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Temporal and differential stabilizability and detectability of piecewise constant rank systems

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SUMMARY

In a past note we drew attention to the fact that time-varying continuous-time linear systems may be *temporarily* uncontrollable and unreconstructable and that this is vital knowledge to both control engineers and system scientists. Describing and detecting the temporal loss of controllability and reconstructability require considering piecewise constant rank (PCR) systems and the differential Kalman decomposition. In this note for conventional as well as PCR systems *measures of temporal and differential stabilizability and detectability* are developed. These measures indicate to what extent the temporal loss of controllability and reconstructability may lead to temporal instability of the closed-loop system when designing a static state or dynamic output feedback controller. It is indicated how to compute the measures from the system matrices. The importance of our developments for control system design is illustrated through three numerical examples concerning LQ and LQG perturbation feedback control of a non-linear system about an optimal control and state trajectory. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Feedback control design and stability analysis of non-linear systems along trajectories are often performed using the linearized dynamics about the trajectory [1]. If the trajectory is time-varying the linearized model is generally *time-varying*. If in addition the non-linear dynamics or the controls are non-smooth, i.e. in the case of bang-bang or digital control, the *structure* of the time-varying linearized system may no longer be constant. If no non-smoothness is present changes of structure do not occur but may *almost* occur. For control system design this is vital information since this structure reveals the *temporal loss* of controllability and reconstructability of the linearized system. They in turn may lead to *temporal instability* of the closed-loop system. This note develops *measures* of temporal stability of time-varying linear systems over arbitrary finite time intervals, notably intervals where controllability or reconstructability is lost temporarily. Associated with this, measures of *temporal and differential stabilizability* and *temporal and differential detectability* are developed. Through two numerical examples the importance of these measures for control system

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design are illustrated. The examples deal with the design of LQ and LQG perturbation feedback controllers, based on linearized dynamics, to control a nonlinear system along a prescribed optimal control and state trajectory.

Temporal stability may sound as a contradiction because formally stability relates to behavior when time tends to infinity. However, in one of his early seminal papers [2] Kalman together with Bertram already proposed measures of stability over finite time intervals (p. 386). Intuitively stability relates to growth of the system state. Intuitively over intervals where the state grows we call the system temporal unstable and over intervals where the state decays, we call the system temporal stable. This intuition is formalized by the temporal stability property proposed in this note. This property is derived from a *measure* of temporal stability also proposed in this note that measures the maximum growth of the state over an arbitrary interval. Our concept of stability over a finite time interval differs from what is called finite-time stability [3, 4]. The reason we make a different choice is that our measures, their computation and the associated control system designs come down to solving standard LQ problems. The standard LQ problems are of a special type called cheap control LQ problems [5, 6]. They are characterized by a control penalty that tends to zero. The computations associated with finite-time stability concern matrix inequalities [3, 4]. These computations are generally much more complicated as are the associated control system designs.

The description and detection of temporal linear system structure require considering conventional as well as *piecewise constant rank systems* (PCR systems) [7, 8]. PCR systems also arise as a result of balanced truncation of time-varying linear systems [9]. The differential Kalman decomposition (d-Kalman decomposition) developed in [7, 8] detects temporal linear system structure. Among other things it detects the intervals where the *temporal loss* of controllability or reconstructability occurs. These intervals are the interesting ones to apply the temporal and differential stabilizability and detectability measures introduced in this paper to. They *measure* to what extend the loss of controllability and reconstructability is problematic when it comes to feedback control system design.

The paper is organized as follows. Section 2 presents an example illustrating an optimal control and state trajectory producing linearized dynamics about the trajectory that are temporal uncontrollable. Section 3 presents important results relating to PCR systems and the associated d-Kalman decomposition, introduced in [7, 8]. They are needed to derive the main results, presented in Section 4. Section 5 presents two motivating examples illustrating how these results decide on the temporal stability obtainable by LQ and LQG perturbation feedback controllers. Section 6 discusses some important and delicate numerical issues whereas Section 7 concludes the paper.

2. ILLUSTRATIVE EXAMPLE

Example 1

Consider the following optimal control problem. Given the non-linear system:

$$\begin{gathered}\dot{x}(t) = f(x(t), u(t)), \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ f(x(t), u(t)) = \begin{bmatrix} -x_1(t) - x_2(t) + u(t) + 0.05 x_1(t) u(t) \\ x_1(t) x_2(t) \end{bmatrix},\end{gathered} \tag{1}$$

with initial conditions,

$$x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}, \tag{2}$$

find the bounded control,

$$-10 \leqslant u(t) \leqslant 10, \tag{3}$$

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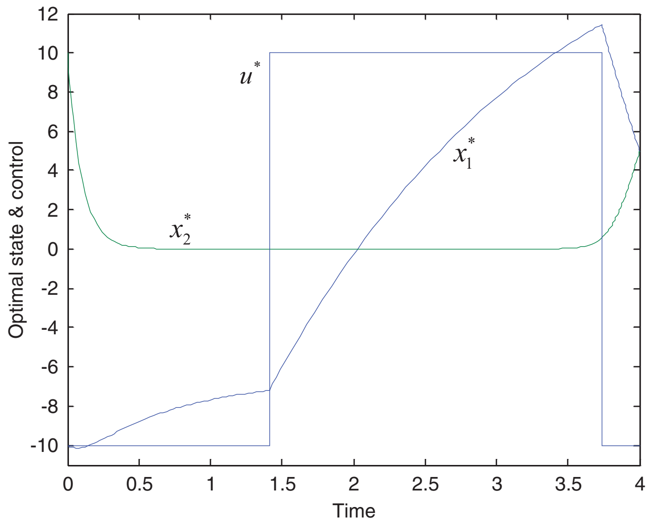

Figure 1. Optimal control and state trajectory Example 1.

that minimizes the performance measure,

$$J = (x_1(4) - 5)^2 + (x_2(4) - 5)^2 + \int_0^4 x_2^2(t)\,\mathrm{d}t. \tag{4}$$

The performance measure (4) promotes $x_2(t)$, $t \in [0, 4]$ to be close to zero while it also promotes x_1 and x_2 to be close to 5 at the final time 4.

The optimal control that switches at times 1.4109 and 3.7372 and the optimal state trajectory will be denoted by the superscript*. They are plotted in Figure 1.

As expected, during most of the time, x_2^* is *very close* to zero. Observe that since $f(x, u)$ in (1) is analytic, over the time intervals $(0, 1.0192)$, $(1.0192, 3.818)$, $(3.818, 4)$ where the optimal bang-bang control is constant $x_2^*(t)$ is analytic. Therefore, $x_2^*(t)$ cannot become *exactly* zero over any interval. Consider the time-varying linearized system about the trajectory,

$$\partial\dot{x}(t) = A(t)\partial x(t) + B(t)\partial u(t), \quad t \in [0, 4], \tag{5}$$

where $\partial x(t) = x(t) - x^*(t)$, $\partial u(t) = u(t) - u^*(t)$ are deviation (perturbation) variables and,

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x(t)=x^*(t)\\u(t)=u^*(t)}} = \begin{bmatrix} -1+0.05u^* & -1 \\ x_2^*(t) & x_1^*(t) \end{bmatrix}, \quad B(t) = \left.\frac{\partial f}{\partial u}\right|_{\substack{x(t)=x^*(t)\\u(t)=u^*(t)}} = \begin{bmatrix} 1+0.05x_1^* \\ 0 \end{bmatrix}. \tag{6}$$

From Equations (5), (6) observe that over intervals where $x_2^*(t) = 0$, according to [8], the linearized system is temporal uncontrollable. Owing to this over intervals where x_2^* is *very close* to zero the system is *almost* temporal uncontrollable. Applying the temporal stability and stabilizability measures introduced in this paper to these intervals reveals to what extend temporal uncontrollability leads to temporal instability of static state and dynamic output perturbation feedback control systems. The description and detection of temporal uncontrollability require considering conventional as well as *PCR systems* [7, 8]. These are introduced in the next section.

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3. CONSTANT RANK AND PCR SYSTEMS

Consider a conventional time-varying continuous-time linear system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \quad t \in (t_0, t_f), \quad -\infty \leqslant t_0 < t_f \leqslant +\infty, \end{aligned} \tag{7}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the bounded input and $y(t) \in R^l$ is the output of the system. $A(t)$, $B(t)$ and $C(t)$ are real bounded matrices of appropriate dimension. Denote the system (7) by $(A(t), B(t), C(t))$. The differential controllability (d-controllability) grammian and the differential reconstructability (d-reconstructability) grammian associated with $(A(t), B(t), C(t))$ are given by, respectively [10],

$$\begin{aligned} W_t &= C_j(t)C_j^{\mathrm{T}}(t), \quad C_j(t) = [P_0(t) \;\; P_1(t) \;\; \ldots \;\; P_j(t)], \\ P_0(t) &= B(t), \quad P_{k+1}(t) = -A(t)P_k(t) + \dot{P}_k(t), \quad k = 0, 1, \ldots, j-1, \end{aligned} \tag{8}$$

with j the smallest value such that $\mathrm{rank}(C_{j+1}(t)) = \mathrm{rank}(C_j(t))$ and

$$\begin{aligned} M_t &= O_k^{\mathrm{T}}(t)O_k(t), \quad O_k = [S_0^{\mathrm{T}}(t) \;\; S_1^{\mathrm{T}}(t) \;\; \ldots \;\; S_k^{\mathrm{T}}(t)]^{\mathrm{T}}, \\ O_0(t) &= C(t), \quad O_{l+1}(t) = O_l(t)A(t) + \dot{O}_l(t), \quad l = 0, 1, \ldots, k-1, \end{aligned} \tag{9}$$

with k the smallest value such that $\mathrm{rank}(O_k(t)) = \mathrm{rank}(O_{k+1}(t))$. In Equation (8) $C_j(t)$ represents the d-controllability matrix and in Equation (9) $O_j(t)$ represents the d-reconstructability matrix.

Definition 1
The system $(A(t), B(t), C(t))$ is called *constant rank* (CR) if W_t, M_t exist and both have constant rank for all $t \in (t_0, t_f)$.

If $(A(t), B(t), C(t))$ results from a linearization about a trajectory where the control is discontinuous, such as bang-bang or digital controls, then $A(t), B(t), C(t)$ may be discontinuous due to the discontinuities in the control. This happens in Example 1 as can be seen from Equation (6). Each moment $A(t), B(t), C(t)$ are discontinuous, from (8), (9), the associated grammians W_t, M_t do not exist.

Definition 2 (*see also, van Willigenburg and De Koning* [8])
PCR systems (PCR systems) are defined for times $t \in \mathrm{T}$, $\mathrm{T} = \{(t_i, t_{i+1}), \;\; i = 0, 1, \ldots, N-1\}$, $t_{i+1} > t_i$, $t_N = t_f$. Within each open interval (t_i, t_{i+1}), $i = 0, 1, \ldots, N-1$ the system is described by Equation (7). Moreover W_t, M_t as defined by Equations (8), (9) exist and have constant rank. The state transitions from one open interval to the next are described by:

$$x(t_i^+) = A_i x(t_i^-), \quad i = 1, 2, \ldots, N-1. \tag{10}$$

A_i, $i = 1, 2, \ldots, N-1$ are real bounded possibly non-square matrices. A PCR system is denoted by $(A(t), B(t), C(t), A_i, t_i, N)$ where $t \in \mathrm{T}$.

From Definition 2 observe that Equation (10) distinguishes PCR systems from ordinary ones. Equation (10) enables the state to be discontinuous at the isolated times t_i, $i = 1, 2, \ldots, N-1$. Moreover, these isolated times are excluded from the PCR systems time-domain T. In addition, since A_i in Equation (10) need not be square, this equation allows for changes of the state dimension of a PCR system. Let I_n denote the identity matrix of dimension n.

Definition 3
A PCR system $(A(t), B(t), C(t), A_i, t_i, N)$ is called *PCR identical* to a conventional system $(A'(t), B'(t), C'(t))$ if $A(t) = A'(t)$, $B(t) = B'(t)$, $C(t) = C'(t)$, $t \in \mathrm{T}$ and $A_i = I_n$, $i = 1, 2, \ldots, N-1$.

Definition 4
A PCR system $(A(t), B(t), C(t), A_i, t_i, N)$ is called *PCR identical* with another PCR system $(A'(t), B'(t), C'(t), A_i', t_i', N')$ if $t_0 = t_0'$, $t_N = t_{N'}'$ and $\forall t \in \mathrm{T} \cap \mathrm{T}'$, $A'(t) = A(t)$, $B(t) = B'(t)$,

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$C(t) = C'(t)$, and $\forall t'_i \in \mathrm{T}$, $A'_i = I_{n'_i}$ and $\forall t_i \in \mathrm{T}'$, $A_i = I_{n_i}$ where n_i and n'_i are the state dimensions at t_i^+ and t'^+_i of the associated PCR system.

Theorem 1
PCR identical systems have time domains and associated *input-state-output* behaviors that are identical except at isolated times.

Proof
According to Definition 2 over each sampling interval (t_i, t_{i+1}) a PCR system behaves as an ordinary conventional time-varying linear constant rank system. Then Theorem 1 follows directly from Definition 3, Definition 4 and the fact that we assumed the input and system matrices to be bounded. □

Let $T(t) \in R^{n \times n}$ represent a non-singular matrix that is defined and continuously differentiable over the intervals (t_i, t_{i+1}), $t_{i+1} > t_i$, $i = 0, 1, \ldots, N-1$ associated with a PCR system $(A(t), B(t), C(t), A_i, t_i)$. Consider the associated state basis transformation,

$$\begin{aligned} x'(t) &= T(t)x(t), \\ A'(t) &= T(t)A(t)T^{-1}(t) + \dot{T}(t)T^{-1}(t), \\ B'(t) &= T(t)B(t), \quad C'(t) = C(t)T^{-1}(t), \quad t \in (t_i, t_{i+1}), \quad i = 0, 1, \ldots, N-1, \\ A'_i &= T(t_i^+)A_i T^{-1}(t_i^-), \quad i = 1, 2, \ldots, N-1, \end{aligned} \tag{11}$$

where the prime denotes system quantities obtained after the transformation.

Definition 5
A PCR system $(A(t), B(t), C(t), A_i, t_i, N)$ is called *PCR equivalent* with another PCR system $(A'(t), B'(t), C'(t), A'_i, t'_i, N')$ if $(A(t), B(t), C(t), A_i, t_i, N)$ can be obtained through a state basis transformation (11) applied to a PCR system that is PCR identical to $(A'(t), B'(t), C'(t), A'_i, t'_i, N')$.

Theorem 2
PCR equivalent systems have time domains and associated *input–output* behaviors that are identical except at isolated times.

Proof
Follows directly from Definition 5, Theorem 1 and its proof. □

Starting from $(A(t), B(t), C(t))$, described by Equation (7), the *differential Kalman decomposition* (d-Kalman decomposition) introduced in [7, 8] *detects* the isolated times t_i, $i = 1, 2, \ldots, N-1$ where the associated grammians W_t and/or M_t change rank. This detection results in a PCR identical system $(A(t), B(t), C(t), A_i, t_i, N)$. Next the d-Kalman decomposition computes a *canonical* system representation [7, 8] that is PCR equivalent. To obtain this equivalent canonical representation, the associated state basis transformation $T(t)$ in (11) is generally discontinuous at t_i, $i = 1, 2, \ldots, N-1$. In general, these discontinuities lead to discontinuities of the state at times t_i, $i = 1, 2, \ldots, N-1$ leading naturally to the definition of PCR systems. The equivalent canonical representation obtained from the d-Kalman decomposition decomposes the state into four parts. The dimensions n_a, n_b, n_c, n_d of these four parts satisfy,

$$n_a + n_b = rank(W_t), \quad n_b + n_d = rank(M_t), \quad n_b = rank(W_t M_t), \quad n_a + n_b + n_c + n_d = n. \tag{12}$$

Over each separate open interval $t \in (t_i, t_{i+1})$, $i = 0, 1, \ldots, N-1$ these dimensions are constant. Assuming a linear system to have this piecewise constant temporal linear structure appears to be the *weakest* assumption to prevent the occurrence of highly pathological linear systems [8]. If $t \in (t_i, t_{i+1})$, rank$(W_t) < n$, the system is temporal uncontrollable over (t_i, t_{i+1}). If $t \in (t_i, t_{i+1})$, rank$(M_t) < n$, the system is temporal unreconstructable over (t_i, t_{i+1}) [8].

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4. TEMPORAL AND DIFFERENTIAL STABILIZABILITY AND DETECTABILITY

After detection of the time-instants t_i, $i=1,2,\ldots,N-1$ where the system structure changes, i.e. where the grammians W_t and/or M_t change rank, we can analyse PCR systems over the open time-intervals (t_i,t_{i+1}), $i=0,1,2,\ldots,N-1$. Over each separate open time-interval (t_i,t_{i+1}), $i=0,1,2,\ldots,N-1$ the system is of the *conventional* linear time-varying type and moreover *is constant rank* implying that its structure does not change. The latter implies that controllability, reachability, d-controllability and d-reachability are all equivalent [7, 8, 10]. Dually reconstructability, observability, d-reconstructability and d-observability are also equivalent. In this sense, the analysis is easy and conventional.

Stabilizability is a property that relates entirely to the uncontrollable part of a system. A general approach to determine stabilizability is to extract this uncontrollable part that is autonomous by means of a Kalman decomposition and to determine its stability. In the case of conventional time-varying linear systems, the extraction faces a numerical problem. This problem relates to the requirement for $T(t)$ in Equation (11) to be continuous and continuously differentiable. This issue we plan to address in another paper. In addition, as will become clear in this section, application of a state basis transformation changes temporal stability and stabilizability properties. To recover them we need to transform back to the original state basis. The stabilizability analysis presented in this section does not face these problems since it does not require transformation of the state basis. It relies fully on well-established standard LQ theory applied to the original system representation. Therefore, the associated numerical computations are also very efficient.

The stabilizability analysis in this section is *unconventional* in the sense that stability, stabilizability and detectability over *finite time intervals* is required. Stability over an interval relates to growth of the magnitude of the state over this interval. Throughout this paper $\|\cdot\|$ denotes the matrix 2 norm. For vectors this amounts to the L2 norm. In the next section we will demonstrate how to compute numerically the temporal and differential stabilizability and detectability measures presented in this section, using only evaluations of the system matrices.

Remark 1
In what follows (t_i,t_{i+1}) will always denote one of the time intervals (t_i,t_{i+1}), $i=0,1,2,\ldots,N-1$ associated with a PCR system $(A(t), B(t), C(t), A_i, t_i)$.

Definition 6
An autonomous PCR system $(A(t), 0, 0, A_i, t_i)$ is called *temporal stable over* (t_i,t_{i+1}) if for any $x(t_i^+)\neq 0$, $\|x(t_{i+1}^-)\|/\|x(t_i^+)\|<1$.

Loosely speaking, according to Definition 6 an autonomous PCR system is called temporal stable over (t_i,t_{i+1}) if for any initial state the magnitude of the associated terminal state is smaller than that of the initial state. An important difference between our definition and other finite-time stability concepts [3] is that ours does not impose any restrictions on the magnitude of the state inside the interval. The advantage of Definition 6 is that it matches LQ control design as opposed to finite-time stability that relates to control system design using matrix inequalities [3] that is generally much more complicated.

Definition 7
Associate with Definition 6 the following *temporal stability measure*:

$$\rho(t_i,t_{i+1})=\max_{x(t_i^+)\neq 0}\left(\frac{\|x(t_{i+1}^-)\|^2}{\|x(t_i^+)\|^2}\right)\geqslant 0. \tag{13}$$

Observe that $\rho(t_i,t_{i+1})$ in Definition 7 is the largest possible ratio $\|x(t_{i+1}^-)\|^2/\|x(t_i^+)\|^2$. This ratio matches the largest possible ratio $\|x(t_{i+1}^-)\|/\|x(t_i^+)\|$ in Definition 6. Therefore, $\rho(t_i,t_{i+1})$ is indeed a measure of temporal stability associated with Definition 6. The smaller $\rho(t_i,t_{i+1})$, the larger temporal stability. It will become clear that the squares in Equation (13) are needed to achieve compatibility with LQ control computations.

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Theorem 3
An autonomous PCR system $(A(t), 0, 0, A_i, t_i)$ is temporal stable over the open time interval (t_i, t_{i+1}) if and only if,

$$\rho(t_i, t_{i+1}) = \|\Phi^{\mathrm{T}}(t_i^+, t_{i+1}^-)\Phi(t_i^+, t_{i+1}^-)\| < 1, \tag{14}$$

where Φ represents the state transition matrix of the associated autonomous system.

Proof
Because Theorem 3 applies to autonomous systems,

$$x(t_{i+1}^-) = \Phi(t_i^+, t_{i+1}^-)x(t_i^+). \tag{15}$$

Using Equation (15) the temporal stability measure (13) becomes,

$$\begin{aligned}
\rho(t_i, t_{i+1}) &= \max_{x(t_i^+)\neq 0}\left(\frac{\|\Phi(t_i^+, t_{i+1}^-)x(t_i^+)\|^2}{\|x(t_i^+)\|^2}\right)\\
&= \max_{x(t_i^+)\neq 0}\left(\frac{x^{\mathrm{T}}(t_i^+)\Phi^{\mathrm{T}}(t_i^+, t_{i+1}^-)\Phi(t_i^+, t_{i+1}^-)x(t_i^+)}{x^{\mathrm{T}}(t_i^+)x(t_i^+)}\right)\\
&= \|\Phi^{\mathrm{T}}(t_i^+, t_{i+1}^-)\Phi(t_i^+, t_{i+1}^-)\|.
\end{aligned} \tag{16}$$

The last equality in Equation (16) holds because $\Phi^{\mathrm{T}}(t_i^+, t_{i+1}^-)\Phi(t_i^+, t_{i+1}^-) > 0$ and symmetric. Theorem 3 now follows from (16), Definition 6, Definition 7 and,

$$\|x(t_{i+1}^-)\|/\|x(t_i^+)\| < 1 \Leftrightarrow \|x(t_{i+1}^-)\|^2/\|x(t_i^+)\|^2 < 1. \tag{17}$$

Stabilizability over a finite time-interval relates to the ability to stabilize the system over that interval by means of control. □

Definition 8
Associate with Definition 6 and Definition 7 the following *temporal stabilizability measure* that applies to PCR systems $(A(t), B(t), C(t), A_i, t_i)$ considered over the interval (t_i, t_{i+1}):

$$\rho_{\min}(t_i, t_{i+1}) = \max_{x(t_i^+)\neq 0}\left(\frac{\min_{u(t)|x(t_i^+)}\|x(t_{i+1}^-)\|^2}{\|x(t_i^+)\|^2}\right) \geqslant 0, \tag{18}$$

where $u(t)|x(t_i^+)$ indicates a control law dependent on $x(t_i^+)$.

Definition 9
$(A(t), B(t), C(t), A_i, t_i)$ is called *temporal stabilizable over* (t_i, t_{i+1}) if $\rho_{\min}(t_i, t_{i+1}) < 1$.

Theorem 4
$(A(t), B(t), C(t), A_i, t_i)$ temporal controllable over $(t_i, t_{i+1}) \Rightarrow \rho_{\min}(t_i, t_{i+1}) = 0 \Rightarrow$ $(A(t), B(t), C(t), A_i, t_i)$ temporal stabilizable over (t_i, t_{i+1}).

Proof
$(A(t), B(t), C(t), A_i, t_i)$ temporal controllable over $(t_i, t_{i+1}) \Leftrightarrow (A(t), B(t), C(t), A_i, t_i)$ d-controllable at all $t \in (t_i, t_{i+1})$ [8, 10]. Hence, at any time $t \in (t_i, t_{i+1})$ the state can be controlled to the zero state in an arbitrarily short time. This implies $\rho_{\min}(t_i, t_{i+1}) = 0$ and, according to Definition 9, temporal stabilizability over (t_i, t_{i+1}). □

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Remark 2
As with ordinary controllability and stabilizability, temporal controllability is a stronger property than temporal stabilizability.

To state the main theorem in this section consider the following parameterized continuous-time finite-horizon time-varying LQ problem [11].

Given the system,

$$\dot{x}(t)=A(t)x(t)+B(t)u(t), \quad t\in(t_i,t_{i+1}), \tag{19}$$

with initial state,

$$x(t_i^+), \tag{20}$$

find the control $u(t)$, $t\in(t_i,t_{i+1})$ that minimizes the cost function,

$$J_{LQ}(\varepsilon)=x^{\mathrm{T}}(t_{i+1}^-)Hx(t_{i+1}^-)+\int_{t_i^+}^{t_{i+1}^-}[x^{\mathrm{T}}(t)Q(t)x(t)+u^{\mathrm{T}}(t)R_\varepsilon(t)u(t)]\,\mathrm{d}t, \tag{21}$$

with,

$$H=I_n, \quad Q(t)=0, \quad R_\varepsilon(t)=\varepsilon I_m, \quad 0\leqslant\varepsilon\ll 1. \tag{22}$$

If $\varepsilon>0$ the Linear Quadratic control problem (19), (20)–(22) satisfies $H\geqslant 0$, $Q(t)\geqslant 0$, $R_\varepsilon(t)>0$. In this standard case it is well known [11] that the optimal control is given by,

$$u(t)=-L_\varepsilon(t)x(t), \quad L_\varepsilon(t)=R_\varepsilon^{-1}(t)B^{\mathrm{T}}(t)S_\varepsilon(t), \quad t\in(t_i,t_{i+1}). \tag{23}$$

and the minimum cost by,

$$J_{LQ}^*(\varepsilon)=x^{\mathrm{T}}(t_i^+)S_\varepsilon(t_i^+)x(t_i^+). \tag{24}$$

where $S_\varepsilon(t)$, $t\in(t_i,t_{i+1})$ is the solution of the matrix Riccati differential equation,

$$-\dot{S}_\varepsilon(t)=A^{\mathrm{T}}(t)S_\varepsilon(t)+S_\varepsilon(t)A(t)-S_\varepsilon(t)B^{\mathrm{T}}(t)R_\varepsilon^{-1}(t)B(t)S_\varepsilon(t)+Q(t), \quad S_\varepsilon(t_{i+1}^-)=H. \tag{25}$$

Theorem 5

$$S^*(t)=\lim_{\varepsilon\downarrow 0}S_\varepsilon(t), \tag{26}$$

exists, where $S_\varepsilon(t)$, $\varepsilon\downarrow 0$ satisfies the Matrix Riccati differential equation (25) with data as specified by Equation (22). Furthermore,

$$\rho_{\min}(t,t_{i+1})=\|S^*(t)\|, \quad t\in(t_i,t_{i+1}). \tag{27}$$

As a special case of (27) we obtain,

$$\rho_{\min}(t_i,t_{i+1})=\|S^*(t_i^+)\|. \tag{28}$$

Proof
First observe that in the parameterized LQ problem (19)–(24) we may replace the initial time t_i^+ by any $t'\in(t_i,t_{i+1})$. This also holds for the stabilizability measure $\rho_{\min}$. Next from Equations (21), (22) observe,

$$\min_{u(t)|x(t')}J_{LQ}(0)=\min_{u(t)|x(t')}(x^{\mathrm{T}}(t_{i+1}^-)x(t_{i+1}^-))=\min_{u(t)|x(t')}\|x(t_{i+1}^-)\|^2. \tag{29}$$

Now the key to proving (26), (27) is to prove that,

$$\min_{u(t)|x(t')}J_{LQ}(0)=\lim_{\varepsilon\downarrow 0}J_{LQ}^*(\varepsilon)=x^{\mathrm{T}}(t')S^*(t')x(t'). \tag{30}$$

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Suppose Equation (30) holds then from Equations (18), (29), (30),

$$\rho_{\min}(t', t_{i+1}) = \max_{x(t_i^+)\neq 0}\left(\frac{\min_{u(t)|x(t')}\|x(t_{i+1}^-)\|^2}{\|x(t')\|^2}\right) = \max_{x(t')}\left(\frac{x^{\mathrm{T}}(t')S^*(t')x(t')}{x^{\mathrm{T}}(t')x(t')}\right) = \|S^*(t')\|. \tag{31}$$

The last equality in Equation (31) holds because $S^*(t')\geqslant 0$ and symmetric. Hence, we are left to prove Equation (30). From Remark 1 recall that the time interval (t_i^+, t_{i+1}^-) is such that over this interval the system (19) has a constant structure. Therefore, according to the d-Kalman decomposition [8], the system (19) can be separated in a part that is d-controllable at all times and a remaining part that is autonomous. The contribution of the d-controllable system part to $\min_{u(t)|x(t')} J_{LQ}(0)$ is zero. The contribution to $J_{LQ}^*(\varepsilon)$ tends to zero as $\varepsilon \downarrow 0$. The contribution of the autonomous part to both $\min_{u(t)|x(t')} J_{LQ}(0)$ and $J_{LQ}^*(\varepsilon)$ is identical and independent of ε and $u(t)$, $t \in (t', t_{i+1}^-)$. Because the system matrices are assumed bounded this contribution is also finite. This proves the existence of the limit (26) as well as the equality (30). □

Remark 3
There are three reasons for considering $0<\varepsilon \ll 1$ in Equation (22), instead of $\varepsilon = 0$. Taking $0<\varepsilon \ll 1$, ε may be used to (1) keep the control within certain bounds that apply in practice and (2) as a numerical tolerance to prevent ill-conditioning of the computation of Equation (23), in which $L(t)\to\infty$ as $\varepsilon\downarrow 0$. In practice the selection of $0<\varepsilon\ll 1$ will be a compromise and $S_\varepsilon(t)$ will *approximate* $S^*(t)$, $t\in(t_i, t_{i+1})$. As a result all computations in this paper involving $S^*(t)$ will be approximations, although generally very good ones. Third $\varepsilon = 0$ leads to a singular LQ problem that is generally much more difficult. In addition, the solution to this singular problem is no longer guaranteed to be unique.

When analyzing control systems the state behavior over the entire interval (t_i, t_{i+1}) is generally of interest, not just the behavior at the initial time t_i^+ and the final time t_{i+1}^-. This behavior is partly considered by Equation (27) of Theorem 5 that determines the stabilizability measure for each sub interval (t, t_{i+1}), $t \in (t_i, t_{i+1})$. The following theorem introduces a differential stabilizability measure that applies to individual time instants.

Theorem 6
$-\mathrm{d}\|S^*(t)\|/\mathrm{d}t$ is a *differential stabilizability measure* (*d-stabilizability measure*) at time $t\in(t_i, t_{i+1})$.

Proof
Using Equation (27),

$$\rho_{\min}(t_i, t_{i+1}) = \int_{t_{i+1}^-}^{t_i^+} -\frac{\mathrm{d}\rho_{\min}(t, t_{i+1})}{\mathrm{d}t}\,\mathrm{d}t + \rho_{\min}(t_{i+1}, t_{i+1}) = \int_{t_{i+1}^-}^{t_i^+} -\frac{\mathrm{d}\|S^*(t)\|}{\mathrm{d}t}\,\mathrm{d}t + \|S^*(t_{i+1}^-)\|. \tag{32}$$

Hence, $-\mathrm{d}\|S^*(t)\|/\mathrm{d}t$ is the contribution at time $t\in(t_i, t_{i+1})$ to the temporal stabilizability measure $\rho_{\min}(t_i, t_{i+1})$ when integrating backward in time from t_{i+1}^- to t_i^+. □

Definition 10
$(A(t), B(t), C(t), A_i, t_i)$ is called *differentially stabilizable* (*d-stabilizable*) at time $t\in(t_i, t_{i+1})$ if $-\mathrm{d}\|S^*(t)\|/\mathrm{d}t<0$.

Theorem 7
$(A(t), B(t), C(t), A_i, t_i)$ temporal controllable over $(t_i, t_{i+1}) \Leftrightarrow (A(t), B(t), C(t), A_i, t_i)$ d-controllable at all times $t\in(t_i, t_{i+1}) \Rightarrow (A(t), B(t), C(t), A_i, t_i)$ d-stabilizable at all times $t\in(t_i, t_{i+1}) \Rightarrow$ $(A(t), B(t), C(t), A_i, t_i)$ temporal stabilizable over (t_i, t_{i+1}).

Proof
The equivalence is proved in the proof of Theorem 4. d-controllability at all times $t\in(t_i, t_{i+1})$ implies that any state can be controlled to the zero state in an arbitrary short time at all times $t\in(t_i, t_{i+1})$. This implies that $-\mathrm{d}\|S^*(t)\|/\mathrm{d}t$ can be given arbitrary large negative values at all times

$t \in (t_i, t_{i+1})$ which implies d-stabilizability at all times $t \in (t_i, t_{i+1})$ and also temporal stabilizability over (t_i, t_{i+1}) because $\|S^*(t_i^+)\|$ can be made arbitrarily small. □

Remark 4
As with ordinary controllability and stabilizability, d-controllability is a stronger property than d-stabilizability.

Because detectability is dual to stabilizability we can use our temporal stabilizability measure (18) as a temporal detectability measure for PCR systems $(A(t), B(t), C(t), A_i, t_i, N)$ when we apply (18) to the following system that is dual to (19):

$$\dot{x}'(t_{i+1}+t_i-t) = A^{\mathrm{T}}(t_{i+1}+t_i-t)x'(t_{i+1}+t_i-t)+C^{\mathrm{T}}(t_{i+1}+t_i-t)u'(t_{i+1}+t_i-t),$$
$$t \in (t_i, t_{i+1}). \tag{33}$$

Theorem 8 (*dual of Theorem 5*)
Dual to the temporal stabilizability measure (27) is the *temporal detectability measure*,

$$\sigma_{\min}(t, t_{i+1}) = \|P^*(t)\|, \quad t \in (t_i, t_{i+1}), \tag{34}$$

where $\sigma_{\min}$ is the dual of $\rho_{\min}$ and $P^*(t)$ the dual of $S^*(t)$, i.e.

$$P^*(t) = \lim_{\varepsilon \downarrow 0} P_\varepsilon(t), \quad t \in (t_i, t_{i+1}), \tag{35}$$

where $P_\varepsilon(t)$ satisfies the following Riccati matrix differential equation that is dual to (25) and therefore runs forward in time:

$$\dot{P}_\varepsilon(t) = A(t)P_\varepsilon(t) + P_\varepsilon(t)A^{\mathrm{T}}(t) - P_\varepsilon(t)C^{\mathrm{T}}(t)R_\varepsilon^{-1}(t)C(t)P_\varepsilon(t) + Q(t), \quad P_\varepsilon(t_i^+) = H. \tag{36}$$

with H, $Q(t)$, $R_\varepsilon(t)$ as in (22).

Definition 11 (*dual of Definition 9*)
$(A(t), B(t), C(t), A_i, t_i)$ is called *temporal detectable over* (t_i, t_{i+1}) if $\sigma_{\min}(t_i, t_{i+1}) < 1$.

Theorem 9 (*dual of Theorem 4*)
$(A(t), B(t), C(t), A_i, t_i)$ temporal reconstructable over $(t_i, t_{i+1}) \Rightarrow \sigma_{\min}(t_i, t_{i+1}) = 0 \Rightarrow (A(t), B(t), C(t), A_i, t_i)$ temporal detectable over (t_i, t_{i+1}).

Remark 5
As with ordinary reconstructability and detectability, temporal reconstructability is a stronger property than temporal detectability.

Theorem 10 (*dual of Theorem 6*)
$\mathrm{d}\|P^*(t)\|/\mathrm{d}t$ is a *differential detectability measure* (*d-detectability measure*) at time $t \in (t_i, t_{i+1})$.

Definition 12 (*dual of Definition 10*)
A PCR system is called *differentially detectable* (*d-detectable*) at time $t \in (t_i, t_{i+1})$ if $(\mathrm{d}\|P^*(t)\|/\mathrm{d}t) < 0$.

Theorem 11 (*dual of Theorem 7*)
$(A(t), B(t), C(t), A_i, t_i)$ temporal reconstructable over $(t_i, t_{i+1}) \Leftrightarrow (A(t), B(t), C(t), A_i, t_i)$ d-reconstructable at all times $t \in (t_i, t_{i+1}) \Rightarrow (A(t), B(t), C(t), A_i, t_i)$ d-detectable at all times $t \in (t_i, t_{i+1}) \Rightarrow (A(t), B(t), C(t), A_i, t_i)$ temporal detectable over (t_i, t_{i+1}).

Remark 6
As with ordinary reconstructability and detectability d-reconstructability is a stronger property than d-detectability.

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Remark 7
Having computed $S^*(t)$, $t \in (t_i, t_{i+1})$ the d-stabilizability measure $-\mathrm{d}\|S^*(t)\|/\mathrm{d}t$ is easily evaluated by computing the associated matrix norms and, e.g., finite differences. Dual arguments hold for $P^*(t)$, $t \in (t_i, t_{i+1})$, $\mathrm{d}\|P^*(t)\|/\mathrm{d}t$.

In the next section issues related to the numerical computation of the temporal and differential stabilizability and detectability measures presented in this section will be presented through a number of motivating examples. The numerical computations will be based entirely on evaluations of the system matrices.

5. MOTIVATING EXAMPLES

Optimal open-loop control combined with optimal LQ or LQG *perturbation* feedback control of non-linear systems [1] motivated our development of, and research into, PCR systems. Optimal LQ and LQG perturbation feedback control concerns the control to zero of state *perturbations* $\partial x(t) = x(t) - x^*(t)$ by means of control *corrections* $\partial u(t) = u(t) - u^*(t)$. The dynamics of these state perturbations and control corrections are approximately described by the linearized dynamic equations (5), (6). As demonstrated by Example 1, these time-varying linearized dynamics can be *temporal* uncontrollable. Since any LQ and LQG perturbation feedback controller design relies fully on these linearized dynamics the question arises whether these controllers can prevent temporal instability of the closed-loop system. Examples 2 and 3 in this section show *how* one can read from the temporal and differential stabilizabilty and detectability measures, proposed and computed in this paper, whether or not LQ and LQG perturbation feedback controllers are able to prevent temporal instability of the closed-loop system.

Example 2
Reconsider Example 1. The linearized dynamics (5), (6) associated with Example 1 are almost temporal uncontrollable when $x_2^*(t)$, plotted in Figure 1, is close to zero. In Example 1 temporal uncontrollability of the linearized dynamics (5), (6) occurs whenever $\mathrm{rank}(W_t) < n = 2$. We computed $\mathrm{rank}(W_t)$ numerically using Matlab. Figure 2 shows the result. The highest straight lines denote rank 2. The lowest straight line denotes rank 1. Over the interval $(1.37, 2.74)$ $\mathrm{rank}(W_t) = 1$ and the linearized dynamics (5), (6) are temporal uncontrollable. Taking $\varepsilon = 10^{-4}$ in Equation (22), see Remark 3, we computed the associated temporal stabilizability measure $\rho_{\min}(1.37, 2.74)$ to be approximately $0.61 < 1$. Therefore, according to Definition 9, the linearized system (5), (6) is temporal stabilizable over $(1.37, 2.74)$. Figure 2 also plots $\|S_\varepsilon(t)\|$, $t \in (1.37, 2.74)$ which, according to (26), (27), approximates the temporal stabilizability measure $\|S^*(t)\| = \rho_{\min}(t, 2.74)$ associated with the subintervals $(t, 2.74)$, $t \in (1.37, 2.74)$. For most subintervals, temporal stabilizability does not hold because $\|S_\varepsilon(t)\| > 1$. From Figure 2, $-\mathrm{d}\|S_\varepsilon(t)\|/\mathrm{d}t$ takes on large negative values within $(1.37, 2.03)$ and large positive values within $(2.03, 2.74)$. According to Definition 10 this implies that the linearized system (5), (6) is highly d-unstabilizable within $(2.03, 2.74)$. This however is compensated for by the high d-stabilizability within the interval $(1.37, 2.03)$ leading overall to temporal stabilizability over $(1.37, 2.74)$. As t increases from 1.37 towards 2.03 this compensation becomes less and the temporal stabilizability measure $\rho_{\min}(t, 2.74)$ associated with the subinterval $(t, 2.74)$ increases leading to temporal unstabilizability.

Observe that the intervals $(1.37, 2.03)$ and $(2.03, 2.74)$ mentioned in Example 2 coincide with $x_1^*(t) < 0$ and $x_1^*(t) > 0$ in Figure 1. This is to be expected because, according to Equation (6), whenever $x_2^*(t) = 0$, $x_1^*(t)$ equals the scalar system matrix of the uncontrollable autonomous subsystem. When $x_1^*(t) > 0$ the autonomous uncontrollable subsystem is unstable and when $x_1^*(t) < 0$ it is stable.

As to the temporal stability of the closed-loop system achievable by means of LQ perturbation feedback control the following may be concluded. Within the interval $(1.37, 2.74)$ temporal instability of the closed-loop system may occur due to temporal uncontrollability. Over the interval $(1.37, 2.03)$ temporal instability of the closed-loop system can be prevented due to d-stabilizability.

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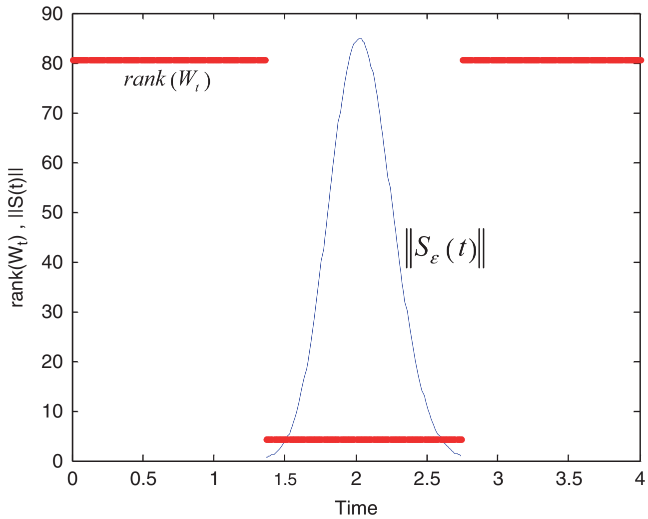

Figure 2. rank(W_t), $t \in (0, 4)$, $\|S_\varepsilon(t)\|$, $t \in (1.37, 2.74)$.

Over the interval $(2.03, 2.74)$ temporal instability cannot be prevented due to d-unstabilizability. Hence, Example 2 demonstrates that although over the full interval $(1.37, 2.03)$ the linearized system may be temporal stabilizable, this neither guarantees d-stabilizability over the entire interval nor temporal stabilizability over subintervals.

Interestingly, at time $t = 1.4109$ u_1 switches and W_t is formally undefined. On the other hand, rank$(W_t) = 1$ just before and after $t = 1.4109$. Therefore, the structure of the linearized system does not change and $t = 1.4109$ is not detected by the d-Kalman decomposition. As argued before this has no practical consequences.

Example 3
Reconsider Example 1. Extend the system (1)–(3) with the following output equation:

$$y = g(x, u) = (x_1 + 12)x_2. \tag{37}$$

The linearized dynamics of the system (1)–(3), (37) about the optimal trajectory $u^*(t)$, $x^*(t)$ are now described by (7) with $t_0 = 0$, $t_f = 4$ where $A(t)$ and $B(t)$ are specified by Equations (5), (6) and,

$$C(t) = \left.\frac{\partial g}{\partial x}\right|_{\substack{x(t)=x^*(t)\\ u(t)=u^*(t)}} = \begin{bmatrix} x_2^*(t) \\ x_1^*(t) + 12 \end{bmatrix}. \tag{38}$$

Observe from Equations (6), (38) that, besides the properties of the linearized system already mentioned in Example 2, the linearized system is now also almost temporal unreconstructable when $x_2^*(t)$, plotted in Figure 1, is close to zero. Then rank$(M_t) < n = 2$. We computed rank(M_t) numerically using Matlab. Figure 3 shows the result. The highest straight lines denote rank 2. The lowest straight line denotes rank 1. Over the interval $(1.63, 2.67)$ rank$(M_t) = 1$ and the linearized system (5), (6), (38) is temporal unreconstructable. Taking $\varepsilon = 10^{-4}$ in (22), see Remark 3, we computed the associated temporal detectability measure $\sigma_{\min}(1.63, 2.67)$ to be approximately $0.35 < 1$. Therefore, according to Definition 11, the linearized system (5), (6), (38) is temporal detectable over $(1.63, 2.67)$. Figure 3 also plots $\|P_\varepsilon(t)\|$, $t \in (1.63, 2.67)$. Since $\|P_\varepsilon(t)\| < 1$ at all times $t \in (1.63, 2.67)$, from (34), (35), over every subinterval $(1.63, t)$, $t \in (1.63, 2.67)$ the system is temporal detectable. In addition, from Figure 3, $\mathrm{d}\|P_\varepsilon(t)\|/\mathrm{d}t$ takes on negative values over the entire interval $(1.63, 2.67)$. From Definition 12 this indicates d-detectability at all times $t \in (1.63, 2.67)$.

As to the temporal stability of the closed-loop system achievable by means of LQG perturbation feedback control the following may be concluded. Within the intervals $(1.63, 2.67)$ and $(1.37, 2.74)$

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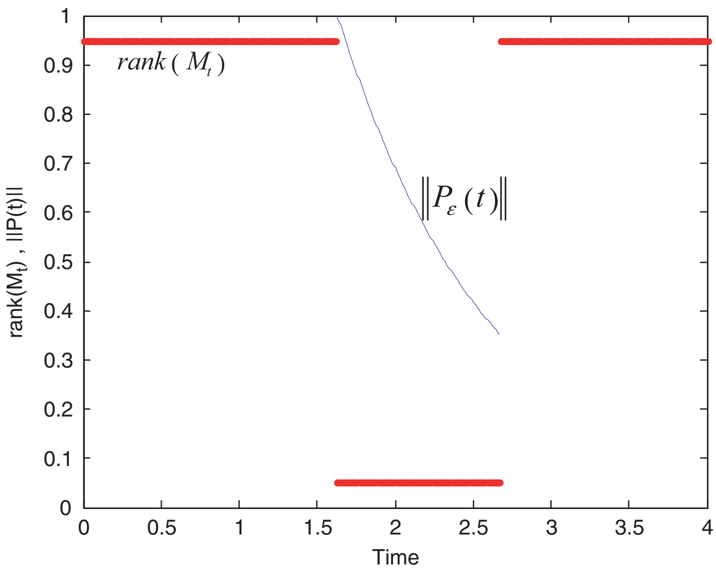

Figure 3. rank(M_t), $t\in(0,4)$, $\|P_\varepsilon(t)\|$, $t\in(1.63, 2.67)$.

temporal instability of the closed-loop system may occur due to temporal unreconstructability and temporal uncontrollability, respectively. Over the interval $(1.37, 2.03)$ temporal instability can be prevented due to d-stabilizability and d-detectability. Within the interval $(2.03, 2.74)$ it cannot due to d-unstabilizability.

The computations made in Examples 2 and 3 raise several numerical issues. These relate to selecting the tolerance used to compute rank(W_t) and dually rank(M_t) as well as the tolerance ε used to compute $\rho_{\min}$ and dually $\sigma_{\min}$, see also Remark 3. Furthermore, the computation of W_t itself, and dually M_t, is not straightforward. These issues are addressed in the next section.

6. NUMERICAL ISSUES

For general non-linear systems, knowing $f(x,u)$, $g(x,u)$ analytically, as in (1), (38), and having computed $u^*(t)$, $x^*(t)$ numerically, as in Example 1, the system matrices $A(t)$, $B(t)$, $C(t)$ of the linearized system (5), (6), (38) can be evaluated very accurately using automatic differentiation [12]. Evaluating time derivatives of $A(t)$, $B(t)$, $C(t)$ needed to compute $\dot{P}_k(t)$ in Equation (8), and $\dot{O}_l(t)$ in Equation (9), cannot be performed in this manner. That would require analytic expressions for $u^*(t)$, $x^*(t)$ that are generally unavailable. Therefore, in general derivatives $\dot{P}_k(t)$ and $\dot{O}_l(t)$ must be computed using finite differences that are less accurate. The accuracy moreover decreases rapidly as the order of the derivatives in Equations (8), (9) becomes higher. This presents a serious numerical problem since the determination of rank(W_t) and rank(M_t), that determine temporal controllability and temporal reconstructability, demand $\dot{P}_k(t)$ and $\dot{O}_l(t)$ to be highly accurate. Therefore, a numerical scheme would be preferred that only uses *evaluations* of $A(t)$, $B(t)$, $C(t)$.

Such a scheme can be obtained if, instead of the d-controllability grammian at time τ, we compute the reachability grammian over a small time-interval $(\tau-\Delta\tau, \tau+\Delta\tau)$ presuming the system structure does not change over $(\tau-\Delta\tau, \tau+\Delta\tau)$. Presuming the latter, d-controllability, controllability, d-reachability and reachability are all equivalent over this time interval [8, 10] and we may compute any of the associated grammians. To compute rank(W_t) in Figure 2 we used,

$$W_\tau = W(\tau-\Delta\tau, \tau+\Delta\tau) \tag{39}$$

where $W(\tau-\Delta\tau, \tau+\Delta\tau)$ is the reachability grammian associated with the interval $(\tau-\Delta\tau, \tau+\Delta\tau)$ with $\Delta\tau$ small. The reachability grammian $W(\tau-\Delta\tau, \tau+\Delta\tau)$ in (39) satisfies,

$$\begin{aligned} \dot{W}(\tau-\Delta\tau, t) &= A(t)W(\tau-\Delta\tau, t)+W(\tau-\Delta\tau, t)A^{\mathrm{T}}(t)+B(t)B^{\mathrm{T}}(t), \\ W(\tau-\Delta\tau, \tau-\Delta\tau) &= 0, \quad t\in(\tau-\Delta\tau, \tau+\Delta\tau) \end{aligned} \tag{40}$$

In Figure 2 we took,

$$\tau=iT_s, \quad \Delta\tau=nT_s, \quad i=n,n+1,n+2,\ldots,t_f/T_s-n, \quad T_s=0.01 \tag{41}$$

We numerically integrated equation (40) using Euler forward integration while guaranteeing symmetry and non-negativeness of $W(\tau-\Delta\tau, \tau+\Delta\tau)$ during each time step,

$$\begin{aligned} W(\tau-\Delta\tau, t+\Delta t) = W(\tau-\Delta\tau, t)&+\Delta t(A(t)W(\tau-\Delta\tau, t)+(A(t)W(\tau-\Delta\tau, t))^{\mathrm{T}} \\ &+B(t)B^{\mathrm{T}}(t)), \quad \Delta t=T_s \end{aligned} \tag{42}$$

The choices of Δt, $\Delta\tau$ and T_s are guided by (1) Δt should be sufficiently small to guarantee sufficient accuracy within $(\tau-\Delta\tau, \tau+\Delta\tau)$. (2) Δt should be sufficiently small so that the number of Euler integration time steps $\Delta\tau/\Delta t$ within $(\tau-\Delta\tau, \tau+\Delta\tau)$ is sufficiently large to allow $\mathrm{rank}(W(\tau-\Delta\tau, \tau+\Delta\tau))$ to grow sufficiently. From (42) it follows that the maximum increase of $\mathrm{rank}(W(\tau-\Delta\tau, t))$ during each time step Δt is $\mathrm{rank}(B(t))$. In addition, $\mathrm{rank}(W(\tau-\Delta\tau, \tau+\Delta\tau))\leqslant n$. In our example $\mathrm{rank}(B(t))=1$. Hence, the minimum number of steps that might allow $\mathrm{rank}(W(\tau-\Delta\tau, \tau+\Delta\tau))=n$ to be reached is n. As can be seen from (41) we take $2n$ time steps. Roughly this choice is appropriate if j associated with Equation (8) is never larger than $2n$ which generally is the case [10]. Since M_t is dual to W_t dual arguments apply to the computation of M_t.

Remark 8
Using the equality (39) and dually $M_\tau=M(\tau-\Delta\tau, \tau+\Delta\tau)$ leads to a maximum error $2\Delta\tau$ in t_i, $i=1,2,\ldots,N-1$ presuming $\mathrm{rank}(W_t)$ and $\mathrm{rank}(M_t)$ change rank at most once over every subinterval $(\tau-\Delta\tau, \tau+\Delta\tau)$, $\tau\in(t_0+\Delta\tau, t_f-\Delta\tau)$.

Computation of the numerical $\mathrm{rank}(W_t)$ and $\mathrm{rank}(M_t)$ also requires the selection of a tolerance. This tolerance is actually a measure of how close a numerical matrix must be to being rank deficient before it is considered to be rank deficient. This tolerance must be small, but large enough to suppress rounding and other numerical errors. In Examples 2 and 3 this tolerance also determines how close $x_2^*(t)$ must be to zero, before the linearized system about the optimal trajectory becomes temporal uncontrollable (Example 2), as well as temporal unreconstructable (Example 3). *Practically* this is partly determined by the available magnitude of state perturbation control (Example 2) as well as the sensitivity of the output for state perturbations (Example 3). Taking into account these considerations we selected a tolerance of 10^{-13} for both $\mathrm{rank}(W_t)$ and $\mathrm{rank}(M_t)$. This may seem a too small value but observe from Equations (8), (9) that W_t, M_t are matrix squares of $C_j(t)$ and $O_k(t)$. The tolerance associated with the latter two matrices is therefore $\sqrt{10^{-13}}=3.33\times10^{-7}$.

The temporal stabilizability measure $\rho_{\min}=\|S^*(t_i^+)\|$ measures the maximum decay of the state achievable through control. Therefore, in the associated LQ problem $\varepsilon\downarrow 0$ is required in Equation (22). Remark 3 states considerations to select $0<\varepsilon\ll 1$. In addition to these, the available magnitude of state perturbation control (Example 2) as well as the sensitivity of the output for state perturbations (Example 3) should again be considered. In Examples 2 and 3 we took $\varepsilon=10^{-4}$. The same value was used to approximate the dual temporal detectability measure $\sigma_{\min}=\|P^*(t_N^-)\|$. The algorithm used to solve the standard finite-horizon continuous time-varying LQ problems was recently published in Van Willigenburg and De Koning ([13], Algorithm 1). The algorithm is designed for linear systems with white stochastic parameters and exploits the delta operator. As a special case systems with deterministic parameters, considered in this paper, can be handled.

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7. CONCLUSIONS

Time-varying linear systems may be temporal uncontrollable and temporal unreconstructable over certain open time intervals that are part of their time-domain. The differential Kalman decomposition is able to detect these properties and the associated open time intervals. As demonstrated in this paper, after introduction of a suitable, simple stability property that applies over finite time intervals, application of ordinary standard LQ theory and algorithms enables the computation of temporal and differential stabilizability measures. Temporal and differential detectability measures are computed exploiting duality. The interesting intervals over which to perform these computations are intervals over which the system is temporal uncontrollable or temporal unreconstructable. The differential Kalman decomposition as well as the concepts of temporal controllability and temporal reconstructability require the consideration of PCR systems (PCR systems) [7, 8].

As an alternative to LQ theory, temporal stabilizability may be determined by extracting the temporal uncontrollable or temporal unreconstructable subsystems and analyzing their temporal stability. In principle, the differential Kalman decomposition is able to extract these subsystems. In practice this extraction faces numerical problems that we will address in another paper. Moreover, the extraction employs state basis transformations that generally change temporal stability properties. The approach presented in this paper does not suffer from these problems because it applies standard LQ theory to the original untransformed system. Although the LQ problems in this paper are singular in principle, they can be very well approximated by non-singular LQ problems, as demonstrated in this paper. The interpretation of $\|S^{*}(t)\|$ as a temporal stabilizability measure is new and highly interesting. The same applies to the interpretation of $-\mathrm{d}\|S^{*}(t)\|/\mathrm{d}t$ as a differential stabilizability measure.

Much of our own research has been inspired by Athans [1], i.e. optimal control combined with optimal perturbation feedback design for non-linear systems. This paper has further explored the design leading to new interpretations and tools for analysis and design of these types of control systems, as demonstrated by the three numerical examples presented in this paper. Developments of the same kind have been started in discrete-time [14]. These are important for digital control system design. Along the lines of this paper we are also currently exploring temporal and differential properties of time-varying linear systems with white stochastic parameters [15]. These enable robust optimal perturbation feedback design for non-linear systems.

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