1) is obtained from equation (2.5) by taking $f(x) = e^x$, $x_0 = 0$. Check this. From equation (2.1) taking $x = j\phi$ we obtain,

$$e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{j^2\phi^2}{2!} + \frac{j^3\phi^3}{3!} + \dots \quad (2.7)$$

Using equation (2.2) this becomes,

$$e^{j\phi} = 1 + j\frac{\phi}{1!} - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + j\frac{\phi^5}{5!} - \dots \quad (2.8)$$

which after rearrangement of the terms on the right-hand side of the equality sign becomes,

$$e^{j\phi} = \underbrace{\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \dots\right)}_{\cos(\phi)} + j \underbrace{\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \dots\right)}_{\sin(\phi)} \quad (2.9)$$

The first term in between brackets in equation (2.9) is a Taylor series expansion of $\cos(\phi)$ while the second term in between the brackets is a Taylor series expansion of $\sin(\phi)$ (verify this using equation (2.5)). So, equation (2.9) is identical to equation (2.3)! The only equations we have used to obtain this result are (2.2) and (2.5)!

2.4 Linearization and linear approximations of functions

Consider equation (2.5) again. Suppose that $x - x_0$ is small. Mathematically speaking small values of $x - x_0$ define a so-called neighbourhood of x_0 . When $x - x_0$ is small from equations (2.5) $f(x)$ may be approximated by,

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) \quad (2.10)$$

because the other terms in equation (2.5) are negligible. The approximation (2.10) may be interpreted as follows, see Figure 2-1.

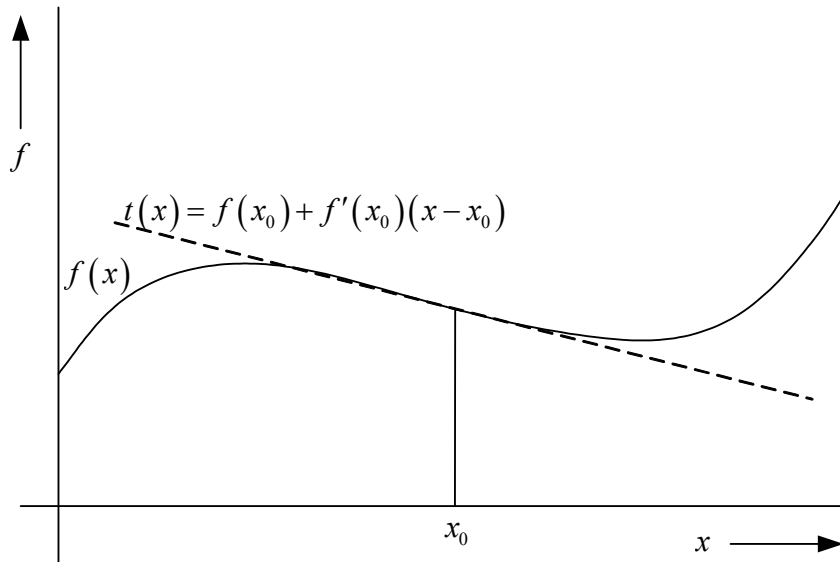


Figure 2-1 Approximation of $f(x)$ near $x = x_0$ by the tangent line $t(x)$.

Near $x = x_0$ the function $f(x)$ may be very well approximated by the straight line described by the right-hand side of equation (2.10). This line is the tangent line of $f(x)$ at $x = x_0$ denoted by $t(x)$. From (2.10) and Figure 2-1,

$$t(x) = f(x_0) + f'(x_0)(x - x_0) \quad (2.11)$$

Example 2.1

$f(x) = (x-2)^3 - 3x + 10$. Find the tangent line for $x_0 = 2$ that approximates $f(x)$ near $x = 2$.

$$t(x) = \underbrace{(2-2)^3 - 3 \cdot 2 + 10}_{f(x_0)} + \underbrace{(3 \cdot (2-2)^2 - 3)}_{f'(x_0)} \underbrace{(x-2)}_{(x-x_0)} = \underbrace{4}_{f(x_0)} \underbrace{- 3}_{f'(x_0)} \underbrace{(x-2)}_{(x-x_0)} = -3x + 10$$

What is so interesting about the straight-line approximation? Well, the approximation is a linear function of x . And linear functions are much easier to analyse and calculate than non-linear functions. Later in this course we shall see that the same holds for linear systems versus non-linear systems. The tangent line described by equation (2.11) is called a linearization of the function $f(x)$ at $x = x_0$. This linearization is an accurate approximation of the function $f(x)$ itself in the neighbourhood of x_0 . Later in this course we will compute linearizations of non-linear systems.

2.5 Complex numbers

Imaginary and complex numbers are very useful to analyse linear systems, as we shall see later in this course. In section 2.2 we already introduced imaginary numbers and complex numbers which are different from the real numbers that we are familiar with. The set of all complex numbers is denoted by C and is given by,

$$C = \{a + jb \mid a \in R, b \in R\}, j^2 = -1 \quad (j = \sqrt{-1}) \quad (2.12)$$

Of the complex numbers $a + jb$ in (2.12) a is called the real part of the complex number and b the imaginary part, which is the part that multiplies j . The set of all imaginary numbers is denoted by I and is given by,

$$I = \{jb \mid b \in R\}, j^2 = -1 \quad (j = \sqrt{-1}) \quad (2.13)$$

Clearly the set of complex numbers C contains the set of imaginary numbers I and also the set of real numbers R ,

$$C \supset I, C \supset R \quad (2.14)$$

We will use complex numbers to make computations which are useful to analyse linear systems. The rules of computation are identical to the rules we know for real numbers. The only two additional things we have to be aware of are the following. (1) any complex number always separates into a real part, which may be zero, and an imaginary part, which may also be zero. In other words, we cannot and may not mix real parts with imaginary parts. For instance, we cannot add the real part to the imaginary part of a complex number. (2) when making computations we must realize that $j^2 = -1$. A few examples of computations with complex numbers are listed below.

$$(3+4j)+(1-2j)=3+1+4j-2j=4+2j \quad (2.15)$$

$$(3+4j)-(1-2j)=3-1+4j+2j=2+6j \quad (2.16)$$

$$\begin{aligned} (3+4j)(1-2j) &= 3 \cdot 1 - 3 \cdot 2j + 1 \cdot 4j - 4 \cdot 2j^2 = \\ &= 3 - 6j + 4j - 8j^2 = 3 - 2j - 8j^2 = \\ &= 3 - 2j + 8 = 11 - 2j \end{aligned} \quad (2.17)$$

When dividing two complex numbers, although the rules do not change, we need a 'trick' to obtain the real and imaginary part of the answer,

$$\frac{(3+4j)}{(1-2j)} = \frac{(3+4j)(1+2j)}{(1-2j)(1+2j)} = \frac{-5+10j}{5} = -1+2j \quad (2.18)$$

The 'trick' is to multiply the numerator and the denominator in (2.18) by $(1+2j)$. Then after multiplying out the denominator, it becomes a complex number with a zero imaginary part. This is needed to specify the real and imaginary part of the answer. Now the question is: "How do we know we have to multiply with $(1+2j)$?" The complex number $1+2j$ is the so-called complex conjugate of the complex number $1-2j$ in the denominator. The complex conjugate of a complex number is the same number except for the sign of the imaginary part, which is reversed. Multiplying a complex number with its complex conjugate gives a complex number with a zero imaginary part,

$$(a+jb)(a-jb) = a^2 - ajb + ajb - j^2b^2 = a^2 + b^2 \quad (2.19)$$

and that is precisely what we need in (2.18). How can you verify in (2.18) that $-1+2j$ is the correct answer?

Complex numbers $a + jb$ can be represented graphically in what is called the complex plane, see Figure 2-2.

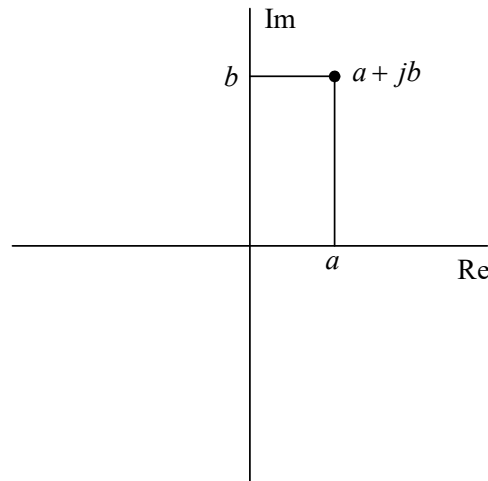


Figure 2-2 The complex plane with the real axis Re and the imaginary axis Im.

Next we turn our attention to the complex numbers, given by equation (2.3),

$$e^{j\phi} = \cos(\phi) + j \sin(\phi) \quad (2.3)$$

From the right-hand side of (2.3) observe that these numbers lie on the unit circle of the complex plane, see Figure 2-3.

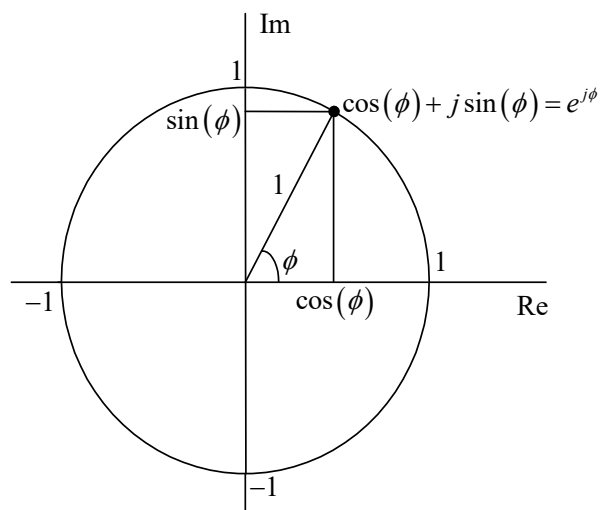


Figure 2-3 Complex numbers on the unit circle.

Combining Figure 2-2 and Figure 2-3, we find that every complex number can be written in two ways.

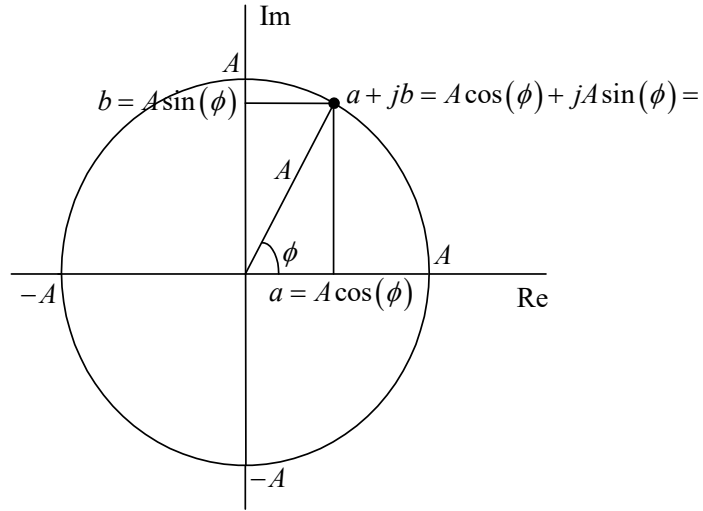


Figure 2-4 Two ways to write complex numbers.

The second way to write a complex number according to Figure 2-4 is $Ae^{j\phi} = A\cos(\phi) + jA\sin(\phi)$. This notation determines the complex number through A and ϕ . A is called the absolute value of the complex number and equals the distance between the complex number and the origin in the complex plane. ϕ is called the argument of the complex number. It is the angle that the line connecting the complex number with the origin makes with the real axis. Knowing the real part a and the imaginary part b of a complex number, according to Figure 2-4, we can compute its absolute value A and argument ϕ as follows,

$$A = \sqrt{a^2 + b^2}, \phi = \arctan\left(\frac{b}{a}\right) \quad (2.20)$$

On the other hand, from A, ϕ we can compute a, b as follows (again see Figure 2-4),

$$a = A\cos(\phi), b = A\sin(\phi) \quad (2.21)$$

The multiplication of complex number has an interesting graphical interpretation. This can be seen if we write complex numbers using their absolute value and argument,

$$A_1 e^{j\phi_1} A_2 e^{j\phi_2} = A_1 A_2 e^{j(\phi_1 + \phi_2)} \quad (2.22)$$

So multiplication of complex numbers comes down to multiplying their absolute values and adding up their arguments. The graphical representation of this is represented by Figure 2-5.

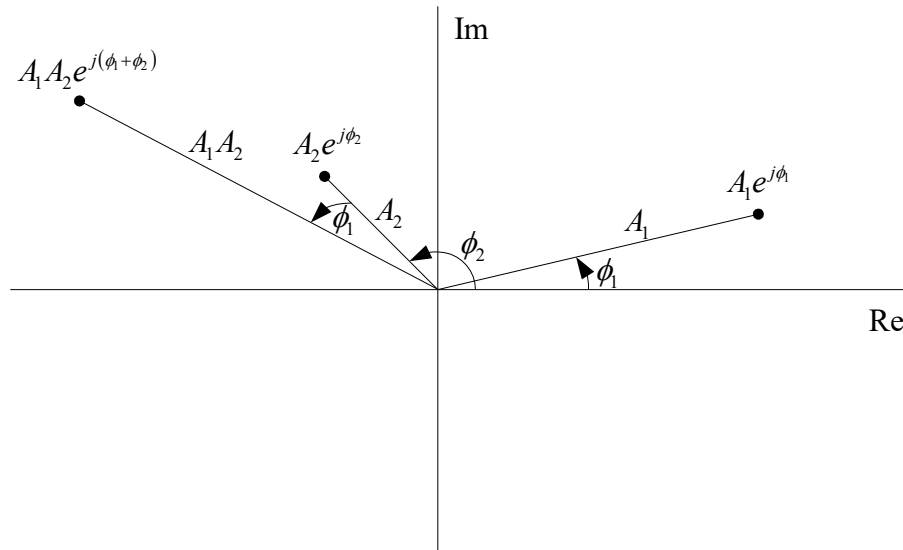


Figure 2-5 Graphical representation of the multiplication of complex numbers.

2.6 Matrices

2.6.1 Introduction

Matrices are rectangular blocks of separate numbers that turn out to be very handy for certain computations.

Example 2.2

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 1 & 3.1 \\ -1.5 & 2.2 \end{bmatrix}$$

Matrix A has 2 rows and 3 columns of numbers. Matrix B has 3 rows and 2 columns. Matrix A is therefore called a 2×3 matrix while matrix B is called a 3×2 matrix. In general, a matrix has n rows and m columns so it is an $n \times m$ matrix, with integers $n, m \geq 0$. If n or m equals zero the matrix is called an empty matrix. The number of rows n and columns m are called the dimensions of the matrix. Matrix A contains 6 numbers called elements of a matrix. The position of each element of a matrix is given by the row and column number or index. For example, the number 8 in matrix A has row index 2 and column index 3. The number 8 is also called element 2,3 of matrix A denoted by $A_{2,3}$. So $A_{2,3} = 8$. Also 2,3 is called the location of the number 8 within matrix A .

Question 2.1

How many rows and columns does matrix B have? $B_{2,2} = ..?$

2.6.2 Matrix addition/subtraction

When adding or subtracting two matrices add/subtract the elements at the same location of both matrices. Thus, the two matrices must have the same dimensions.

Example 2.3

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2 \times 3} + \begin{bmatrix} -1 & 2 & 2.5 \\ 0 & -2 & -3.1 \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 0 & 5 & 7.5 \\ 2 & 2 & 2.9 \end{bmatrix}_{2 \times 3} \quad (2.23)$$

$$\begin{bmatrix} 2 & 4 \\ 1 & 3.1 \\ -1.5 & 2.2 \end{bmatrix}_{3 \times 2} - \begin{bmatrix} -1 & 3 \\ 10.4 & -7 \\ 1.5 & 2.1 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 3 & 1 \\ -9.4 & 10.1 \\ -3 & 0.1 \end{bmatrix}_{3 \times 2} \quad (2.24)$$

2.6.3 Matrix multiplication

To define how matrices are multiplied we first define the multiplication of a row vector times a column vector with the same length. We start with an example:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1 \times 3} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_{3 \times 1} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32 \quad (2.25)$$

In general:

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}_{1 \times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1} = \underbrace{x_1 y_1 + x_2 y_2 + \dots + x_n y_n}_{1 \times 1} = \sum_{k=1}^n x_k y_k \quad (2.26)$$

Multiplying two matrices comes down to repeated multiplication of row vectors with column vectors having the same length. Two matrices A, B can only be multiplied with one another if the number of columns of A equals the number of rows of B . The general rule for multiplication is stated below.

$$\underbrace{A}_{n \times m} \underbrace{B}_{m \times k} = \underbrace{C}_{n \times k}, \quad \underbrace{C_{i,j}}_{\substack{\text{element } i,j \\ \text{of } C}} = i^{\text{th}} \text{ row of } A \text{ times } j^{\text{th}} \text{ column of } B \quad (2.27)$$

Example 2.4

$$A = \underbrace{\begin{bmatrix} 1 & 4 & -2 \\ 3 & -1 & 2 \end{bmatrix}}_{2 \times 3} \underbrace{\begin{bmatrix} 2 & 1 & -1 & -3 \\ 3 & -2 & 0 & 1 \\ 5 & -3 & 2 & 6 \end{bmatrix}}_{3 \times 4} = \underbrace{\begin{bmatrix} 4 & 1 & -5 & -11 \\ 13 & -9 & 1 & 2 \end{bmatrix}}_{2 \times 4} \quad (2.28)$$

Exercise 2.1

What about $\underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3 \times 1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1 \times 3}$? Can they be multiplied? If yes, what comes out?

As opposed to numbers, multiplying matrices is not commutative because in general $AB \neq BA$. Explain why by using a suitable example.

2.6.4 Square matrix, identity matrix and matrix inverse.

A matrix is called square if it has the same number of rows and columns.

For example: $\underbrace{A}_{3 \times 3} = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \\ -2 & -6 & 0 \end{bmatrix}}_{3 \times 3}$

To indicate the dimensions of matrix A as well as its type of elements we may write $A \in R^{3 \times 3}$. Here $\in$ means “is an element of” and $R^{3 \times 3}$ means “the set of 3x3 matrices with real numbers as elements”. The square matrix A is said to have dimension 3.

The identity matrix is a square matrix with elements all equal to zero, except on the diagonal where each element equals one. The identity matrix is often denoted by I . So the identity matrix with dimension 3 is:

$$\underbrace{I}_{3 \times 3} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3 \times 3} \in R^{3 \times 3} \quad (2.29)$$

The identity matrix I has more or less the same role as the number 1. Pre and post multiplying a matrix with I of compatible dimensions results in the same matrix:

$$\underbrace{I}_{n \times n} \underbrace{A}_{n \times m} = \underbrace{A}_{n \times m}, \quad \underbrace{A}_{n \times m} \underbrace{I}_{m \times m} = \underbrace{A}_{n \times m} \quad (2.30)$$

Check this by doing an example with $n \neq m$.

Most square matrices (except for singular square matrices) have an inverse matrix. The inverse of square matrix A has the same dimensions and is denoted by A^{-1} . When a square matrix is pre or post multiplied with its inverse the identity matrix is obtained:

$$\underbrace{A}_{n \times n} \underbrace{A^{-1}}_{n \times n} = \underbrace{I}_{n \times n} = \underbrace{A^{-1}}_{n \times n} \underbrace{A}_{n \times n} \quad (2.31)$$

Also,

$$(A^{-1})^{-1} = A \quad (2.32)$$

Finally,

$$\left(\underbrace{\begin{bmatrix} \underbrace{A}_{n \times n} & \underbrace{B}_{n \times n} \end{bmatrix}}_{n \times n} \right)^{-1} = \underbrace{B^{-1}}_{n \times n} \underbrace{A^{-1}}_{n \times n} \quad (2.33)$$

Note the reversed order in (2.33).

For $n=2$ the inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Note that

matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is pre-multiplied by the scalar $\frac{1}{ad-bc}$. By definition this means that all

elements of matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ are multiplied by $\frac{1}{ad-bc}$. So we obtain

$A^{-1} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix}$. Check that the expression for A^{-1} is correct by doing

an example. A problem occurs if $ad-bc=0$. Then the inverse equals infinity and so formally does not exist. $ad-bc$ is called the determinant of square matrix A . So square matrix A has no inverse when its determinant equals zero. The determinant of a square matrix equals zero when the square matrix is singular. A square matrix is singular when one of its columns (rows) is a linear combination of the other columns (rows). That means that this column (row) is not independent of the others. The rank of a square matrix is the number of independent columns (rows). If the rank of square matrix A is identical to its dimension n the matrix is called full rank. Only in this case all rows and columns of A are independent, the determinant of A is non-zero and the inverse A^{-1} exists. Otherwise, the square matrix A is

singular. For $n > 2$ the determinant of A and A^{-1} can be calculated using the Matlab function `det`.

One of the benefits of matrices is that they allow to quickly solve a set of linear equations by computing a matrix inverse.

Example 2.5

$$\begin{aligned} \left. \begin{aligned} 1x_1 + 2x_2 &= 5 \\ -3x_1 - 4x_2 &= 1 \end{aligned} \right\} &\Leftrightarrow \underbrace{\begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_y \xRightarrow{\substack{\text{pre-multiply} \\ \text{by } A^{-1}}} \underbrace{A^{-1}A}_I x = A^{-1}y \Leftrightarrow x = A^{-1}y \\ A^{-1} &= \frac{1}{-4 - -6} \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2 & -1 \\ 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 8 \end{bmatrix} \end{aligned} \quad (2.34)$$

2.6.5 Matrix division

Dividing by a number a is the same as multiplying by the number a^{-1} . So division can be defined using multiplication. For matrices left and right division by square matrix A is defined as pre and post multiplication by matrix A^{-1} respectively:

$$\underbrace{A \setminus B}_{\substack{\text{left divide} \\ B \text{ by } A}} = A^{-1}B, \quad \underbrace{B / A}_{\substack{\text{right divide} \\ B \text{ by } A}} = BA^{-1} \quad (2.35)$$

Why - in general - are left and right division not identical?

Note that if matrix A is singular in (2.35) then A^{-1} does not exist. In that case, instead of using A^{-1} , a matrix denoted by A^+ is used that exists for any matrix A , even if A is singular and even if A is not square. This matrix A^+ is called the Moore-Penrose pseudo-inverse of A . A^+ has the same dimensions as A . When the matrix A is square and non-singular $A^+ = A^{-1}$. So A^+ is a generalized inverse of matrix A . Using A^+ we may replace (2.35) by,

$$\underbrace{A \setminus B}_{\substack{\text{left divide} \\ B \text{ by } A}} = A^+B, \quad \underbrace{B / A}_{\substack{\text{right divide} \\ B \text{ by } A}} = BA^+ \quad (2.36)$$

2.6.6 Interchanging rows and columns: matrix transpose

The transpose of $n \times m$ matrix $\underline{A}_{n \times m}$ is denoted by $\underline{A}^T_{m \times n}$ (A transposed) which has dimensions $m \times n$. The columns of $\underline{A}_{n \times m}$ are equal to the rows of $\underline{A}^T_{m \times n}$. Alternatively, one may say that the rows of $\underline{A}_{n \times m}$ are the columns of $\underline{A}^T_{m \times n}$.

Example 2.6

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$$

The following rule that is similar to equation (2.33) applies to two matrices of compatible but otherwise arbitrary dimensions:

$$\left(\begin{bmatrix} \underline{A} & \underline{B} \end{bmatrix}_{n \times m \quad m \times k} \right)^T = \begin{bmatrix} \underline{B}^T & \underline{A}^T \end{bmatrix}_{k \times m \quad m \times n} \quad (2.37)$$

You can check this rule by doing a simple example.

2.7 Rules of differentiation

Computation of derivatives of functions, mostly first derivatives, plays a central role in modelling. Therefore, rules for differentiating functions are of primary importance.

2.7.1 Notation

A scalar function f of scalar x will be denoted by,

$$f(x), f \in R^1, x \in R^1 \quad (2.38)$$

Instead of $f(x)$ you may write y .

Example

$$f(x) = x^3 - 3x + 4 \quad (2.39)$$

$$y = x^3 - 3x + 4 \quad (2.40)$$

The first derivative of $f(x)$ with respect to x is written in two different ways,

$$f'(x) = \frac{df}{dx} = 3x^2 - 3 \quad (2.41)$$

The value of this derivative function for $x = 2$ is also written in two different ways,

$$f'(2) = \left. \frac{df}{dx} \right|_{x=2} = 3 \cdot 2^2 - 3 = 9 \quad (2.42)$$

Using expression (2.40) instead of (2.39), expressions (2.41), (2.42) become,

$$y' = \frac{dy}{dx} = 3x^2 - 3 \quad (2.43)$$

$$y'(2) = \left. \frac{dy}{dx} \right|_{x=2} = 3 \cdot 2^2 - 3 = 9 \quad (2.44)$$

When writing down mathematical models in this course we will generally use f to denote a function and so we will use expressions (2.41), (2.42) not (2.43), (2.44). Moreover, the first derivative of function f will mostly be written as $\frac{df}{dx}$ not $f'(x)$. Summarizing in this course we mostly write,

$$f(x) = x^3 - 3x + 4, \quad \frac{df}{dx} = 3x^2 - 3, \quad \left. \frac{df}{dx} \right|_{x=2} = 3 \cdot 2^2 - 3 = 9 \quad (2.45)$$

2.7.2 Power rule

$$f(x) = x^n, \quad \frac{df}{dx} = nx^{n-1} \quad \text{with } f, x \in R^1, n \in I^1 \quad (2.46)$$

In expression (2.46) $x, f \in R^1, n \in I^1$ means that x and f are real scalars while n is an integer scalar.

Example

$$f(x) = x^{-3}, \quad \frac{df}{dx} = -3x^{-4} \quad (2.47)$$

2.7.3 Function times a constant

$$f(x) = c \cdot g(x), \quad \frac{df}{dx} = c \frac{dg}{dx}, \quad c, f, g, x \in R^1, c: \text{constant} \quad (2.48)$$

Example

$$f(x) = 2x^{-3}, \quad c = 2, \quad g(x) = x^{-3}, \quad \frac{df}{dx} = 2 \cdot (-3x^{-4}) = -6x^{-4} \quad (2.49)$$

2.7.4 Sum (difference)

$$f(x) = g(x) + h(x), \quad \frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx}, \quad f, g, h, x \in R^1 \quad (2.50)$$

Example

$$f(x) = x^{-3} - x^2, g(x) = x^{-3}, h(x) = -x^2, \frac{df}{dx} = -3x^{-4} - 2x \quad (2.51)$$

2.7.5 Product rule

$$f(x) = g(x) \cdot h(x), \frac{df}{dx} = \frac{dg}{dx} h(x) + \frac{dh}{dx} g(x), f, g, h, x \in \mathbb{R}^1 \quad (2.52)$$

Example

$$\begin{aligned} f(x) &= 3x^2 \cdot x^{-4}, g(x) = 3x^2, h(x) = x^{-4}, \\ \frac{df}{dx} &= 6x \cdot x^{-4} + -4x^{-5} \cdot 3x^2 = 6x^{-3} - 12x^{-3} = -6x^{-3} \end{aligned} \quad (2.53)$$

2.7.6 Quotient rule

$$f(x) = \frac{g(x)}{h(x)}, \frac{df}{dx} = \frac{\frac{dg}{dx} h(x) - \frac{dh}{dx} g(x)}{h^2(x)}, f, g, h, x \in \mathbb{R}^1 \quad (2.54)$$

Example

$$\begin{aligned} f(x) &= \frac{3x^2}{x^{-4}}, g(x) = 3x^2, h(x) = x^{-4}, \\ \frac{df}{dx} &= \frac{6x \cdot x^{-4} - -4x^{-5} \cdot 3x^2}{(x^{-4})^2} = \frac{6x^{-3} + 12x^{-3}}{x^{-8}} = \frac{18x^{-3}}{x^{-8}} = 18x^5 \end{aligned} \quad (2.55)$$

Exercise 2.2 (advanced)

Derive the quotient rule from the product and power rule.

2.7.7 Chain rule

$$f(x) = g(h(x)), \frac{df}{dx} = \frac{dg}{dh} \frac{dh}{dx}, f, g, h, x \in \mathbb{R}^1 \quad (2.56)$$

Example

$$\begin{aligned} f(x) &= \sin^3(x), g(x) = h^3(x), h(x) = \sin(x), \frac{df}{dx} = 3h^2(x) \cdot \cos(x) = \\ &= 3\sin^2(x) \cdot \cos(x) \end{aligned} \quad (2.57)$$

2.7.8 Combining rules in the right order

Example

$$f_1(x) = \frac{\sin^4(x^2)}{\cos(x) \cdot x^{-3}} \quad (2.58)$$

First: quotient rule.

$$f_1(x) = \frac{g_1(x)}{h_1(x)} = \frac{\sin^4(x^2)}{\cos(x) \cdot x^{-3}} \Rightarrow g_1(x) = \sin^4(x^2), h_1(x) = \cos(x) \cdot x^{-3} \quad (2.59)$$

$$\frac{df_1}{dx} = \frac{\frac{dg_1}{dx} h_1 - \frac{dh_1}{dx} g_1}{h_1^2} \quad (2.60)$$

To compute $\frac{dg_1}{dx}$, use the chain rule applied to $g_1(x) = f_2(h_2(x))$, $f_2 = h_2^4$, $h_2(x) = \sin(x^2)$,

$$\frac{dg_1}{dx} = \frac{df_2}{dh_2} \frac{dh_2}{dx} = 3h_2^3 \cdot \underbrace{\cos(x^2) \cdot 2x}_{dh_2/dx \text{ using chain rule}} = 3\sin^3(x^2) \cdot \cos(x^2) \cdot 2x \quad (2.61)$$

To compute $\frac{dh_1}{dx}$, use the product rule applied to $h_1(x) = f_3(x) \cdot h_3(x)$, $f_3 = \cos(x)$

$$h_3(x) = x^{-3},$$

$$\frac{dh_1}{dx} = \frac{df_3}{dx} g_3(x) + \frac{dg_3}{dx} f_3(x) = -\sin(x) x^{-3} + -3x^{-4} \cdot \cos(x) \quad (2.62)$$

Using expressions (2.61), (2.62) we obtain from (2.60),

$$\begin{aligned} \frac{df_1}{dx} &= \frac{3\sin^3(x^2) \cos(x^2) 2x \cos(x) x^{-3} - (-\sin(x) x^{-3} - 3x^{-4} \cos(x)) \sin^4(x^2)}{(\cos(x) x^{-3})^2} = \\ &= \frac{6\sin^3(x^2) \cos(x^2) \cos(x) x^{-2} + (\sin(x) x^{-3} + 3x^{-4} \cos(x)) \sin^4(x^2)}{\cos^2(x) x^{-6}} = \dots \end{aligned} \quad (2.63)$$

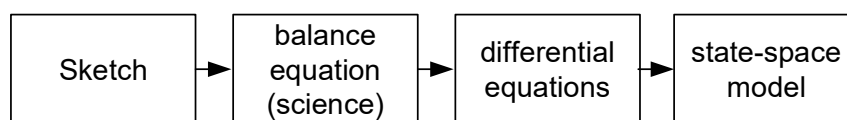
3 From a sketch and science towards a mathematical model of a dynamic system

3.1 Introduction

In chapter 1 we explained roughly what a system is and what a mathematical model of a system is. Looking at the title of this chapter there is one question left. What is a dynamic system? It is a system with "memory" meaning that its future behaviour depends on the current content of its "memory". For instance, the current position and velocity of a car determine the future position and velocity of the car, so position and velocity are the "memory" of the car. The "memory" may also be called the "state". The current state determines future states. On the other hand, the current "state" ("memory" content) depends on past states. So, one could say that the current "state" ("memory" content) is what separates the future from the past. From a mathematical modelling point of view a dynamic system is a system described by differential equations. Actually, almost any phenomenon that occurs in nature is dynamical; it has a "memory" (a "state") and is mathematically described by differential equations. Therefore, building mathematical models of dynamic systems has a large potential.

After reading this chapter the student:

- Is able to make simple drawings of simple systems.
- Is able to derive differential equations from these drawings and first principles, like physical, chemical or mechanical laws.
- Is able to derive differential equations from verbal system descriptions and first principles, like physical, chemical or mechanical laws.
- Is able to put the differential equations in state-space form.



3.2 A simple example: the temperature in a greenhouse

Suppose you are an engineer who is asked to select a proper heating system for a greenhouse. In addition, the greenhouse climate control system manufacturer asks you to deliver to him a computer program to predict the temperature in the greenhouse when the heating system is turned on. What do you do?

You build a mathematical model of the greenhouse. The greenhouse is a dynamic system the greenhouse temperature being the memory. The future greenhouse temperature depends on the current greenhouse temperature (the current memory content) which in turn depends on the past greenhouse temperature.

To build a mathematical model it is usually very helpful to start by making a simple drawing of the system, see Figure 3-1.

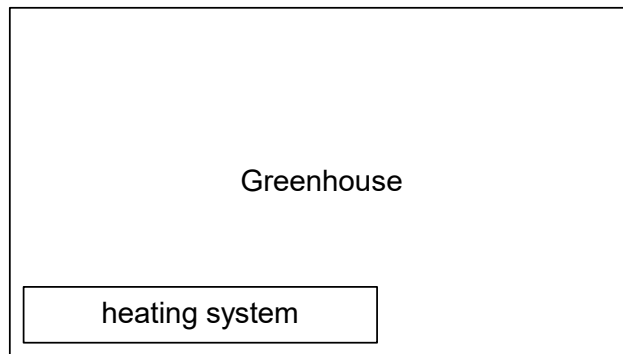


Figure 3-1 A simple drawing of the system.

The next step towards building a mathematical model is by putting relevant phenomena, associated quantities and corresponding mathematical symbols in the drawing.

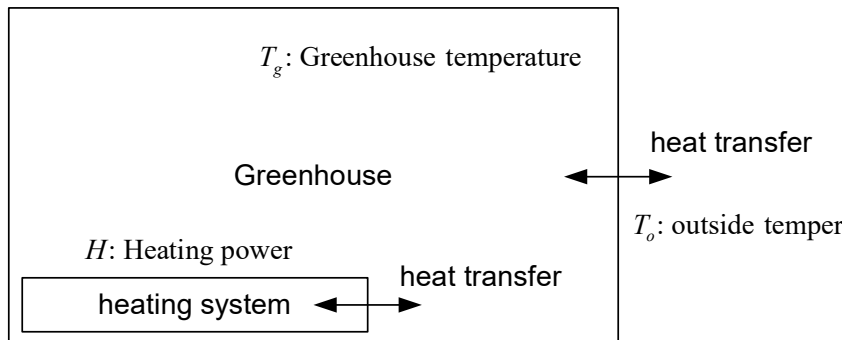


Figure 3-2 Adding relevant processes, quantities and mathematical symbols.

Next applying scientific knowledge (first principles), in this case physical laws describing the heat transport, leads to the following differential equation,

$$\frac{dT_g}{dt} = c_1(T_o - T_g) + c_2H \quad (3.1)$$

In equation (3.1) c_1, c_2 are two constants that depend on several physical properties of the system. The constant c_1 depends on the volume of the greenhouse, the heat transfer

coefficient and surface of the walls of the greenhouse and the heat capacity of the air inside the greenhouse. In the next section we will be more explicit about how to obtain exactly, from physical laws, these constants.

Equation (3.1) is a simple mathematical model of the greenhouse. To interpret the model the meaning and units of all the quantities appearing in the model must be specified. Without this specification the mathematical model is incomplete. The names and units are also determined by scientific knowledge (first principles) and the units, which can be selected in different ways, should be consistent. In the case of equation (3.1) the following names and units apply,

T_g	Greenhouse temperature	$[^{\circ}C]$
T_o	Outside temperature	$[^{\circ}C]$
$\frac{dT_g}{dt}$	Greenhouse temperature change	$[^{\circ}Cs^{-1}]$
H	Heating power	$[W = Js^{-1}]$
c_1	Constant related to heat transfer	$[s^{-1}]$
c_2	Constant related to heating	$[^{\circ}CJ^{-1}]$

The consistency of the units is checked as follows,

$$\begin{aligned} \frac{dT_g}{dt} &= c_1(T_o - T_g) + c_2 H \\ [^{\circ}Cs^{-1}] &= [s^{-1}][^{\circ}C] + [^{\circ}CJ^{-1}][Js^{-1}] \end{aligned} \quad (3.2)$$

Having the mathematical model (3.1) with its associated description and units we can start to address the problem of selecting an appropriate heating system for the greenhouse and the computation of the greenhouse temperature.

To select the heating system the most important thing is the maximum heating power. To compute it we can use the mathematical model (3.1) with its associated description and units. However, we also need to know the worst case. The required heating power is maximum when the outside temperature T_o is the lowest and the greenhouse temperature T_g is the highest. Suppose the lowest outside temperature we want to accommodate for is

$-20^\circ C$ and that in this case we want the greenhouse temperature not to go below $+5^\circ C$. Then from the mathematical model (3.1) with its associated description and units we can compute the maximum heating power required. To do this we must consider the situation where the outside temperature and the greenhouse temperature remain constant. This implies,

$$\frac{dT_g}{dt} = 0, T_g = +5^\circ C, T_o = -20^\circ C \Rightarrow H = \frac{c_1}{c_2} 25 W \quad (3.3)$$

Of course there are usually other considerations, such as the price, to select the heating system. We will not consider these because our primary objective here is to show the importance of the mathematical model in making the selection.

The second issue, computing the temperature when we know the heating power, also requires the mathematical model (3.1). Our system, described by equation (3.1), is a dynamical system because it is described by a differential equation. Remember that we stated that a dynamical system has a memory and to compute its future behaviour we need to know the current contents of its memory. The memory of our dynamic system is the greenhouse temperature T_g . Say we start the computation at time t_0 then we need to know $T_g(t_0)$. Also, we need to know the heating power $H(t), t \geq t_0$ where t denotes time. To compute the greenhouse temperature $T_g(t), t \geq t_0$ we furthermore need to know the outside temperature $T_o(t), t \geq t_0$. To see how we compute the greenhouse temperature $T_g(t), t \geq t_0$ from the mathematical model we need to consider a Taylor series expansion of $T_g(t)$,

$$T_g(t) = T_g(t_0) + \frac{T'_g(t_0)}{1!}(t-t_0) + \frac{T''_g(t_0)}{2!}(t-t_0)^2 + \dots \quad (3.4)$$

Observe that $T'_g(t_0) = \left. \frac{dT_g}{dt} \right|_{t_0}$ and using the mathematical model (3.1),

$$\left. \frac{dT_g}{dt} \right|_{t_0} = c_1(T_o(t_0) - T_g(t_0)) + c_2 H(t_0) \quad (3.5)$$

The right hand side of equation (3.5) is entirely known and therefore $T'_g(t_0) = \left. \frac{dT_g}{dt} \right|_{t_0}$ in equation (3.4) may be computed. Now if $t-t_0$ in equation (3.4) is small the higher order

terms $\frac{T_g''(t_0)}{2!}(t-t_0)^2 + \dots$ may be neglected. As we have seen in section 2.4, this comes down to approximating the function locally by its tangent line, see Figure 2-1. Using this tangent line approximation, we may compute $T_g(t)$ for values t in a neighbourhood of t_0 by,

$$T_g(t) \approx T_g(t_0) + \left. \frac{dT_g}{dt} \right|_{t_0} (t-t_0) \text{ when } (t-t_0) \text{ small} \quad (3.6)$$

Equation (3.6) is graphically represented by Figure 3-3.

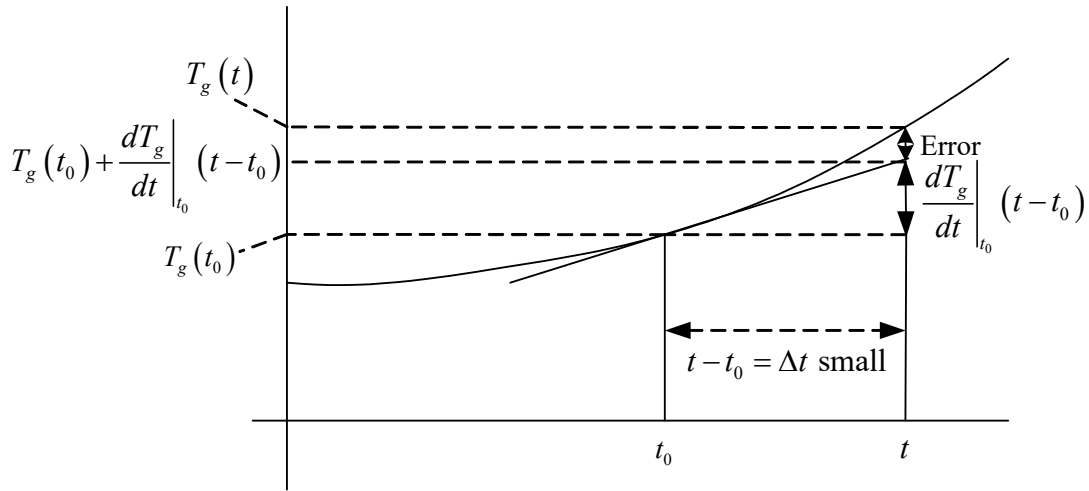


Figure 3-3 Tangent line approximation $T_g(t)$ by $T_g(t_0) + \left. \frac{dT_g}{dt} \right|_{t_0} (t-t_0)$.

In Figure 3-3 we have introduced,

$$\Delta t = t - t_0 \quad (3.7)$$

that represents the step forward in time we made using the tangent line approximation. For this approximation to be accurate the time step Δt should be sufficiently small. Using equation (3.7) rewrite tangent line approximation (3.6) as,

$$T_g(t_0 + \Delta t) \approx T_g(t_0) + \left. \frac{dT_g}{dt} \right|_{t_0} \Delta t \text{ when } \Delta t \text{ small} \quad (3.8)$$

Using (3.8) and the next tangent line approximation at $t_0 + \Delta t$ we can compute,

$$T_g(t_0 + 2\Delta t) \approx T_g(t_0 + \Delta t) + \left. \frac{dT_g}{dt} \right|_{t_0 + \Delta t} \Delta t, \Delta t \text{ small} \quad (3.9)$$

Verify that indeed we can compute the right-hand side of equation (3.9) using the mathematical model (3.1). Then it is easy to see that we can compute $T_g(t_0 + k\Delta t)$, $k = 1, 2, 3, \dots$ recursively as follows,

$$T_g(t_0 + k\Delta t) \approx T_g(t_0 + (k-1)\Delta t) + \left. \frac{dT_g}{dt} \right|_{t_0 + (k-1)\Delta t} \Delta t, \quad k = 1, 2, 3, \dots \quad (3.10)$$

There is a trade off in the selection of Δt . Taking it smaller increases the accuracy of the computation, see Figure 3-3, but at the expense of computation time because more computational steps are required over a fixed time interval $t_0 \leq t \leq t_f$. Making Δt smaller increases the accuracy up to a certain point where rounding errors of computer computations start to dominate the computation. Then the accuracy decreases again.

The recursive procedure to compute $T_g(t)$, $t = t_0 + k\Delta t$, $k = 1, 2, 3, \dots$ is called numerical integration. The reason is that to obtain $T_g(t)$, $t \geq t_0$ which is the solution of the differential equation (3.1) we have to integrate this differential equation. The solution consists of numerical values $T_g(t_0 + k\Delta t)$ and is called a numerical solution of the differential equation. On the other hand, if we can find an analytical function description of $T_g(t)$, $t \geq t_0$, for example,

$$T_g(t) = c_1 \sin(t - t_0) T_o(t) + c_2 (H(t) - H(t_0)) + T_g(t_0) \quad (3.11)$$

This is called an analytical solution of the differential equation. Note: equation (3.11) is meant as an example; it is not actually the solution of the differential equation (3.1). Analytical solutions are to be preferred because they can be evaluated easier and quicker, and they provide more insight. In equation (3.11), if we change $T_g(t_0)$ or $H(t)$, we can see immediately how this affects the solution. Using numerical integration, we can only recompute the solution for different values $T_g(t_0)$, but the result and changes are not a-priori obvious and have to be recomputed for every value $T_g(t_0)$. Unfortunately for most differential equations analytical solutions cannot be obtained. Luckily for many applications, such as choosing an appropriate heating system and computing the greenhouse temperature, numerical solutions, which can always be obtained, suffice!

More advanced numerical integration procedures (ode23, ode45 in Matlab) are commonly used in practice. They approximate in some way part of the higher order terms in the Taylor series expansion which we have neglected. As a result of this the time steps Δt can be chosen larger and the accuracy and computational efficiency increases. The numerical integration method presented in this section is called explicit Euler numerical integration. It is well known for its simplicity and numerical stability.

3.3 Obtaining mathematical models from first principles systematically

The way in which we obtained the mathematical model (3.1) of the greenhouse from first principles, was not entirely transparent or systematic. In this section we will derive this model in a systematic way and furthermore consider a second example.

A systematic way to obtain a mathematical model from first principles is by means of a so-called balance equation. The balance equation is represented below, followed by a simplified version,

$$\begin{bmatrix} \text{Rate of mass} \\ \text{or energy} \\ \text{or momentum} \\ \text{or electrical charge} \\ \text{accumulation in the system} \end{bmatrix} = \begin{bmatrix} \text{Rate of mass} \\ \text{or energy} \\ \text{or momentum} \\ \text{or electrical charge} \\ \text{entering the system} \end{bmatrix} - \begin{bmatrix} \text{Rate of mass} \\ \text{or energy} \\ \text{or momentum} \\ \text{or electrical charge} \\ \text{leaving the system} \end{bmatrix} \quad (3.12)$$

$$\begin{bmatrix} \text{Rate of} \\ \text{change} \end{bmatrix} = \begin{bmatrix} \text{Rate of} \\ \text{input} \end{bmatrix} - \begin{bmatrix} \text{Rate of} \\ \text{output} \end{bmatrix} \quad (3.13)$$

The balance equation represents fundamental general scientific principles from which we can derive laws and formulas that apply to specific systems, as we shall see. It is important to note that equations (3.12), (3.13) are formulated in terms of rates. Rates are changes per unit of time.

Example 3.1: The greenhouse

For the greenhouse equation (3.12) applies where the energy is heat energy. So we formulate our mathematical model in terms of heat energy rates. From first principles the heat energy in the greenhouse equals,

$$\rho c_p V T_g \quad (3.14)$$

where,

$$\rho \quad \text{greenhouse air density} \quad [kgm^{-3}]$$

c_p	greenhouse air heat capacity	$\left[J (kg \text{ } ^\circ C)^{-1} \right] = \left[J kg^{-1} \text{ } ^\circ C^{-1} \right]$
V	greenhouse volume	$\left[m^3 \right]$
T_g	greenhouse air temperature	$\left[^\circ C \right]$

Therefore,

$$\left[\begin{array}{c} \text{Rate of heat energy} \\ \text{accumulation in the greenhouse} \end{array} \right] = \frac{d\rho c_p V T_g}{dt} \quad (3.15)$$

Furthermore,

$$\left[\begin{array}{c} \text{Rate of heat energy input} \\ \text{in the greenhouse} \end{array} \right] = H \quad (3.16)$$

$$\left[\begin{array}{c} \text{Rate of heat energy} \\ \text{leaving the greenhouse} \end{array} \right] = h_w s_w (T_g - T_o) \quad (3.17)$$

where,

H	Effective heating power	$\left[W = Js^{-1} \right]$
h_w	heat transfer coefficient of the walls	$\left[J (^\circ C m^2 s)^{-1} \right] = \left[J ^\circ C^{-1} m^{-2} s^{-1} \right]$
s_w	total surface of the walls	$\left[m^2 \right]$
T_o	outside temperature	$\left[^\circ C \right]$

From equations (3.15)-(3.17), after dividing each equation by $\rho c_p V$ which is a constant number, we obtain the following mathematical model including the associated unit check.

$$\frac{dT_g}{dt} = \frac{H}{\rho c_p V} - \frac{h_w s_w (T_g - T_o)}{\rho c_p V} \quad (3.18)$$

$$\left[^\circ C s^{-1} \right] = \left[Js^{-1} \right] \left[J^{-1} \text{ } ^\circ C \right] - \left[J ^\circ C^{-1} m^{-2} s^{-1} \right] \left[m^2 \right] \left[^\circ C \right] \left[J^{-1} \text{ } ^\circ C \right]$$

In equation (3.18) the unit of $\frac{1}{\rho c_p V}$ is written directly as $\left[J^{-1} \text{ } ^\circ C \right]$. This unit is easily

obtained from the units of ρ, c_p, V mentioned previously. Comparing the mathematical models (3.18) and (3.2) we see that indeed they are identical and,

$$c_1 = \frac{h_w s_w}{\rho c_p V}, c_2 = \frac{1}{\rho c_p V} \quad (3.19)$$

The model (3.18) which is derived in a systematic manner, is to be preferred over (3.2) because it is built entirely out of known physical quantities. Observe that we require (3.18) to obtain (3.19), which is needed to specify the constants in the model (3.2).

Example 3.2: Newton's law

Consider a mass with a force acting on it. The force may vary but is always pointing in the same direction. Our aim is to determine the motion of the mass. Again, to build a mathematical model we start to make a drawing of the system including relevant processes, quantities and mathematical symbols.

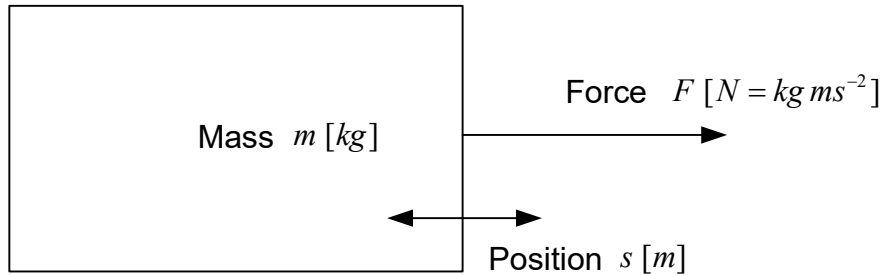


Figure 3-4 The system; a mass with a force acting on it.

Is this a dynamic system? Yes, because the future position depends on the current position and so the position is (part?) of the memory of the system. The systematic way to obtain a mathematical model of this system is to use a balance equation and first principles. But actually, you should be familiar with the mathematical model of this system. It is Newton's law,

$$F = m a \quad (3.20)$$

$$[N = kg\ ms^{-2}] = [kg][ms^{-2}]$$

Although we have a dynamic system the mathematical model (3.20) doesn't look like a differential equation. However, it actually is a differential equation. To see this observe the relationship between the position s in Figure 3-4 and a in equation (3.20).

$$a = \frac{d^2 s}{dt^2} \quad (3.21)$$

Using (3.21) equation (3.20) becomes,

$$\begin{aligned}\frac{d^2s}{dt^2} &= \frac{F}{m} \\ [ms^{-2}] &= [kg\ ms^{-2}][kg^{-1}]\end{aligned}\quad (3.22)$$

Next let us derive the mathematical model (3.22) from the balance equation (3.12) and first principles. The energy in equation (3.12) in our system is kinetic energy associated to the motion of the mass. Mind you the derivation is a bit complicated. That's one of the reasons why Newton became famous! Consider the labour A associated to the movement of a constant force F over a distance s ,

$$\begin{aligned}A &= F\ s \\ [kgm^2s^{-2}] &= [kgms^{-2}][m]\end{aligned}\quad (3.23)$$

The labour A in our system is used to increase the kinetic energy E_k of the mass,

$$\Delta E_k = A = Fs \quad (3.24)$$

However, in our system the force is not necessarily constant. Let δs denote an increment (that is a very small change) of the position s . During this increment the force is (approximately) constant and the associated kinetic energy increment δE_k , according to equation (3.24), equals,

$$\delta E_k = F\delta s \quad (3.25)$$

Let δt denote the time increment during which the increments δs and δE_k occur. Then from equation (3.25) we obtain,

$$\frac{\delta E_k}{\delta t} = F \frac{\delta s}{\delta t} \quad (3.26)$$

and in the limit $\delta t \rightarrow 0$ this becomes,

$$[\text{Rate of kinetic energy change}] = \frac{dE_k}{dt} = F \frac{ds}{dt} = Fv = [\text{Kinetic energy input rate}] \quad (3.27)$$

where $v = \frac{ds}{dt}$ is the velocity.

On the other hand, from first principles we know,

$$E_k = \frac{1}{2}mv^2 \quad (3.28)$$

Differentiating (3.28) with respect to time by applying the chain rule gives,

$$[\text{Rate of kinetic energy change}] = \frac{dE_k}{dt} = \frac{dE_k}{dv} \frac{dv}{dt} = mv \frac{dv}{dt} \quad (3.29)$$

Combining equations (3.27) and (3.29) we obtain,

$$[\text{Rate kin. en. change}] = mv \frac{dv}{dt} = Fv = [\text{Kin. en. input rate}] \Rightarrow F = m \frac{dv}{dt} = ma \quad (3.30)$$

which is Newton's law (3.20) from which we obtain our mathematical model (3.22) again.

3.4 Representing models of dynamic systems in the state-space form

Example 3.3: Newton's law

Reconsider the system represented by Figure 3-4 and its mathematical model (3.22),

$$\begin{aligned} \frac{d^2 s}{dt^2} &= \frac{F}{m} \\ [ms^{-2}] &= [kg \, ms^{-2}] [kg^{-1}] \end{aligned} \quad (3.22)$$

Now suppose we want to compute the position $s(t)$, $t \geq t_0$ from our mathematical model.

What do we need to know? Of course, we need to know "where we start" that is the initial position $s(t_0)$. Clearly, we also need to know $F(t)$, $t \geq t_0$ the force acting on the mass. Is this enough? No. Hopefully you remember from your physics lessons that to compute the motion of a mass, knowing all the forces acting on it, you need to know both the initial position $s(t_0)$ and the initial velocity $s'(t_0) = \left. \frac{ds}{dt} \right|_{t=t_0}$. What you also learned is an analytical

solution of the differential equation (3.22) which holds if the force is constant: $F(t) = F$. Then

$a = \frac{F}{m}$ is also constant and,

$$s(t) = s(t_0) + s'(t_0)(t - t_0) + \frac{1}{2} a (t - t_0)^2, \quad t \geq t_0 \quad (3.31)$$

If the force is not constant, we cannot find an analytical solution and we can only calculate numerical solutions. In section 3.2 we learned how to numerically integrate the differential equation (3.2). This differential equation is a so called first-order differential equation because the first derivative is the highest one appearing in this differential equation. This first derivative could be used to approximate locally the function we want to calculate by a straight

line, see Figure 2-1. This local approximation by a straight line forms the basis for numerical integration. However, now we have the second-order differential equation (3.31) from which we cannot obtain first derivatives to approximate the function $s(t)$. To resolve this problem, we will write higher-order differential equations as a set of first-order differential equations. This, we shall see, is always possible. After we have done this, we can numerically integrate all the first-order differential equations in the same manner as equation (3.2) in section 3.2.

The procedure to obtain the first-order differential equations is as follows. This procedure also applies to sets of coupled higher order differential equations.

- Introduce so called state variables. A state variable is a variable of which derivatives appear in the differential equation(s). In the case of (3.22) s is a state variable. However, all time-derivatives of s up to the highest one appearing in the differential equation(s) are also state variables. So in the case of (3.22) $s' = \frac{ds}{dt}$ is also a state variable because the highest derivative appearing in the differential equation(s) is the second derivative $s'' = \frac{d^2s}{dt^2}$. State variables we will denote by $x_1, x_2, \dots$ so in the case of (3.22).

$$x_1 = s, x_2 = \frac{ds}{dt} \quad (3.32)$$

- Using the state variables write each original n^{th} -order differential equation as a set of n first-order differential equations with the first derivative of each equation appearing solemnly and explicitly on the left hand side. In the case of this example, these two differential equations are,

$$\frac{dx_1}{dt} = x_2 \quad (3.33)$$

$$\frac{dx_2}{dt} = \frac{F}{m} \quad (3.34)$$

From equations (3.33), (3.34) it is immediately obvious now that to compute $x_1(t), x_2(t), t \geq t_0$ we require both $x_1(t_0)$ and $x_2(t_0)$ the initial value of both position and speed. State variables of dynamic systems models constitute precisely the "memory" or "state" of the dynamic system. Furthermore, the system behaviour equals the memory content as a function of time (the state response), that is $x_1(t), x_2(t), t \geq t_0$ for the systems

model (3.33), (3.34). Apart from state variables, systems models written in state-space form contain input variables (inputs) denoted by $u_1, u_2, \dots$ and parameters denoted by $p_1, p_2, \dots$

Parameters in a systems model are all the constants in the model. For the systems model (3.22),

$$p_1 = m \quad (3.35)$$

Input variables are the variables that are left over. For the systems model (3.22) we obtain,

$$u_1 = F \quad (3.36)$$

The state-space form of the systems model (3.22) equals,

$$\frac{dx_1}{dt} = x_2 \quad (3.37)$$

$$\frac{dx_2}{dt} = \frac{u_1}{p_1} \quad (3.38)$$

Associated with a model represented in state-space form is a so-called state-space mapping. This mapping is given by equations (3.32), (3.35), (3.36) which map the variables in the original differential equation(s) onto:

- the state variables (3.32)
- the parameters (3.35)
- the inputs (3.36)

State variables (x), parameters (p) and inputs (u) as well as the state-space representation are generally used by systems and control scientists/engineers. The original variable names (e.g. s, F, m) are generally used by people working in the field of application. To communicate among them the state-space mapping (3.32), (3.35), (3.36) is very important. Therefore, the state-space mapping should always be clearly stated including the meaning and units of the variables, as is done in Table 3-1.

Table 3-1: State-space mapping

Notation			
State-space	Field of application	Meaning	Unit
x_1	s	position	$[m]$
x_2	ds / dt	velocity	$[ms^{-1}]$

p_1	m	mass	$[kg]$
u_1	F	Force	$[N] = [kgms^{-2}]$

Example 3.4: A set of higher-order differential equations

Consider the following set of coupled higher-order differential equations,

$$\frac{d^3 p}{dt^3} + c_1 \frac{d^2 q}{dt^2} + r + s = 0 \quad (3.39)$$

$$\frac{d^2 q}{dt^2} = c_2 \frac{d^2 p}{dt^2} + q + c_3 s \quad (3.40)$$

where c_1, c_2, c_3 are constants. To write the set of differential equations (3.39), (3.40) in state-space form we first have to recognise, respectively, the state variables, the parameters and the inputs. To determine the state variables, note that the highest derivative of p that appears is $\frac{d^3 p}{dt^3}$ so,

$$x_1 = p, \quad x_2 = \frac{dp}{dt}, \quad x_3 = \frac{d^2 p}{dt^2} \quad (3.41)$$

are state variables. Similarly, since the highest derivative of q that appears is $\frac{d^2 q}{dt^2}$,

$$x_4 = q, \quad x_5 = \frac{dq}{dt} \quad (3.42)$$

are also state variables. To decide about the parameters and inputs in equations (3.39), (3.40) the constants in the model must be known. The constants are c_1, c_2, c_3 . Therefore, the parameters are,

$$p_1 = c_1, \quad p_2 = c_2, \quad p_3 = c_3 \quad (3.43)$$

and the input variables (inputs) are,

$$u_1 = r, \quad u_2 = s \quad (3.44)$$

Knowing the state variables, the parameters and the inputs the systems model (3.39), (3.40) can be written in state-space form,

$$\frac{dx_1}{dt} = x_2 \quad (3.45)$$

$$\frac{dx_2}{dt} = x_3 \quad (3.46)$$

$$\frac{dx_3}{dt} = -p_1(p_2x_3 + x_4 + p_3u_2) - u_1 - u_2 \quad (3.47)$$

$$\frac{dx_4}{dt} = x_5 \quad (3.48)$$

$$\frac{dx_5}{dt} = p_2x_3 + x_4 + p_3u_2 \quad (3.49)$$

Equations (3.45), (3.46), (3.48) are obtained almost trivially. Equation (3.49) is obtained directly from (3.40). Equation (3.47) is obtained from (3.40) using (3.49). To see this observe that we may rewrite equation (3.40),

$$\frac{d^3p}{dt^3} = -c_1 \frac{d^2q}{dt^2} - r - s \quad (3.50)$$

In equation (3.50) the problem is that $\frac{d^2q}{dt^2}$ is not a state variable but the first derivative of the state variable $\frac{dq}{dt} = x_5$. Therefore, to obtain a first-order differential equation with the first derivative appearing solemnly and explicitly on the left-hand side, we need to substitute the right hand side of equation (3.49) for $\frac{d^2q}{dt^2}$ in (3.50). In this way we obtain (3.47).

Exercise 3.1

Which previous equations determine the state-space mapping in example 3.4? Put them in a table, like Table 3-1. Which column of this table cannot be determined in this example? Why?

The procedure outlined in this section to write systems described by differential equations in state-space form can be applied to any set of coupled differential equations except when derivatives of inputs appear in the differential equations. This does not happen when differential equations are properly obtained from first principles. If it does happen the situation can still be handled. This however is beyond the scope of this course. For a thorough mathematical treatment of the representation of systems in state-space form and by differential equations we refer to the book: "Introduction to mathematical systems theory" by Jan Willem Polderman and Jan C. Willems.

3.5 The importance of writing dynamic systems models in state-space form

In the previous section we have actually witnessed several reasons why it is important to write systems in state-space form:

- Only models represented in state-space form are suitable for numerical integration for which standard software is available.
- The systems memory may be identified with the systems state variables.
- The system behaviour is identical to the contents of the systems memory as a function of time which is identical to the time response of the state variables. System behaviour is computed by numerical integration of the state-space model.
- Cause and effect are clearly distinguished in systems models written in state-space form. The cause being the inputs and the initial values of the state variables, the effect being the systems behaviour.
- Inputs, state variables and parameters, which are fundamentally different, are distinguished in the state-space form.

3.6 Writing systems models in state-space form using vector notation: the state equation

To simplify the state-space notation the state variables $x_1, x_2, \dots, x_{n_x}$ are collected in the state vector x ,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_x} \end{bmatrix} \in R^{n_x} \quad (3.51)$$

The total number of state variables is denoted by n_x and is called the order of the systems state-space model / order of the system. Therefore n_x is also the dimension of the state vector x .

Similarly, the inputs are collected in the input vector,

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_u} \end{bmatrix} \in R^{n_u} \quad (3.52)$$

The total number of inputs is denoted by n_u and thus is equal to the dimension of input vector u .

The parameters are collected in the parameter vector,

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_p} \end{bmatrix} \in R^{n_p} \quad (3.53)$$

The total number of parameters is denoted by n_p and thus equals the dimension of parameter vector p .

Using these vectors the general state-space notation of the systems model becomes,

$$\frac{dx}{dt} = f(x, u, p) \quad (3.54)$$

and is called the state equation where,

$$\frac{dx}{dt} \stackrel{def}{=} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_{n_x}}{dt} \end{bmatrix} \in R^{n_x}, f(x, u, p) = \begin{bmatrix} f_1(x, u, p) \\ f_2(x, u, p) \\ \vdots \\ f_{n_x}(x, u, p) \end{bmatrix} \in R^{n_x}, x \in R^{n_x}, u \in R^{n_u}, p \in R^{n_p} \quad (3.55)$$

Because the outcome of $f(x, u, p)$ in (3.55) is a vector, $f(x, u, p)$ is called a vector function. Note that x, u, p, f are all column vectors.

Example 3.5: Newton's law

Consider the state-space model (3.37), (3.38). In vector form, this model is described by:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{u_1}{p_1} \end{bmatrix} = \begin{bmatrix} f_1(x, u, p) \\ f_2(x, u, p) \end{bmatrix} = f(x, u, p)$$

Exercise 3.2

For the state-space systems model (3.45)-(3.49), determine the vector function $f(x, u, p)$.

Determine the system order n_x and the dimensions n_u, n_p .

3.7 Obtaining state-space models from verbal system descriptions

Example 3.6: Biochemical reactor

Consider the following verbal system description. A biochemical reactor contains fluid with two components: biomass and substrate. The biomass increases because it consumes (eats) the substrate. An example is a wastewater treatment plant where the biomass is used to eat waste chemicals which form the substrate.

Consider a perfectly mixed biochemical reactor where the fluid flowing in, which contains variable concentrations of substrate and biomass, can be regulated with a valve. The fluid flowing out of the biochemical reactor can also be regulated with a valve. Furthermore, the consumption of substrate and the associated biomass generation per m^3 in the reactor are both proportional to the concentration of biomass.

To build a mathematical model in state-space form from this description we will follow the procedure described below.

- (1) Using the description above draw a picture (sketch) of the biochemical reactor.
- (2) Put relevant processes, quantities and mathematical symbols in the picture (sketch) to help you build a mathematical model of this system.
- (3) Using balance equations (scientific knowledge) derive differential equations which describe this system.
- (4) Attach units to all the variables appearing in the differential equations.
- (5) Determine the state variables, parameters and inputs of this systems model
- (6) Write the systems model in state-space form

Below the results of these six steps are listed.

(1),(2)

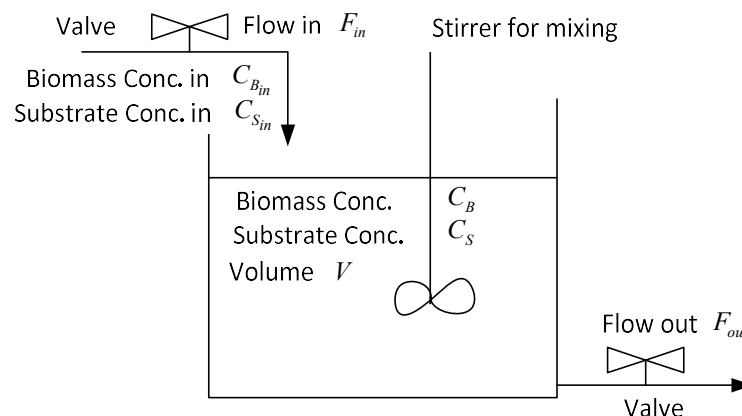


Figure 3-5 Sketch of the biochemical reactor.

- (3) The consumption of substrate r_s per m^3 is proportional to the biomass concentration

$$B : r_s = c_1 C_B$$

The generation of biomass r_B per m^3 is proportional to the biomass concentration B :

$$r_B = c_2 C_B$$

Balance equations:

[rate of change of biomass] =

[generation rate biomass] + [input rate biomass] - [output rate biomass]

$$\frac{d(C_B V)}{dt} = r_s V + F_{in} C_{B_{in}} - F_{out} C_B = c_1 B V + F_{in} C_{B_{in}} - F_{out} C_B \quad (3.56)$$

[rate of change of substrate] =

[generation substrate] + [input substrate] - [output substrate]

$$\frac{d(C_S V)}{dt} = r_B V + F_{in} C_{S_{in}} - F_{out} C_S = c_1 B V + F_{in} C_{S_{in}} - F_{out} C_S \quad (3.57)$$

[rate of change of mass/volume] =

[input rate mass/volume] - [output rate mass/volume]

$$\frac{dV}{dt} = F_{in} - F_{out} \quad (3.58)$$

$$(4) \quad C_B, C_{B_{in}}, C_S, C_{S_{in}} : [\text{kg m}^{-3}], F_{in}, F_{out} : [\text{m}^3 \text{ s}^{-1}], \frac{d(C_B V)}{dt}, \frac{d(C_S V)}{dt} : [\text{kg s}^{-1}], c_1, c_2 : [\text{s}^{-1}]$$

$$(5) \quad \frac{d(C_B V)}{dt} = C_B \frac{dV}{dt} + V \frac{dC_B}{dt} \Rightarrow \frac{dC_B}{dt} = \frac{1}{V} \left(\frac{d(C_B V)}{dt} - C_B \frac{dV}{dt} \right) = c_2 C_B + \frac{F_{in} (C_{B_{in}} - C_B)}{V}$$

$$\frac{d(C_S V)}{dt} = C_S \frac{dV}{dt} + V \frac{dC_S}{dt} \Rightarrow \frac{dC_S}{dt} = \frac{1}{V} \left(\frac{d(C_S V)}{dt} - C_S \frac{dV}{dt} \right) = c_1 C_B + \frac{F_{in} (C_{S_{in}} - C_S)}{V}$$

State variables:

$$x_1 = C_B, x_2 = C_S, x_3 = V \quad (3.59)$$

Parameters:

$$p_1 = c_1, p_2 = c_2 \quad (3.60)$$

Input variables (inputs):

$$u_1 = F_{in}, u_2 = F_{out}, u_3 = C_{B_{in}}, u_4 = C_{S_{in}} \quad (3.61)$$

Then the state-space model becomes,

$$\frac{dx_1}{dt} = p_2 x_1 + \frac{u_1 (u_3 - x_1)}{x_3} \quad (3.62)$$

$$\frac{dx_2}{dt} = p_1 x_1 + \frac{u_1 (u_4 - x_2)}{x_3} \quad (3.63)$$

$$\frac{dx_3}{dt} = u_1 - u_2 \quad (3.64)$$

Exercise 3.3

Which previous equations determine the state-space mapping in example 3.6? Put them in a table, like Table 3-1.

3.8 Control inputs, external inputs, measurements, output variables and the output equation

So far mathematical systems models in state-space form are represented by the state equation,

$$\frac{dx}{dt} = f(x, u, p), \quad x, f \in R^{n_x}, \quad u \in R^{n_u}, \quad p \in R^{n_p} \quad (3.65)$$

The state equation describes the dynamic phenomena responsible for the system behaviour. Knowing the initial state $x(t_0)$ and the inputs $u(t)$, $t \geq t_0$ the system behaviour $x(t)$, $t \geq t_0$ is computed by means of numerical integration. The inputs of the system may be classified into control inputs and external inputs. Control inputs can be manipulated (changed) by the control engineer at each time $t \geq t_0$ and are therefore suitable to control the system. The heat supplied by the heating system in the greenhouse is an example of a control input. Often control inputs have an upper and lower bound, like the heat supplied by the greenhouse heating system,

$$u_i^{\min} \leq u_i \leq u_i^{\max}, \quad i = 1, 2, \dots, n_u \quad (3.66)$$

External inputs on the other hand are fully determined by external conditions and therefore cannot be used for control purposes at all. The outside temperature of the greenhouse is an example of an external input. To distinguish external inputs from control inputs external inputs are denoted by d instead of u . With this distinction the state equation (3.65) becomes,

$$\frac{dx}{dt} = f(x, u, d, p), \quad x, f \in R^{n_x}, \quad u \in R^{n_u}, \quad d \in R^{n_d}, \quad p \in R^{n_p} \quad (3.67)$$

In chapter 1 we explained that for control purposes and to improve the model (system identification) measurements are needed. So far, our mathematical systems model in state-space form (3.67) does not describe measurements. Variables that are measured will be denoted by $y_1, y_2, \dots, y_{n_y}$ and are called outputs of the system. They are collected in the output vector,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_y} \end{bmatrix} \in R^{n_y} \quad (3.68)$$

where n_y denotes the total number of system outputs.

Let us reconsider the mass on which a force is acting, see Figure 3-4 and the associated state-space model (3.37), (3.38). Suppose that we measure the position of the mass. This is described by,

$$y_1 = x_1 \quad (3.69)$$

If we also measure the speed this is described by,

$$y_2 = x_2 \quad (3.70)$$

Using the vector notation (3.68) equations (3.69), (3.70) may be replaced with,

$$y = x \quad (3.71)$$

Equation (3.71) states that all the state variables are measured. For control purposes this is the most favourable situation because we measure (observe) the complete behaviour of the system. If equation (3.71) applies control engineers call this 'having complete state information'. Usually however not all the state variables can be measured. In general measured variables $y_1, y_2, \dots, y_{n_y}$ are functions of the state variables, the inputs and the parameters,

$$y = g(x, u, d, p), \quad y, g \in R^{n_y} \quad (3.72)$$

where $g \in R^{n_y}$ is a vector function,

$$g = \begin{bmatrix} g_1(x, u, d, p) \\ g_2(x, u, d, p) \\ \vdots \\ g_{n_y}(x, u, d, p) \end{bmatrix} \in R^{n_y} \quad (3.73)$$

When analysing systems, we may not just be interested in the system behaviour $x(t), t \geq t_0$ but also in the behaviour of variables that are functions of the state variables, the parameters and possibly the inputs. Such variables are also described by equation (3.72) and are called output variables or outputs of the system. Therefore equation (3.72) is called the output equation of the system. Recall that equation (3.67) is called the state equation of the system. Together the state and output equation (3.67), (3.72) determine the state-space model. This model is fully determined by the two vector functions f, g . Mathematicians therefore sometimes just speak about f and g when they talk about a mathematical systems model in state-space form. If $n_u > 1, n_y > 1$ the system is called a multi-input multi-output or MIMO system. If $n_u = 1, n_y = 1$ the system is called a single-input single-output or SISO system. Similarly, we have SIMO and MISO systems. Also, systems may just be called single-input or multi-input or single-output or multi-output.

As an example of an output variable, consider the kinetic energy associated to the motion of the mass. According to equations (3.33), (3.34), (3.28) the associated output equation (3.72) is,

$$y_1 = \frac{1}{2} p_1 x_2^2 \quad (3.74)$$

If we also consider the impulse of the mass then,

$$y_2 = p_1 x_2 \quad (3.75)$$

becomes another model output.

3.9 State-space models: complete but not unique

The general state-space model of systems including outputs is described by the state equation (3.67) and the output equation (3.72). Representing a systems model in the state-space form (3.67), (3.72) can be done in different ways. To see this observe that the numbering of the state variables, the input variables and the parameters can be performed in different ways. Renumbering is actually a special case of changing the basis used to represent the variables (Mathematics at work Part 1, Chapter 1: Vectors). Changing the

basis used to represent variables is actually not the only way in which different state-space models of the same system can be obtained. The next example demonstrates this.

Example 3.7: Biochemical reactor

Reconsider the bioreactor described in example 3.6 (section 3.7). Assume that the cross-sectional area of the reactor is everywhere constant. Then in the differential equations (3.56), (3.57) we may replace the volume $V \left[m^3 \right]$ by $hA \left[m^3 \right]$ where $h \left[m \right]$ is the height of the liquid in the tank and $A \left[m^2 \right]$ is the constant cross-sectional area. Doing so A becomes a parameter and instead of the state variable V we now obtain the state variable h .

Exercise 3.4

Determine the state-space model when the substitution suggested above is made. Check that the order and interpretation of this state-space model is the same as that of the original state-space model.

Although there are different ways to represent the system in state-space form, each representation is complete, meaning that it completely describes the system behaviour. So, if you and someone else build a state-space model of a system from the same first principles your results may be different. However, the interpretation of all models in terms of the original system is unique.

Exercise 3.5

Suppose we measure the height of the liquid in the tank and the substrate concentration. For each of the two state-space models write the output equation.

3.10 Summary

From verbal or other types of system descriptions we can usually make a drawing of a system. Then after adding the relevant processes, quantities and mathematical symbols, from balance equations and first principles we may derive differential equations including the units of each quantity appearing in these differential equations. These differential equations describe the dynamic behaviour of the system.

Next, we can write the differential equations in state-space form as follows:

- Identify the state variables (states) $x_1, x_2, \dots, x_{n_x}$. State variables are the variables of which time derivatives appear in the differential equations. Also, all time-derivatives of the state

variables are state variables up to (so not including) the highest time-derivative appearing in the differential equations.

- Identify the parameters $p_1, p_2, \dots, p_{n_p}$. Parameters are all the constants appearing in the differential equations.
- Identify the input variables (inputs) $u_1, u_2, \dots, u_{n_u}$. Usually these are the variables that are left over. In exceptional cases derivatives of inputs may appear in the differential equations. To recognise this, we first have to identify the inputs instead of the state variables. Inputs are variables that are not constant and which influence the system, while they are not influenced by the system.
- Identify the output variables (outputs) of the system $y_1, y_2, \dots, y_{n_y}$. Outputs are either measured variables or variables of interest. These variables in general depend on the state variables, the inputs and the parameters.
- Having determined the state variables, parameters, inputs and outputs the state-space form of systems can be written in vector form as follows,

State equation:

$$\frac{dx}{dt} = f(x, u, p), \quad \frac{dx}{dt}, x, f \in R^{n_x}, u \in R^{n_u}, p \in R^{n_p} \quad (3.76)$$

Output equation:

$$y = g(x, u, p), \quad y, g \in R^{n_y} \quad (3.77)$$

- Inputs may be distinguished into control inputs $u_1, u_2, \dots, u_{n_u}$ and external inputs $d_1, d_2, \dots, d_{n_d}$. In this case the state-space form becomes,

State equation:

$$\frac{dx}{dt} = f(x, u, d, p), \quad \frac{dx}{dt}, x, f \in R^{n_x}, u \in R^{n_u}, d \in R^{n_d}, p \in R^{n_p} \quad (3.78)$$

Output equation:

$$y = g(x, u, d, p), \quad y, g \in R^{n_y} \quad (3.79)$$

From a mathematical point of view the system is fully described by the two vector functions f, g . So all the properties of the system are determined by these two vector functions. The order of the system equals the number of state variables n_x . If $n_u = 1, n_y = 1$ the system is called a single-input single-output or SISO system. If $n_u > 1, n_y > 1$ the system is called a multi-input multi-output or MIMO system. Similarly, we have SIMO and MISO systems. Also systems may just be called single-input or multi-input or single-output or multi-output. A

systems model represented in the state-space form (3.76), (3.77) or (3.78), (3.79) is complete but not unique. The interpretation of different state-space models in terms of the original system is unique.

The main reasons to represent systems models in the state-space form (3.76), (3.77) or (3.78), (3.79) are the following:

- Only models represented in state-space form are suitable for numerical integration for which standard software is available.
- The systems memory equals the n_x systems state variables collected in state vector x . The dimension n_x of state vector x is the order of the system.
- System behaviour equals: $x(t), t \geq t_0$ or in other words, the time response of the system state, called state response. System behaviour is computed by numerical integration of the state-space model, see (1).
- Cause and effect are clearly distinguished in systems models written in state-space form. The cause being the inputs $u(t), t \geq t_0$ and the initial states $x(t_0)$; the effect being the system behaviour $x(t), t \geq t_0$, also called state response, and the model output $y(t), t \geq t_0$, also called output response.
- Input variables, state variables, output variables and parameters, which are fundamentally different, are distinguished in the state-space form.

The state-space model is entirely determined by the two vector functions f, g in equations (3.76), (3.77) or (3.78), (3.79). Note that system outputs need not necessarily be specified. In that case the system is entirely determined by the vector function f .

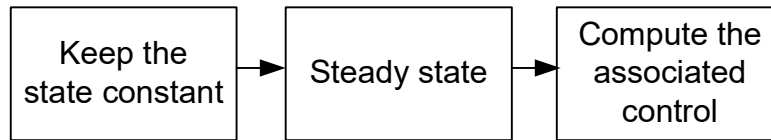
4 A simple control problem: steady states

4.1 Introduction

How can we maintain the concentrations in a chemical reactor? How can we maintain the temperature in a greenhouse? How can we maintain the speed of a car? For many systems the control objective is to keep several or all of the state variables at a fixed value. If all of the state variables remain constant this constant state is called a steady state.

After reading this chapter the student should:

- Recognise that realising a steady state is a control problem.
- Be able to compute the control that realises a steady state from the state equation.



4.2 Computation of steady states

Example 4.1: The greenhouse

Consider the greenhouse systems model (3.18), (3.19). Assume the outside temperature T_o to be constant. Then the associated state-space model is,

$$\begin{aligned} \frac{dx_1}{dt} &= p_1(p_3 - x_1) + p_2 u_1 \\ [{}^o C s^{-1}] &= [s^{-1}] [{}^o C] + [J s^{-1}] [J^{-1} {}^o C] \end{aligned} \quad (4.1)$$

with,

$$x_1 = T_g, \quad u_1 = H, \quad p_1 = \frac{h_w s_w}{\rho c_p V}, \quad p_2 = \frac{1}{\rho c_p V}, \quad p_3 = T_o \quad (4.2)$$

Suppose we want to maintain a fixed greenhouse temperature. Is this possible? Observe that actually this is a simple control problem where we desire,

$$x(t) = x_1(t) = x^{ss} \quad (4.3)$$

by selecting appropriately, the control inputs, where x^{ss} is the desired steady state. Observe that (4.3) implies,

$$\frac{dx_1}{dt} = 0 \quad (4.4)$$

Substitute (4.4) and (4.3) into the state-space model (4.1) to obtain,

$$0 = p_1(p_3 - x^{ss}) + p_2 u_1 \Rightarrow u_1 = \frac{p_1(x^{ss} - p_3)}{p_2} \quad (4.5)$$

So, the control (4.5) realises the steady state (4.3). Actually, the control (4.5) also steers the state from any initial value $x(t_0)$ to the value x^{ss} ultimately. Try to explain this, using equations (4.1), (4.2). The next example is similar but applies to all systems that can be represented in state-space form.

Example 4.2

Consider the state equation,

$$\frac{dx}{dt} = f(x, u, p), \quad x, f \in R^{n_x}, \quad u \in R^{n_u}, \quad p \in R^{n_p} \quad (4.6)$$

Suppose we want,

$$x(t) = x^{ss} \quad (4.7)$$

Again, this is a control problem because we may use the control input $u(t)$ to achieve (4.7).

Like (4.3), equation (4.7) implies,

$$\frac{dx}{dt} = 0 \quad (4.8)$$

Equations (4.6), (4.7), (4.8) together imply,

$$f(x^{ss}, u, p) = 0, \quad x^{ss} \in R^{n_x}, \quad u \in R^{n_u}, \quad p \in R^{n_p} \quad (4.9)$$

So, we should select the control input u to satisfy (4.9). Therefore (4.9) represents a set of n_x non-linear algebraic equation with n_u unknown control inputs. In general, a solution only exists if $n_u = n_x$, meaning that the number of control inputs must be equal to the number of states. In example 4.1 (the greenhouse) both were equal to 1. If $n_u = n_x$ is not satisfied, in general, arbitrary steady states cannot be realised. However, there may still exist some steady states for certain constant control inputs.

4.3 Solving sets of non-linear algebraic equations

From example 4.2 we learned that to realise an arbitrary steady state, we have to solve a set of n_x non-linear algebraic equations in n_x unknowns. In this section we will present and explain a numerical method to solve this problem. The method is called Newton's method.

Because the parameters and the steady state in (4.9) are fixed we will write $f(u)$ instead of $f(x^{ss}, u, p)$. We solve,

$$f(u) = 0, f, u \in R^{n_x} \quad (4.10)$$

The recursive numerical procedure to obtain the solution of (4.10) is described by,

$$u(k+1) = u(k) - \left(f'(u(k))\right)^{-1} f(u(k)), k = 1, 2, \dots \quad (4.11)$$

where,

$$f'(u(k)) = \left. \frac{df}{du} \right|_{u=u(k)} \quad (4.12)$$

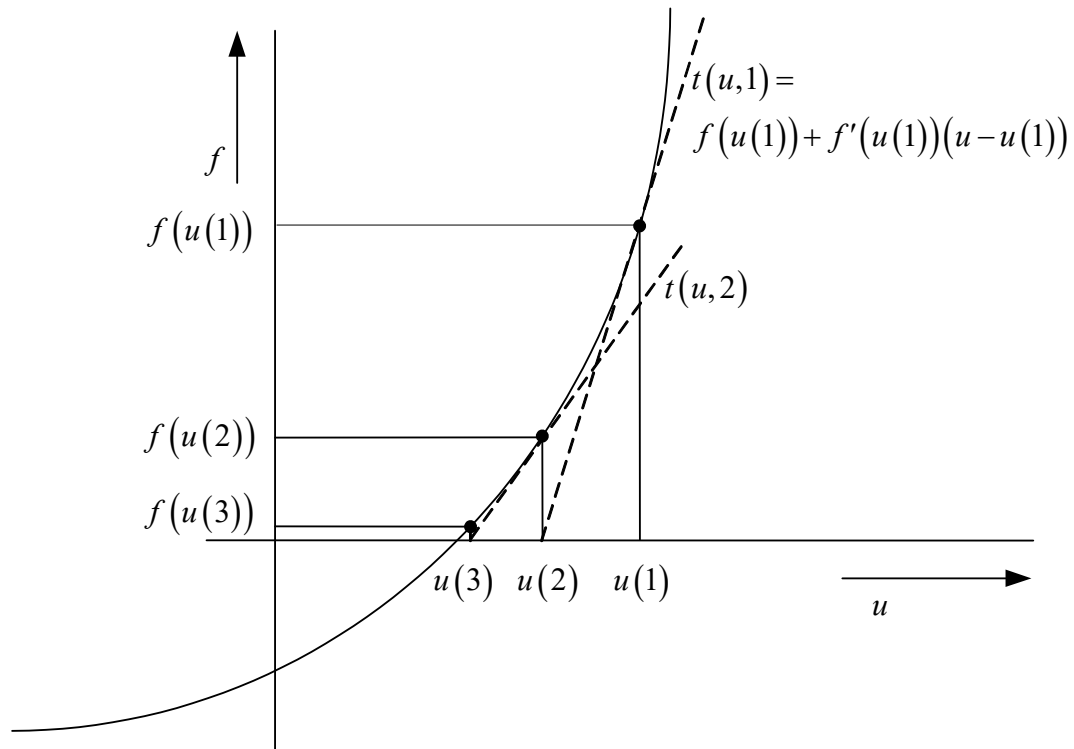


Figure 4-1 Graphical representation of Newton's method.

The recursion (4.11) is started with the value $u(1)$ which should preferably, but not necessarily, be close to the solution. From (4.11) and $u(1)$ we obtain $u(2)$. This computation is graphically represented in Figure 4-1 when we solve one non-linear equation with one unknown.

At $u(k)$ the function is approximated by the tangent line $t(u, k)$ described by,

$$t(u, k) = f(u(k)) + f'(u(k))(u - u(k)) \quad (4.13)$$

Note that (4.13) is identical to (2.11) with $x_0 = u(k)$ and $x = u$. For $k = 1, 2$ these lines (---) are plotted in Figure 4-1. Now $u(k+1)$ is computed to be the zero of the line $t(u, k)$,

$$\begin{aligned} f(u(k)) + f'(u(k))(u - u(k)) &= 0 \Rightarrow \\ f(u(k)) + f'(u(k))u - f'(u(k))u(k) &= 0 \Rightarrow \\ f'(u(k))u &= f'(u(k))u(k) - f(u(k)) \Rightarrow \\ u &= u(k) - (f'(u(k)))^{-1} f(u(k)) \end{aligned} \quad (4.14)$$

The final result of (4.14) is identical to (4.11) when $f, u \in R^1$ (when f and u are both scalars) as they are in Figure 4-1. As you can see from Figure 4-1 the zeros $u(k)$ of the tangent lines $t(u, k)$, $k = 1, 2, \dots$ quickly converge to the zero of the function $f(u)$. Observe that equation (4.11) only requires the computation of $f(u(k))$ and $f'(u(k))$ the value of the function and its first derivative at $u(k)$.

For complicated functions $f(u)$, an analytical description of the derivative function $f'(u)$ may be hard to obtain. Without such an analytical description we can approximate the function value $f'(u(k))$ by,

$$f'(u(k)) \approx \frac{f(u(k) + \Delta u) - f(u(k))}{\Delta u} \quad (4.15)$$

where Δu is an increment of $u(k)$ which should be sufficiently small, but not too small, since then rounding errors destroy the accuracy of the computation. The approximation is based on the definition of the derivative function. This definition is equal to the right-hand

side of equation (4.15) when we take the limit $\Delta u \rightarrow 0$. Computing the derivative according to equation (4.15) is called numerical differentiation by means of computing finite differences.

Now without proof we state that equation (4.11) also applies, and converges quickly, when $f, u \in R^{n_x}$, $n_x \geq 1$. To interpret equation (4.11) in this case we need to introduce the derivative of a vector function of a vector. The derivative of a vector function of a vector $f(u)$, $f \in R^n$, $u \in R^m$, is the following $n \times m$ matrix,

$$f'(u) = \frac{df}{du} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix} \in R^{n \times m} \quad (4.16)$$

This matrix is a generalised derivative and is called the Jacobian. A compact way to write the Jacobian is,

$$[f'(u)]_{i,j} = \left[\frac{df}{du} \right]_{i,j} = \frac{\partial f_i}{\partial u_j} \quad (4.17)$$

where the indices ij refer to matrix element i, j . Because in equations (4.10), (4.11) both f and u have dimension n_x the Jacobian is a square matrix (a square matrix has as many rows as columns and has an inverse). Therefore $(f'(u(k)))^{-1}$ in (4.11) is well defined; it is the inverse of the square Jacobian matrix $f'(u(k))$.

Exercise 4.1

Consider the following state-space model of a system,

$$\frac{dx_1}{dt} = 2u_1^2 - u_2 + x_1, \quad \frac{dx_2}{dt} = -u_1 - u_2^2 + x_2 \quad (4.18)$$

Compute the control u that realises the steady state $x^{ss} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$. To do this, determine the set of two non-linear algebraic equations that have to be solved. To solve these equations,

use the recursive numerical procedure (4.11). Start with $u(1) = \begin{bmatrix} 2 \\ 1.7 \end{bmatrix}$ and compute $u(2), u(3)$. If you make no mistakes $u(3)$ is very close to the solution $\begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}$.

4.4 Summary

- Steady states have the property $f(x, u) = 0$, $f, x \in R^{n_x}$, $u \in R^{n_u}$. To keep a system at an arbitrary desired steady state $x(t) = x^{ss}$, we must solve u^{ss} such that $f(x^{ss}, u^{ss}) = 0$ which constitutes n_x equations in n_u unknowns. In general, a solution only exists if $n_u = n_x$. If this is the case, we can use Newton's method to find a solution.
- Newton's method to solve $f(u) = 0$, $f, u \in R^{n_x}$ is based on the following iteration:

$$u(k+1) = u(k) - \left(f'(u(k)) \right)^{-1} f(u(k)), \quad k = 1, 2, \dots$$
The iteration $u(k)$, $k = 1, 2, \dots$ converges to a solution. Several solutions may exist. The closer $u(1)$ is to the solution, the quicker the iteration converges.
- $f'(u(k)) = \frac{\partial f}{\partial u} \Big|_{u=u(k)} \in R^{n_x \times n_x}$, $\left[\frac{\partial f}{\partial u} \right]_{i,j} = \frac{\partial f_i}{\partial u_j}$.

5 System properties and control

5.1 Introduction

Chemical reactors sometimes explode. Ill understood motorbike accidents are investigated. The steer of a motorbike always shakes a little. Sometimes however – for no apparent reason – this shaking increases, keeps increasing and cannot be stopped.

Dynamic systems may exhibit unexpected behaviours. Roughly speaking unexpected behaviour means that if you change the inputs or the initial state gradually, the behaviour does not change gradually. A chain reaction is another example of unexpected behaviour. Linear systems do not exhibit unexpected behaviour, non-linear systems do. In terms of their behaviour non-linear systems are therefore more interesting (and sometimes dangerous). Almost any biological system is non-linear. Because of their unexpected behaviour, in general, non-linear systems are more difficult to analyse and control than linear systems. This explains why control engineers and people performing system analysis prefer to work with linear systems models. Under certain conditions, which we will explore in this chapter, a linear systems model is an acceptable approximation of a non-linear systems model. For this reason, linearity is probably the most important system property. This property will be introduced and investigated in several ways in this chapter.

Often, we want to realise a steady state. For example, in a chemical reactor one wants to bring the concentrations, which are the state variables, at a certain level and then keep them at that level,

$$x(t) = x^{ss}, t \geq t_1 > t_0, x, x^{ss} \in R^{n_x} \quad (5.1)$$

In equation (5.1), x^{ss} represents the desired steady state and t_1 the time this state should be reached. An important question that arises here is whether small deviations (perturbations) of the state from the steady state, due to internal or external disturbances, tend to zero after a while, or whether they increase. In the latter case our chemical reactor might explode! If the small perturbations tend to zero after a while the steady state x^{ss} is called a stable steady state of the system. If the steady state is unstable, we may use the control inputs to control the state deviations to zero. If, using the control inputs, it is possible to control the state perturbations to zero, the steady state x^{ss} is called stabilisable. Clearly exploding chemical reactors and the motorbike accidents are due to instability of some kind. In this chapter to determine whether or not a steady state is stable or stabilisable we will analyse a linear

systems model. This linear systems model is obtained by linearizing the original, possibly non-linear systems model, around the steady state x^{ss} .

From what we have just mentioned it should be clear that linearity, non-linearity, stability and stabilisability are important system properties. Furthermore, the linearization of systems models turns out to be important to analyse the system. The most important system properties are introduced in this chapter. To simplify the notation in this chapter we will not distinguish external from control inputs. In this section all inputs are assumed to be collected in the input vector u .

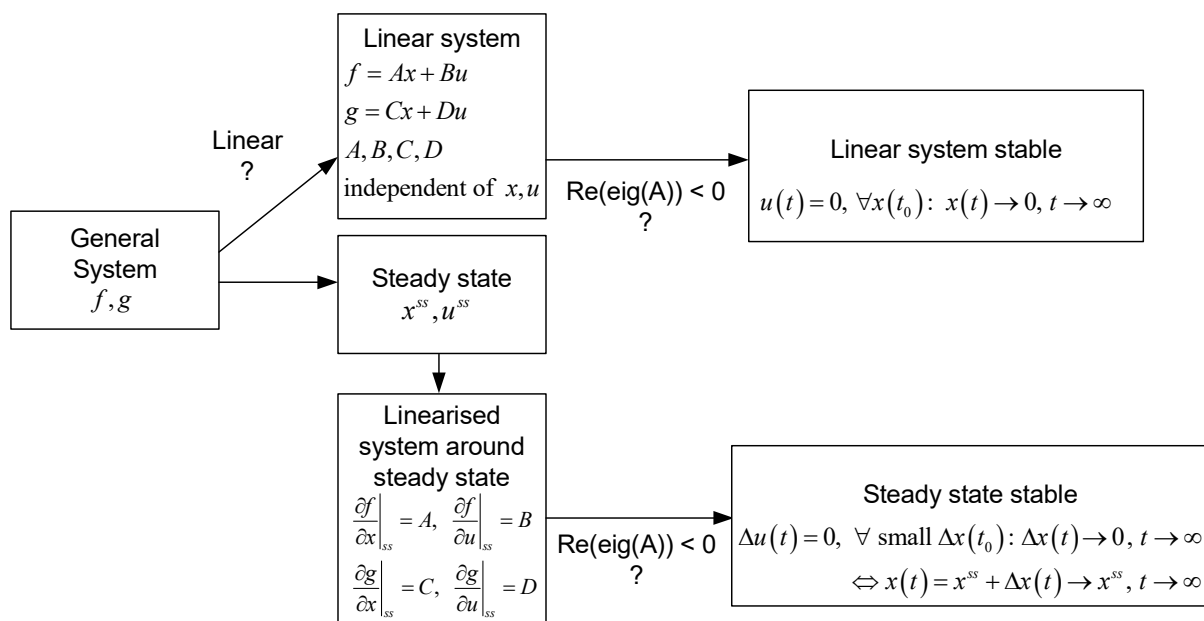


Figure 5-1 Linear systems analysis.

After reading this section the student should:

- Be able to determine from a state-space model whether a system is linear, linear in the state, linear in the control inputs, linear in the parameters, linear in the external inputs.
- Be able for a linear system to calculate from f, g the matrices A, B, C, D which determine the general state-space form of a linear system, see Figure 5-1.
- Be able to determine whether a steady state is stable or not by computing the linearized model around the steady state and determining whether this model is stable, see Figure 5-1.
- Know the interpretation of stable and unstable steady states from the behaviour and meaning of the associated deviation (perturbation) variables.
- Determine the linearized model around optimal state and control trajectories Know the meaning, interpretation and advantage of this model for control system design.

- Be able to determine from a state-space model whether a system is time invariant or time-varying.

5.2 Taylor series expansion and linearization of systems

From the introduction it has become clear that the linearization of systems around steady states is needed to investigate stability and stabilisability of the steady state. To understand the linearization of systems we have to look into Taylor series expansions of the vector functions f, g which fully determine our state-space model. Because the constant parameters do not influence the results in this section, to simplify the notation, the parameter vector is dropped from our notation. This implies that we write $f(x, u)$ instead of $f(x, u, p)$ which we have written so far to stress the dependence on the constant parameters.

Consider the general state equation (3.76) with $n_x = n_u = 1$. The Taylor series expansion of $f(x, u)$ around the values $x = x^{ss}$, $u = u^{ss}$ is,

$$f(x, u) = f(x^{ss}, u^{ss}) + \left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} (x - x^{ss}) + \left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} (u - u^{ss}) + O(2) \quad (5.2)$$

In equation (5.2) $O(2)$ represents all the remaining terms in the Taylor series expansion which contain second and higher order terms in $x - x^{ss}$ and $u - u^{ss}$. If $x - x^{ss}$ and $u - u^{ss}$ are small, that is in a neighbourhood of x^{ss}, u^{ss} , neglecting $O(2)$ is acceptable. Neglecting $O(2)$ means approximating the function by the tangent line as in Figure 2-1 and Figure 4-1. If the system is linear, as we shall see later in section 5.4, the function is this tangent line and $O(2)$ is equal to zero.

Next consider equation (3.76) with $n_x = n_u = 2$. Then we obtain,

$$\begin{aligned}
f(x, u) &= \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \end{bmatrix} = \begin{bmatrix} f_1(x^{ss}, u^{ss}) \\ f_2(x^{ss}, u^{ss}) \end{bmatrix} + \\
&\begin{bmatrix} \left. \frac{\partial f_1}{\partial x_1} \right|_{x=x^{ss}, u=u^{ss}} (x_1 - x_1^{ss}) + \left. \frac{\partial f_1}{\partial x_2} \right|_{x=x^{ss}, u=u^{ss}} (x_2 - x_2^{ss}) \\ \left. \frac{\partial f_2}{\partial x_1} \right|_{x=x^{ss}, u=u^{ss}} (x_1 - x_1^{ss}) + \left. \frac{\partial f_2}{\partial x_2} \right|_{x=x^{ss}, u=u^{ss}} (x_2 - x_2^{ss}) \end{bmatrix} + \\
&\begin{bmatrix} \left. \frac{\partial f_1}{\partial u_1} \right|_{x=x^{ss}, u=u^{ss}} (u_1 - u_1^{ss}) + \left. \frac{\partial f_1}{\partial u_2} \right|_{x=x^{ss}, u=u^{ss}} (u_2 - u_2^{ss}) \\ \left. \frac{\partial f_2}{\partial u_1} \right|_{x=x^{ss}, u=u^{ss}} (u_1 - u_1^{ss}) + \left. \frac{\partial f_2}{\partial u_2} \right|_{x=x^{ss}, u=u^{ss}} (u_2 - u_2^{ss}) \end{bmatrix} + O(2)
\end{aligned} \tag{5.3}$$

Observe that,

$$\begin{bmatrix} f_1(x^{ss}, u^{ss}) \\ f_2(x^{ss}, u^{ss}) \end{bmatrix} = f(x^{ss}, u^{ss}) \tag{5.4}$$

$$\begin{bmatrix} \left. \frac{\partial f_1}{\partial x_1} \right|_{x=x^{ss}, u=u^{ss}} (x_1 - x_1^{ss}) + \left. \frac{\partial f_1}{\partial x_2} \right|_{x=x^{ss}, u=u^{ss}} (x_2 - x_2^{ss}) \\ \left. \frac{\partial f_2}{\partial x_1} \right|_{x=x^{ss}, u=u^{ss}} (x_1 - x_1^{ss}) + \left. \frac{\partial f_2}{\partial x_2} \right|_{x=x^{ss}, u=u^{ss}} (x_2 - x_2^{ss}) \end{bmatrix} = \begin{bmatrix} \left. \frac{\partial f_1}{\partial x_1} \right|_{x=x^{ss}, u=u^{ss}} & \left. \frac{\partial f_1}{\partial x_2} \right|_{x=x^{ss}, u=u^{ss}} \\ \left. \frac{\partial f_2}{\partial x_1} \right|_{x=x^{ss}, u=u^{ss}} & \left. \frac{\partial f_2}{\partial x_2} \right|_{x=x^{ss}, u=u^{ss}} \end{bmatrix} (x - x^{ss}) \tag{5.5}$$

and similarly,

$$\begin{bmatrix} \left. \frac{\partial f_1}{\partial u_1} \right|_{x=x^{ss}, u=u^{ss}} (u_1 - u_1^{ss}) + \left. \frac{\partial f_1}{\partial u_2} \right|_{x=x^{ss}, u=u^{ss}} (u_2 - u_2^{ss}) \\ \left. \frac{\partial f_2}{\partial u_1} \right|_{x=x^{ss}, u=u^{ss}} (u_1 - u_1^{ss}) + \left. \frac{\partial f_2}{\partial u_2} \right|_{x=x^{ss}, u=u^{ss}} (u_2 - u_2^{ss}) \end{bmatrix} = \begin{bmatrix} \left. \frac{\partial f_1}{\partial u_1} \right|_{x=x^{ss}, u=u^{ss}} & \left. \frac{\partial f_1}{\partial u_2} \right|_{x=x^{ss}, u=u^{ss}} \\ \left. \frac{\partial f_2}{\partial u_1} \right|_{x=x^{ss}, u=u^{ss}} & \left. \frac{\partial f_2}{\partial u_2} \right|_{x=x^{ss}, u=u^{ss}} \end{bmatrix} (u - u^{ss}) \tag{5.6}$$

Using (5.4), (5.5), (5.6) and the definition of the Jacobian (4.16), equation (5.3) becomes,

$$f(x, u) = f(x^{ss}, u^{ss}) + \left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} (x - x^{ss}) + \left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} (u - u^{ss}) + O(2) \tag{5.7}$$

Equation (5.7) is completely identical to equation (5.2)! So, equation (5.2) also applies to vector functions of vectors when we define the derivative of a vector function according to (4.16). We might also say that the definition (4.16) is a direct consequence of the generalisation of Taylor series expansion to vector functions of vectors. Although we have

only considered the case $n_x = n_u = 2$ for other values n_x, n_u which need not necessarily be the same, the derivation is along exactly the same line.

Equation (5.7) without $O(2)$ is a linear approximation of the vector function $f(x, u)$ around $x = x^{ss}, u = u^{ss}$. Without $O(2)$ all components of the vector function $f(x, u)$ are approximated by tangent lines as in Figure 2-1 and Figure 4-1. This approximation is accurate in a neighbourhood of $x = x^{ss}, u = u^{ss}$ where $O(2)$ is negligible. Similarly,

$$\frac{dx}{dt} \approx f(x^{ss}, u^{ss}) + \left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} (x - x^{ss}) + \left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} (u - u^{ss}) \quad (5.8)$$

is a linear approximation of the state equation,

$$\frac{dx}{dt} = f(x, u) \quad (5.9)$$

Also,

$$y \approx g(x^{ss}, u^{ss}) + \left. \frac{\partial g}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} (x - x^{ss}) + \left. \frac{\partial g}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} (u - u^{ss}) \quad (5.10)$$

is a linear approximation of the output equation,

$$y = g(x, u) \quad (5.11)$$

Both (5.8) and (5.10) are accurate approximations only in the neighbourhood of the linearization point $x = x^{ss}, u = u^{ss}$.

Remember that in the introduction an important question was whether deviations from the steady state would tend to zero or grow. To verify this, introduce so called deviation (or perturbation) variables,

$$\Delta x = x - x^{ss}, \Delta u = u - u^{ss}, \Delta y = y - y^{ss}, y^{ss} = g(x^{ss}, u^{ss}) \quad (5.12)$$

Observe that,

$$\frac{d\Delta x}{dt} = \frac{d(x - x^{ss})}{dt} = \frac{dx}{dt} - \frac{dx^{ss}}{dt} = f(x, u) - f(x^{ss}, u^{ss}) \quad (5.13)$$

From (5.12), (5.7), (5.13) we obtain,

$$\frac{d\Delta x}{dt} \approx \left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} \Delta u \quad (5.14)$$

Equation (5.14) is called the linearized state equation around $x = x^{ss}$, $u = u^{ss}$ or a linearization of the state equation (5.9) around $x = x^{ss}$, $u = u^{ss}$. This linearization describes approximately the dynamic behaviour of small state deviations Δx as a result of small input deviations Δu . As we shall see, equation (5.14) is a linear systems model which enables us to analyse stability and stabilisability properties. Similarly, from (5.12), (5.10) we obtain,

$$\Delta y \approx \left. \frac{\partial g}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} \Delta x + \left. \frac{\partial g}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} \Delta u \quad (5.15)$$

which is called the linearized output equation around $x = x^{ss}$, $u = u^{ss}$, $y = y^{ss} = g(x^{ss}, u^{ss})$ or a linearization of the output equation (5.11) around $x = x^{ss}$, $u = u^{ss}$, $y = y^{ss} = g(x^{ss}, u^{ss})$.

Example 5.1

Recall the state-space model from exercise 4.1,

$$\frac{dx_1}{dt} = 2u_1^2 - u_2 + x_1, \quad \frac{dx_2}{dt} = -u_1 - u_2^2 + x_2 \quad (4.18)$$

In exercise 4.1 we realised the steady state $x^{ss} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ by computing the associated control

$u^{ss} = \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}$. To compute the linearization (5.14) of the state equation (4.18) around this

steady state we must compute $\left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}}$ and $\left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}}$ where $f = \begin{bmatrix} 2u_1^2 - u_2 + x_1 \\ -u_1 - u_2^2 + x_2 \end{bmatrix}$. Using

the definition (4.16) we compute,

$$\begin{aligned} \left. \frac{\partial f}{\partial x} \right|_{x=x^{ss}, u=u^{ss}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{x=\begin{bmatrix} -5 \\ 4 \end{bmatrix}, u=\begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \left. \frac{\partial f}{\partial u} \right|_{x=x^{ss}, u=u^{ss}} &= \begin{bmatrix} 4u_1 & -1 \\ -1 & -2u_2 \end{bmatrix}_{x=\begin{bmatrix} -5 \\ 4 \end{bmatrix}, u=\begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}} = \begin{bmatrix} 7.2016 & -1 \\ -1 & -2.9662 \end{bmatrix} \end{aligned} \quad (5.16)$$

Therefore the linear state equation,

$$\frac{d\Delta x}{dt} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta x + \begin{bmatrix} 7.2016 & -1 \\ -1 & -2.9662 \end{bmatrix} \Delta u \quad (5.17)$$

describes approximately the behaviour of small deviation variables,

$$\Delta x = x - x^{ss} = x - \begin{bmatrix} -5 \\ 4 \end{bmatrix}, \Delta u = u - u^{ss} = u - \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix} \quad (5.18)$$

When considering stability of steady states, $u(t) = u^{ss}$ implying $\Delta u(t) = 0$. Then from equation (5.17) matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ fully determines the linear state equation. In section 5.5 we learn how to determine the stability of steady states from the eigenvalues of this matrix. When considering stabilisability of steady states, $\Delta u(t) \neq 0$ since it is used to possibly stabilize an unstable steady state. Then both matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7.2016 & -1 \\ -1 & -2.9662 \end{bmatrix}$ in (5.17), determine the outcome. Matlab functions to determine both stability (`eig`) and stabilisability (`stblzb`) of steady states are available and will be demonstrated during the lectures.

Exercise 5.1

Recall the state-space model (3.62), (3.63), (3.64) in the biochemical reactor of example 3.6

(section 3.7). Write $f(x, u, p)$ of this state-space model. Let $x^{ss} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}, u^{ss} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Show that

x^{ss} is not a steady state. Although it is not a steady state, we may still obtain the linearized state equation around this temporary state (as opposed to steady state) in exactly the same manner. Compute this linearized state equation. You may wonder about the use and meaning of this linearized state equation. Both will be explained later in section 5.7.

5.3 Linearity

In the previous section we learned how to compute linearized state equations which are important to analyse the stability and stabilisability of steady states of (non-linear) systems. The term linearized relates to the fact that we are approximating functions by straight lines as in Figure 2-1 and Figure 4-1.

Actually, two definitions of linearity, which are slightly different, are used. These two definitions are visualised in Figure 5-2.

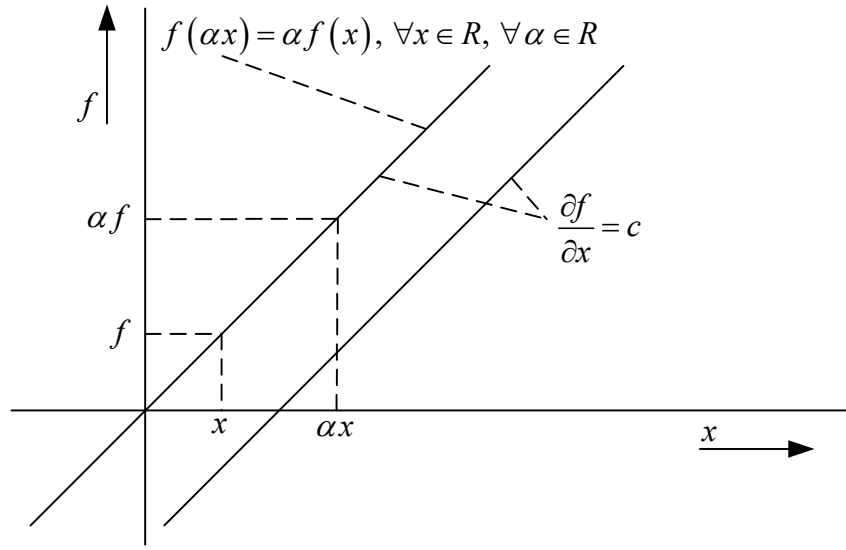


Figure 5-2 Two definitions of linearity.

The formal (strong) definition of linearity states that a scalar function of one variable $f(x)$, $f, x \in R^1$ is linear if,

$$f(\alpha x) = \alpha f(x), \forall x \in R, \forall \alpha \in R \quad (5.19)$$

This definition demands that the function is represented by a straight line that runs through the origin. A weaker definition of linearity is officially called affine. The function $f(x)$, $f, x \in R^1$ is called affine if,

$$\frac{\partial f}{\partial x} = c, \forall x \in R \quad (5.20)$$

where c does not depend on x and therefore is constant. So affine only demands that the function is described by a straight line. The line does not necessarily run through the origin and therefore (5.19) is not necessarily satisfied. Mathematically speaking $(5.19) \Rightarrow (5.20)$ while the reverse is not true. Therefore (strong) linearity (5.19) is a stronger property than affine (5.20). Also observe,

$$(5.20) \text{ and } f(0) = 0 \Leftrightarrow (5.19) \quad (5.21)$$

The simplest way to check whether (5.19) is satisfied is usually to check (5.20) and $f(0) = 0$

Example 5.2

The function $f(x) = x + p$, $f, x, p \in R^1$ with p a constant parameter is affine, but is not linear, unless $p = 0$.

For functions of several variables $f(x)$, $f \in R^1$, $x \in R^{n_x}$, $n_x \geq 1$ the definitions are identical. Only now c in (5.20) becomes a constant row vector. In this case the function is represented by a straight line in the n_x dimensional space. In the case of definition (5.19) this line runs through the origin.

Example 5.3

$f(x) = x_1^2 + x_2$, $f \in R^1$, $x \in R^2$ is not a linear function because $\frac{\partial f}{\partial x} = [2x_1 \quad 1]$ depends on x .

For vector functions $f(x)$, $f \in R^{n_f}$, $x \in R^{n_x}$, $n_f \geq 1$, $n_x \geq 1$ again the definitions are the same while c now is a matrix (a Jacobian). Each component of the vector function f is now represented by a straight line in the n_x dimensional space. In the case of definition (5.19) all these lines run through the origin.

Example 5.4

$f(x) = \begin{bmatrix} 2x_1 - x_2 + p \\ -x_1 \end{bmatrix}$, $f, x \in R^2$ with p a constant parameter is affine but not linear, unless

$p = 0$. The reason is that $\frac{\partial f}{\partial x} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ does not depend on x and the line $2x_1 - x_2 + p$ only runs through the origin if $p = 0$.

5.4 Linear systems

Within the state-space description of systems,

$$\frac{dx}{dt} = f(x, u), \quad y = g(x, u) \quad (5.22)$$

we are dealing with the vector functions,

$$f(x, u), g(x, u), f \in R^{n_x}, g \in R^{n_y}, x \in R^{n_x}, u \in R^{n_u} \quad (5.23)$$

A system is called linear if its state-space model (5.22) satisfies,

$$\left. \begin{aligned} f(\alpha x, \alpha u) &= \alpha f(x, u) \\ g(\alpha x, \alpha u) &= \alpha g(x, u) \end{aligned} \right\} \forall x \in R^{n_x}, \forall u \in R^{n_u}, \forall \alpha \in R \quad (5.24)$$

Observe that definition (5.24) corresponds to the strong definition of linearity (5.19) when we add all the components of the vector u to the vector x . Definition (5.24) implies that for a linear system all the components of both f and g are represented by straight lines in the $n_x + n_u$ dimensional space that run through the origin. Therefore,

$$f(0,0) = 0, g(0,0) = 0 \quad (5.25)$$

must be satisfied. Similar to equation (5.21) the simplest way to check whether (5.24) is satisfied is to compute,

$$f(0,0), g(0,0), A = \frac{\partial f}{\partial x}, B = \frac{\partial f}{\partial u}, C = \frac{\partial g}{\partial x}, D = \frac{\partial g}{\partial u} \quad (5.26)$$

and to check whether $f(0,0) = 0$ and $g(0,0) = 0$, and if the matrices A, B, C, D are not functions of x, u .

Example 5.5: Newton's law

Recall the state-space model of the mass with a force acting on it and with the kinetic energy and the impulse as outputs.

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = \frac{u_1}{p_1}, y_1 = \frac{1}{2} p_1 x_2^2, y_2 = p_1 x_2 \quad (5.27)$$

To check whether this system is linear we compute,

$$f(x, u) = \begin{bmatrix} x_2 \\ u_1 \\ p_1 \end{bmatrix}, g(x, u) = \begin{bmatrix} \frac{1}{2} p_1 x_2^2 \\ p_1 x_2 \end{bmatrix} \quad (5.28)$$

and,

$$\begin{aligned} \frac{\partial f}{\partial x} = A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \frac{\partial f}{\partial u} = B = \begin{bmatrix} 0 \\ 1 \\ p_1 \end{bmatrix}, f(0,0) = 0 \\ \frac{\partial g}{\partial x} = C &= \begin{bmatrix} 0 & p_1 x_2 \\ 0 & p_1 \end{bmatrix}, \frac{\partial g}{\partial u} = D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, g(0,0) = 0 \end{aligned} \quad (5.29)$$

Because element C_{12} of matrix C is a function of x the system is not linear.

Exercise 5.2

Change the output equation $g(x, u)$ so that element C_{12} of matrix C becomes independent of x, u .

If a system is linear, it can be written as,

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du \quad (5.30)$$

where $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ are all independent of x, u . The fact that these matrices are constant does not necessarily imply that the system is linear. We also require $f(0, 0) = 0$ and $g(0, 0) = 0$. Only then the system is represented by equation (5.30).

Summarising: to check whether a system is linear, and if it is, write it in the form (5.30), proceed as follows:

- Write the system in state-space form. That means determine $f(x, u)$, $g(x, u)$.
- Check whether $f(0, 0) = 0$, $g(0, 0) = 0$. If not, the system is not linear.
- Compute $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ and check whether these are not functions of x, u . If so, (5.30) represents this linear system. As soon as one of the elements of the matrices $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ does depend on x or u the system is not linear and is not represented by (5.30).

Exercise 5.3

Modify the linear systems model obtained from exercise 5.2 such that it is no longer linear,

while $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$ are still independent of x, u .

A system is called linear in the state x if,

$$A = \frac{\partial f}{\partial x}, \quad C = \frac{\partial g}{\partial x} \quad (5.31)$$

do not depend on the state x .

A system is called linear in the input u if,

$$B = \frac{\partial f}{\partial u}, D = \frac{\partial g}{\partial u} \quad (5.32)$$

do not depend on the control input u . The control system design for systems that are linear in the control input or linear in the state in general is easier than the control system design for non-linear systems that are not. Note that systems that are both linear in the state and linear in the control input need not necessarily be linear systems.

Exercise 5.4

Write down a state-space model of an arbitrary system that is linear in the state, linear in the control, has the property $f(0,0) = 0$, $g(0,0) = 0$, but still is not linear.

A system is called linear in the parameters p if the derivatives,

$$E = \frac{\partial f}{\partial p}, F = \frac{\partial g}{\partial p} \quad (5.33)$$

do not depend on p . The property linear in the parameters is important for system identification. For systems that are linear in the parameters many well established techniques exist to improve the model parameters through the use of measurement data.

Exercise 5.5

Write down a state-space model of an arbitrary system that is not linear in the state, not linear in the control, and linear in the parameters.

Linear systems have an important property which is sometimes used to define linear systems. This property is called superposition and can be represented mathematically as follows,

$$\left. \begin{aligned} u^1(t) &\rightarrow x^1(t), y^1(t), t \geq t_0, x(t_0) = 0 \\ u^2(t) &\rightarrow x^2(t), y^2(t), t \geq t_0, x(t_0) = 0 \end{aligned} \right\} \Rightarrow \quad (5.34)$$

$$\alpha u^1(t) + \beta u^2(t) \rightarrow \alpha x^1(t) + \beta x^2(t), \alpha y^1(t) + \beta y^2(t), t \geq t_0, x(t_0) = 0, \forall \alpha, \beta \in R$$

The meaning of the notation in (5.34) is as follows. To the input on the left of the arrow the state and output response of the system on the right correspond if the initial state is zero. Equation (5.34) states that if the input is an arbitrary linear combination of two arbitrary inputs the state and output response is the same linear combination of the two associated individual state and output responses if the initial state is zero.

5.5 Stability of steady states and linear systems

The linearized system about a steady state (5.17), (5.12) is a linear system with a state $\Delta x = x - x^{ss}$ and input $\Delta u = u - u^{ss}$ which represent state and input deviation (perturbation) variables. The steady state x^{ss}, u^{ss} is called stable if for *any small* initial perturbation $\Delta x(t_0)$, the state perturbation $\Delta x(t)$ tends to zero as t tends to infinity if the input perturbation $\Delta u(t) = 0$. Mathematically this is represented as follows,

$$\Delta u(t) = 0, \forall \Delta x(t_0), \|\Delta x(t_0)\|_2 < \varepsilon \Rightarrow \Delta x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (5.35)$$

where ε is a small positive number and $\|\Delta x(t_0)\|_2$ is the length (2-norm) of the vector $\Delta x(t_0)$ which together define a small neighbourhood around $\Delta x(t_0) = 0$.

If the inputs are control inputs note that we might use $\Delta u(t)$ to control $\Delta x(t)$ to zero as t tends to infinity. Such a $\Delta u(t)$, instead of a control perturbation, is called a control correction. So, a steady state is stable if without the use of control corrections $\Delta u(t)$, the state $\Delta x(t)$ tends to zero as t tends to infinity for arbitrary small initial state perturbations $\Delta x(t_0)$.

A steady state is called stabilisable if by making use of control corrections $\Delta u(t)$ we can control the state deviations to zero ultimately,

$$\forall \Delta x(t_0), \|\Delta x(t_0)\|_2 < \varepsilon, \exists \Delta u(t) \text{ such that } \Delta x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (5.36)$$

In this course we will only consider the stability of steady states because stabilisability is actually a kind of control problem because we have to find a suitable control correction. Stabilisability is closely related to controllability. These properties are treated in the course Systems and Control Theory (BCT-31306).

To investigate the stability of steady states we investigate the stability of the linearized system about the steady state which is a linear system. This linear system only describes state and input perturbations accurately as long as these remain small. Note that in the definitions (5.35), (5.36) of stability and stabilisability of a steady state, small initial state perturbations are mentioned. Therefore, we may investigate the stability of the linearized

system to decide about the stability of steady states. Observe that to analyse the stability (of steady states) output equations are irrelevant.

Because no control corrections are applied ($\Delta u = 0$), we will investigate the stability of linear systems if the input is zero. When performing this investigation, we drop the notation of Δ 's. But keep in mind that the state and input x, u of a linearized system about a steady state x^{ss}, u^{ss} of a nonlinear system, represent $x - x^{ss} = \Delta x$, $u - u^{ss} = \Delta u$. In the latter two expressions x, u represent the state and input of the nonlinear system. If a linear system,

$$\frac{dx}{dt} = Ax + Bu, \quad x \in R^{n_x}, \quad u \in R^{n_u} \quad (5.37)$$

has zero input then it can be described by,

$$\frac{dx}{dt} = Ax, \quad x \in R^{n_x} \quad (5.38)$$

Consider the particularly simple case where the state x is a scalar. Then the matrix A also reduces to a scalar (a 1x1 matrix). In this case the scalar differential equation (5.38) has the following analytical solution,

$$x(t) = e^{At} x(t_0), \quad t \geq t_0 \quad (5.39)$$

Now without proof we state that the solution (5.39) to (5.38) also applies if $n_x \geq 1$. In this case A is a square matrix and the question in (5.39) arises as to the meaning of e^{At} . To answer this question, recall the series (2.1),

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad x \in R^1 \quad (2.1)$$

Now similar to (2.1) the so-called matrix exponential is defined,

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots, \quad I, A \in R^{n_x \times n_x} \quad (5.40)$$

where I denotes the identity matrix. So e^A and e^{At} are both $n_x \times n_x$ dimensional matrices which complies with equation (5.39).

To compute the matrix exponential and to answer the question regarding the stability of linear systems we require the definition of eigenvalues and eigenvectors (see Mathematics at work, Part 1, Chapter 4) of the square matrix A in (5.38), (5.39). The n_x eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_{n_x}$ of the square matrix $A \in R^{n_x \times n_x}$ and the associated n_x eigenvectors $v_1, v_2, \dots, v_{n_x}$ satisfy,

$$Av_i = \lambda_i v_i, A \in R^{n_x \times n_x}, v_i \in R^{n_x}, \lambda_i \in R^1, i = 1, 2, \dots, n_x \quad (5.41)$$

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{n_x}$ are either real or complex numbers. Complex eigenvalues always appear in pairs of complex conjugate numbers. That is: if $a + jb$ is an eigenvalue, then so is $a - jb$. Also, the components of the associated eigenvectors are each other's complex conjugate.

Observe that,

$$\left. \begin{array}{l} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \\ \vdots \\ Av_{n_x} = \lambda_{n_x} v_{n_x} \end{array} \right\} \Leftrightarrow A \begin{bmatrix} v_1 & v_2 & \dots & v_{n_x} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \dots & v_{n_x} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_{n_x} \end{bmatrix} \quad (5.42)$$

Let,

$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_{n_x} \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_{n_x} \end{bmatrix} \in R^{n_x \times n_x} \quad (5.43)$$

then (5.43) becomes,

$$AV = VD \quad (5.44)$$

The square $n_x \times n_x$ matrix V that has as columns the eigenvectors of A . The matrix V is invertible if all eigenvalues of A are distinct. Then through post multiplication by V^{-1} we obtain from (5.44),

$$A = V D V^{-1} \quad (5.45)$$

Now substitute (5.45) in,

$$e^{At} = I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \dots \quad (5.46)$$

to obtain,

$$e^{At} = VIV^{-1} + \frac{VDV^{-1}t}{1!} + \frac{VDV^{-1}VDV^{-1}t^2}{2!} + \dots =$$

$$V\left(I + \frac{Dt}{1!} + \frac{D^2t^2}{2!} + \dots\right)V^{-1} = Ve^{Dt}V^{-1} \quad (5.47)$$

Now from (5.40) and (5.43) verify that,

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & e^{\lambda_{n_x} t} \end{bmatrix} \quad (5.48)$$

Observe that to compute e^{Dt} we only need to compute the scalar exponentials on the diagonal which is no problem. Then if we are able to compute the eigenvalues and eigenvectors of the matrix A which we can do using standard software (the function `eig` within Matlab) then we can compute e^{At} according to (5.47), (5.48).

Now from (5.39) and (5.47) observe that the solution to (5.38) may be written as,

$$x(t) = Ve^{Dt}V^{-1}x(t_0), \quad t \geq t_0 \quad (5.49)$$

Then from (5.49) observe that if all eigenvalues $\lambda_i, i=1,2,\dots,n_x$ are real and smaller than zero $e^{Dt} \rightarrow 0$ as $t \rightarrow \infty$ and as a result $x(t) \rightarrow 0$ as $t \rightarrow \infty$ in (5.49) for any $x(t_0)$. Otherwise, this does not hold. Note that the eigenvalues may also appear in complex conjugate pairs. We will not consider this case. Without proof we summarise the results which we have only proved for real eigenvalues.

- The systems (5.37) and (5.38) are stable if the eigenvalues of the matrix A all have a real part which is negative. Then for the system (5.38) and for any initial state $x(t_0)$ the system state $x(t) \rightarrow 0$ as $t \rightarrow \infty$. The same thing applies to the system (5.37) if $u(t) = 0, t \geq t_0$. Knowing all the eigenvalues and the associated eigenvectors of the matrix A , the matrix exponential e^A is easily computed using equations (5.43), (5.47), (5.48).
- To check whether a steady state is stable we check the stability of the linearized system (5.14) about the steady state $u(t) = u^{ss}, x(t) = x^{ss}$. If this linearized system is stable, so

is the steady state and small arbitrary (initial) state perturbations tend to zero. If not, then the steady state is unstable and small (initial) state perturbations do not tend to zero in general.

Exercise 5.6

Recall the state-space model (4.18) and its linearized model (5.17) about the steady state

$x^{ss} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$, $u^{ss} = \begin{bmatrix} 1.8004 \\ 1.4831 \end{bmatrix}$. Determine from the linearized model (5.17) whether this steady state is stable.

5.6 Quasi steady state approximation; static and dynamic models

Example 5.6: The greenhouse

Consider again the dynamic model of the greenhouse,

$$\frac{dT_g}{dt} = c_1(T_o - T_g) + c_2 H \quad (5.50)$$

When written into state-space form the model becomes (check yourself),

$$\frac{dx_1}{dt} = p_1(d_1 - x_1) + p_2 u_1 \quad (5.51)$$

Suppose heating is turned off : $u_1(t) = 0$. Then equation (5.51) becomes,

$$\frac{dx_1}{dt} = p_1(d_1 - x_1) \quad (5.52)$$

Also suppose the outside temperature varies slowly according to $d_1(t) = 10 - 5 \cos(2\pi t / 24)$ where the unit of t is [hrs]. Finally suppose the greenhouse is not very well isolated. From equations (3.15)-(3.19) this implies that p_1 is large and the outside temperature largely influences the greenhouse temperature. For two different values of p_1 the system behaviour, i.e. the greenhouse temperature, as well as the outside temperature are plotted in Figure 5-3. The initial greenhouse temperature is $15 \text{ } [^{\circ}\text{C}]$. As you can see, the larger p_1 , the quicker the greenhouse temperature “jumps” towards the outside temperature and the more accurate it follows the outside temperature after that.

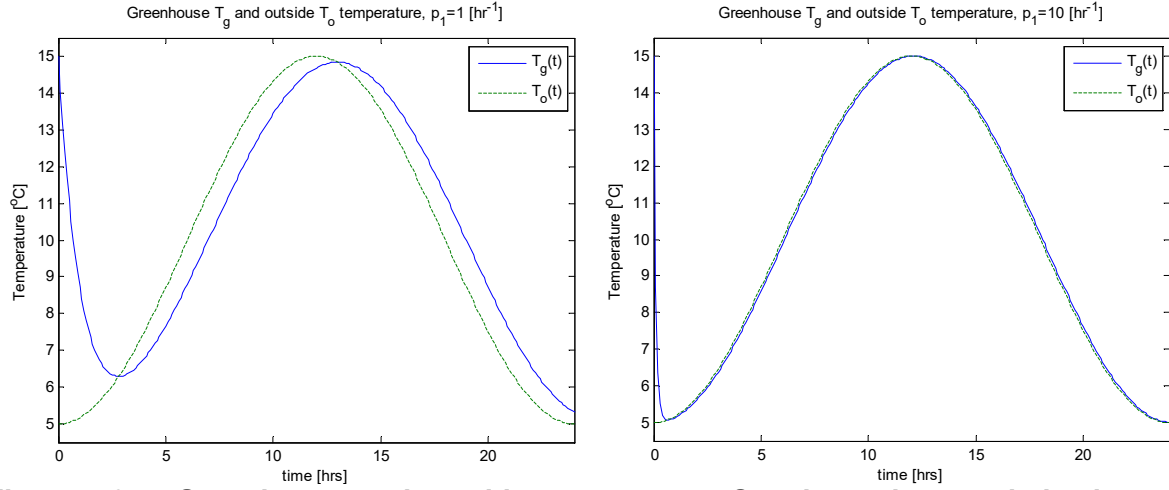


Figure 5-3 Greenhouse and outside temperature. Quasi steady state behaviour on the right

When p_1 is large ($p_1 \geq 10$), the greenhouse temperature jumps very fast from its initial value $15[^\circ C]$ to the outside temperature and then becomes almost equal to it. This is an example of what is called “quasi steady state behaviour”. If a system is in steady state all state derivatives are zero. If we set $dx_1 / dt = 0$ in equation (5.52) we obtain,

$$p_1(d_1 - x_1) = 0 \quad (5.53)$$

from which we obtain,

$$x_1 = d_1 \quad (5.54)$$

which is approximately the behaviour of system (5.52) if p_1 is large. Therefore, this behaviour is called “quasi steady state behaviour”. It occurs if inputs or other states that change slowly have a strong influence on a certain state. This state can then be computed from a quasi steady state approximation that assumes the derivative of this state to be zero, like in (5.52)-(5.54). The quasi steady state approximation results in an algebraic equation, (5.54) in this example, that replaces a differential equation, (5.52). Models of systems that only contain algebraic equations are called static models. They are often obtained from quasi steady state approximations of dynamic models.

5.7 State-space models of interconnected dynamic systems

Consider two dynamic systems, both represented in state-space form. The superscripts a, b are used to distinguish between the two state-space models,

$$\frac{dx^a}{dt} = f^a(x^a, u^a, p^a) \quad (5.55)$$

$$y^a = g^a(x^a, p^a) \quad (5.56)$$

$$\frac{dx^b}{dt} = f^b(x^b, u^b, p^b) \quad (5.57)$$

$$y^b = g^b(x^b, p^b) \quad (5.58)$$

Suppose the outputs of system a are connected with the inputs of system b ,

$$u^b = y^a \quad (5.59)$$

and the outputs of system b are connected with inputs of system a ,

$$u^a = y^b \quad (5.60)$$

The interconnected system has no inputs and outputs anymore and therefore is an autonomous system. Given the importance of representing systems models in state-space form a natural question to ask is: "What is the state-space model of this interconnected autonomous system?" This state-space model is found as follows:

- State variables: State variables of both systems a and b :

$$x = \begin{bmatrix} x^a \\ x^b \end{bmatrix} \quad (5.61)$$

- No inputs, so no u .
- No outputs, so no y and $g(x)$.
- Parameters: parameters of both systems a and b :

$$p = \begin{bmatrix} p^a \\ p^b \end{bmatrix} \quad (5.62)$$

To find $f(x, p)$ of the interconnected autonomous system observe from equations (5.61), (5.55), (5.57) that,

$$\frac{dx}{dt} = \begin{bmatrix} dx^a / dt \\ dx^b / dt \end{bmatrix} = \begin{bmatrix} f^a(x^a, u^a, p^a) \\ f^b(x^b, u^b, p^b) \end{bmatrix} = f \quad (5.63)$$

Note that $f = \begin{bmatrix} f^a(x^a, u^a, p^a) \\ f^b(x^b, u^b, p^b) \end{bmatrix}$ in equation (5.63) still contains the inputs u^a, u^b of the

original separate systems. They have to be eliminated because they are no longer inputs of the interconnected autonomous system. The elimination is achieved using equations (5.56),

(5.58), (5.59), (5.60). Using (5.56), (5.59) u_i^b may be replaced by $g_i^a(x^a, p^a)$, $i = 1, 2, \dots, n_u^b$, an expression containing only states and parameters of system a . According to equations (5.61), (5.62) these are also states and parameters of the autonomous system. Similarly, u_i^a may be replaced by $g_i^b(x^b, p^b)$, $i = 1, 2, \dots, n_u^a$, also an expression containing only states and parameters of the autonomous system. Substituting these expressions into (5.63) provides $f(x, p)$ of the interconnected autonomous system.

Exercise 5.7

Write down two arbitrary state-space models, each one having two inputs and two outputs. Interconnect these two systems as mentioned above and write down the state-space model of the interconnected system.

Exercise 5.8

Consider the two systems described by equations (5.55)-(5.58). Assume the only interconnection is described by equation (5.59). This is called a series interconnection (as opposed to parallel interconnection) of the two systems. Explain the name series interconnection. Does the interconnected system have inputs? Explain. Does it have outputs? Explain. How do we obtain the state-space model of the interconnected system?

Exercise 5.9

Consider again the two systems described by equations (5.55)-(5.58). Write down equation(s) that describe a parallel interconnection of the two systems. Also describe how to obtain the state-space model of the parallel interconnected system.

5.8 Linearization about optimal state, input and output trajectories

What is the optimal trajectory of a rocket flying to the moon? What is the cheapest way to start up a chemical or biochemical reactor? What is the best trajectory of a racing car on the race circuit? In all these cases the control objective is not to keep the system in a steady state. In the second part of the course Systems and Control Theory (BCT-31306), using state-space models of systems we will compute such optimal state trajectories $x^*(t)$ and the associated optimal control $u^*(t)$ and output trajectories $y^*(t)$. Furthermore, we will design controllers to compensate for small modelling errors and disturbances. The control system obtained in this way is represented in Figure 5-4.

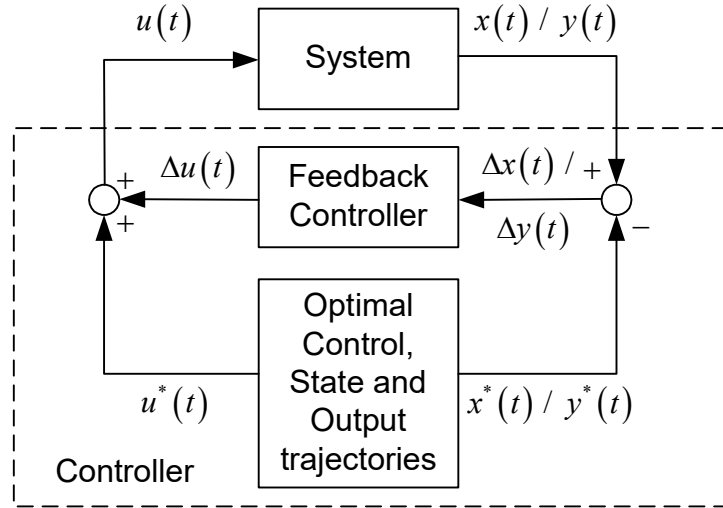


Figure 5-4 An optimal control system.

Apart from the computation of the optimal state, control and output trajectories, $x^*(t)$, $u^*(t)$, $y^*(t)$ another important issue is the design of the feedback controller to compensate for small disturbances and modelling errors. The feedback controller design is concerned with the problem of computing control corrections $\Delta u(t)$ from the measured state or output deviations $\Delta x(t) / \Delta y(t)$ with the objective to keep them small. Observe that,

$$\Delta x(t) = x(t) - x^*(t), \Delta u(t) = u(t) - u^*(t), \Delta y(t) = y(t) - y^*(t) \quad (5.64)$$

Since the deviations and corrections are small their dynamic behaviour can be very well approximated by a linearized model about the optimal trajectories $x^*(t)$, $u^*(t)$, $y^*(t)$. This linearized model is used for the feedback controller design because the design for linear systems is far easier. Similar to (5.14), (5.15) this linearized model is mathematically described by,

$$\frac{d\Delta x}{dt} \approx \left. \frac{\partial f}{\partial x} \right|_{x=x^*(t), u=u^*(t)} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{x=x^*(t), u=u^*(t)} \Delta u \quad (5.65)$$

$$\Delta y \approx \left. \frac{\partial g}{\partial x} \right|_{x=x^*(t), u=u^*(t)} \Delta x + \left. \frac{\partial g}{\partial u} \right|_{x=x^*(t), u=u^*(t)} \Delta u \quad (5.66)$$

Although the mathematical notation is very similar there is an important difference between equations (5.65), (5.66) and (5.14), (5.15). In equation (5.14), (5.15) at each time t the state and control around which we linearize the model is the same, namely, $x(t) = x^{ss}$, $u(t) = u^{ss}$

and so $A = \frac{\partial f}{\partial x} \Big|_{x(t)=x^{ss}, u(t)=u^{ss}}$, $B = \frac{\partial f}{\partial u} \Big|_{x(t)=x^{ss}, u(t)=u^{ss}}$, $C = \frac{\partial g}{\partial x} \Big|_{x(t)=x^{ss}, u(t)=u^{ss}}$, $D = \frac{\partial g}{\partial u} \Big|_{x(t)=x^{ss}, u(t)=u^{ss}}$ are

the same matrices at each time t . In (5.65) and (5.66), at each time t the point of linearization is different, namely $x(t) = x^*(t)$, $u(t) = u^*(t)$ and therefore at each time t the

matrices $A = \frac{\partial f}{\partial x} \Big|_{x(t)=x^*(t), u(t)=u^*(t)}$, $B = \frac{\partial f}{\partial u} \Big|_{x(t)=x^*(t), u(t)=u^*(t)}$, $C = \frac{\partial g}{\partial x} \Big|_{x(t)=x^*(t), u(t)=u^*(t)}$,

$D = \frac{\partial g}{\partial u} \Big|_{x(t)=x^*(t), u(t)=u^*(t)}$ are different. To mathematically denote this difference, the

linearization around the optimal state, control and output trajectory is described by,

$$\frac{d\Delta x}{dt} = A(t)\Delta x(t) + B(t)\Delta u(t) \quad (5.67)$$

$$\Delta y(t) = C(t)\Delta x(t) + D(t)\Delta u(t) \quad (5.68)$$

Equations (5.67), (5.68) represent a linear system which is time varying. Time varying systems are the subject of the next section. In summary time-varying linear systems are important for the feedback controller design which is part of the optimal control system design in Figure 5-4.

Example 5.7: Linearization

Consider the system described by,

$$\frac{dx}{dt} = -x^2 + u, \quad y = xu, \quad x, u, y \in R^1 \quad (5.69)$$

Suppose the optimal state control and output trajectories are,

$$\begin{aligned} x^*(t) &= \sin(t), \quad u^*(t) = \cos(t) + \sin^2(t), \\ y^*(t) &= \sin(t)(\cos(t) + \sin^2(t)) \end{aligned} \quad (5.70)$$

By substituting (5.70) in (5.69) we may verify that the optimal input, state and output response satisfy the systems model (5.69), as they should. The time-varying linearized state equation (5.67) about these trajectories is completely determined by the two matrices,

$$\begin{aligned} A(t) &= \frac{\partial f}{\partial x} \Big|_{x(t)=\sin(t), u(t)=\cos(t)+\sin^2(t)} = [-2x]_{x(t)=\sin(t), u(t)=\cos(t)+\sin^2(t)} = -2\sin(t) \\ B(t) &= \frac{\partial f}{\partial u} \Big|_{x(t)=\sin(t), u(t)=\cos(t)+\sin^2(t)} = [1]_{x(t)=\sin(t), u(t)=\cos(t)+\sin^2(t)} = 1 \end{aligned} \quad (5.71)$$

Exercise 5.10

Determine the two matrices $C(t)$, $D(t)$ which determine the linearized output equation associated with example 5.7.

5.9 Time varying and time-invariant systems

In the previous section time-varying systems have been introduced. From the state-space description of systems models we can immediately see whether a system is time-varying or not. If the time t appears explicitly in the function description of $f(x, u)$ or $g(x, u)$ then the system is time-varying, otherwise it is time-invariant.

Example 5.8

Consider the linearized state equation (5.67), (5.71) from which we obtain $f(\Delta x, \Delta u) = -2 \sin(t) \Delta x + \Delta u$. Because t explicitly appears in the description of f , due to the term $\sin(t)$, the linearized state equation, and so the associated system, is time varying. All the other systems considered so far, except for those in the previous section, are time-invariant.

Observe that in general x, u are time-functions. So instead of $f(x, u)$, $g(x, u)$ we may write $f(x(t), u(t))$, $g(x(t), u(t))$. From this notation it is clear that f, g are also time-functions. The vector functions f, g are said to depend implicitly on time and explicitly on x, u . The function $f(\Delta x, \Delta u) = -2 \sin(t) \Delta x + \Delta u$ in this example depends explicitly on Δx , Δu and also explicitly on t . To properly indicate this, we should now actually write $f(\Delta x, \Delta u, t) = -2 \sin(t) \Delta x + \Delta u$. Similarly, the general state-space notation of time-varying systems becomes,

$$\frac{dx}{dt} = f(x, u, t), \quad y = g(x, u, t), \quad f, x \in R^{n_x}, \quad u \in R^{n_u}, \quad g, y \in R^{n_y} \quad (5.72)$$

Exercise 5.11

A very simple example of a time-varying system is a rocket that burns a fixed amount of fuel per time unit due to which its mass is decreasing linearly with time. Presuming the rocket moves straight it may be seen as a mass with a force acting on it, see Figure 3-4. Note that the mass is not constant now but decreasing linearly with time,

$$m = c_1 - c_2 t \quad (5.73)$$

where c_1, c_2 are two constants. Modify the original state-space model (3.37), (3.38), (3.74), (3.75), to describe this time dependence of the mass and show that the resulting state-space model is time-varying.

5.10 Summary

- To determine several system properties first write the system in state-space form. In other words, determine $f(x, u)$, $g(x, u)$.
- To determine whether a system is linear first check whether $f(0, 0) = 0$, $g(0, 0) = 0$. If not the system is not linear. Next compute $A = \frac{\partial f}{\partial x}$, $B = \frac{\partial f}{\partial u}$, $C = \frac{\partial g}{\partial x}$, $D = \frac{\partial g}{\partial u}$. Check whether all these matrices do not depend on x, u . As soon as one of the matrix elements does depend on x, u the system is not linear. If the system is linear then it is represented by $\frac{dx}{dt} = Ax + Bu$, $y = Cx + Du$. If the linear system is time-varying at least one of the matrices A, B, C, D depends explicitly on time.
- To check whether a steady state x^{ss}, u^{ss} is stable, compute the A matrix of the linearized system around this steady state: $A = \frac{\partial f}{\partial x} \Big|_{x=x^{ss}, u=u^{ss}} \in R^{n_x \times n_x}$. Compute the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{n_x}$ of this matrix and check whether the real part of each eigenvalue is negative. If so both the linearized system and the associated steady state are stable, otherwise they are not. If the steady state is stable without the use of any control corrections small arbitrary state deviations (perturbations) tend to zero ultimately. If a linear system is stable, then with zero control any state ultimately converges to the zero state.
- A steady state is called stabilisable if with the use of control corrections small arbitrary state deviations (perturbations) can be controlled to zero ultimately.
- A system is called linear in the state x if the matrices $A = \frac{\partial f}{\partial x}$, $C = \frac{\partial g}{\partial x}$ do not depend on x . Similarly, a system is called linear in the control u if the matrices $B = \frac{\partial f}{\partial u}$, $D = \frac{\partial g}{\partial u}$ do not depend on u . A system is called linear in the parameters p if the matrices $E = \frac{\partial f}{\partial p}$, $F = \frac{\partial g}{\partial p}$ do not depend on p .

- A system is called time-varying if the time t appears explicitly in the function description of f or g .

- The linearized system about an optimal state and control trajectory $x^*(t), u^*(t)$

characterised by $A(t) = \frac{\partial f}{\partial x} \Big|_{x(t)=x^*(t), u=u^*(t)}$, $B(t) = \frac{\partial f}{\partial u} \Big|_{x(t)=x^*(t), u=u^*(t)}$, $C(t) = \frac{dg}{dx} \Big|_{x(t)=x^*(t), u=u^*(t)}$,

$D(t) = \frac{dg}{du} \Big|_{x(t)=x^*(t), u=u^*(t)}$ is a time-varying linear system that is used for the design of a

feedback controller in an optimal control system, see Figure 5-4. Any linearized system has the same order as the original system because their state vectors $\Delta x, x$ have the same dimension.

- A steady state (static) approximation of a state variable x_i is obtained by setting to zero it's associated state derivative dx_i / dt . This turns the state x_i into an algebraic function of other states, parameters and inputs.
- The state-space representation of interconnected systems can be obtained from the state-space representations of the systems that are interconnected.

