www.ietdl.org

Published in IET Control Theory and Applications
Received on 28th August 2012
Accepted on 26th October 2012
doi: 10.1049/iet-cta.2012.0678

ISSN 1751-8644

Brief Paper

Temporal and one-step stabilisability and detectability of discrete-time linear systems

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Abstract: In a past study the authors drew attention to the fact that time-varying discrete-time linear systems may be temporarily uncontrollable and unreconstructable and that this is vital knowledge for both control engineers and system scientists. Describing and detecting the temporal loss of controllability and reconstructability requires considering discrete-time systems with variable dimensions and the *j*-step, *k*-step Kalman decomposition. In this study for linear discrete-time systems with variable dimensions measures of temporal and one-step stabilisability and detectability are developed. These measures indicate to what extent the temporal loss of controllability and reconstructability may lead to temporal instability of the closed-loop system when designing a static state or dynamic output feedback controller. The measures are calculated by solving specific linear quadratic cheap control problems by means of standard linear quadratic control algorithms. The importance of our developments for control system design is illustrated by means of two numerical examples.

1 Introduction

Perturbation feedback control design and stability analysis of non-linear systems along trajectories is often performed using linearised dynamics about the trajectory [1, 2]. If the trajectory is time-varying, the linearised model is ‘time-varying’. If in addition the non-linear dynamics or the controls are non-smooth, that is, in the case of bang–bang or digital control, the ‘structure’ of the time-varying linearised system may no longer be constant. For control system design, this is vital information since this structure reveals the ‘temporal loss’ of controllability and reconstructability of the linearised system. They in turn may lead to ‘temporal instability’ of a closed-loop control system [3, 4]. Recently, we investigated these issues for continuous-time systems assuming continuous-time control. This investigation leads to the introduction of the properties temporal and differential stabilisability and detectability for continuous-time linear systems [5]. In addition ‘measures’ of these properties were introduced and calculated by solving specific ‘linear quadratic cheap control problems’ [6, 7] by means of standard linear quadratic control algorithms [5].

Computer control systems are digital. The associated control problems are digital control problems (sampled-data control problems). They concern the control of continuous-time systems by means of piecewise constant controls using sampled measurements. A common approach is to transform such control problems into equivalent discrete-time control problems [8, 9]. Following this approach, feedback control system design is performed in discrete-time. This motivates the discrete-time development in this paper that on the one hand parallels, but on the other is also very different from the one in continuous-time. The fact that discrete-time is not dense, as opposed to continuous-time, causes some major differences. In continuous-time our investigation required the introduction of piecewise constant rank systems and the differential Kalman decomposition [3, 4]. In discrete-time their counterparts are discrete-time linear systems with variable state dimensions (VDD systems) and the *j*-step, *k*-step Kalman decomposition [10].

This paper develops ‘measures of temporal stability’ of time-varying linear discrete-time systems over arbitrary finite-time intervals, notably intervals where controllability or reconstructability is lost temporarily. Associated with this, measures of ‘temporal and one-step stabilisability and detectability’ are developed. Through a numerical example the importance of these measures for control system design is illustrated. The example considers possible stabilisation by means of LQG output perturbation feedback control of a non-linear system around an open-loop optimal control and state trajectory. The practical scope of this example is wide since LQG output perturbation feedback control can be used to stabilise any non-linear system around any optimal open-loop trajectory obtained from general, not necessarily quadratic, performance criteria [1].

Temporal stability may sound as a contradiction, because formally, stability relates to the behaviour when time tends to infinity. However, in one of his early seminal papers [11] Kalman together with Bertram has already proposed measures of stability over finite-time intervals (p. 386). Intuitively, stability relates to growth of the system state. Intuitively over intervals where the state grows, we call the system temporarily unstable, and over intervals where the state decays, we call the system temporarily stable. This intuition is formalised by the temporal stability property proposed in this note. This property is derived from a 'measure' of temporal stability also proposed in this note that measures the maximum growth of the state over an arbitrary interval. Our concept of stability over a finite-time interval differs from what is called finite-time stability [12, 13]. The reason we make a different choice is that our measures, their computation and the associated control system designs, come down to solving standard LQ problems. The standard LQ problems are of a special type called cheap control LQ problems [6, 7]. They are characterised by a control penalty that tends to zero. Computations and control system design associated with finite-time stability concern matrix inequalities [12, 13]. These are much more difficult to solve.

The developments in this paper relate to some extent to stabilisability and detectability developments for switched linear systems [14–16], because these systems have a variable structure too. They are also significantly different because of the different nature of the discrete-time systems considered in this paper.

2 Discrete-time systems with variable dimensions and the *j*-step, *k*-step Kalman decomposition

Although early theoretical results suggested linear systems having variable dimensions [17–19] in continuous-time, these have hardly been considered in the literature until recently [3, 4]. Discrete-time linear systems with variable dimensions have been considered since 1992. At that time they were used to describe, analyse and design computational networks [20]. Later, they appeared naturally as part of the solution of the discrete-time optimal reduced-order finite-horizon LQG problem [21]. Finally, they were also used for discrete-time reduced-order modelling [22]. These discrete-time linear systems are described by

$$\begin{aligned} x_{i+1} &= \Phi_i x_i + \Gamma_i u_i \\ y_i &= C_i x_i, \quad i = i_0, i_0+1, \dots, i_N, \quad i_0, i_N \in Z \end{aligned} \tag{1}$$

where $x_i \in R^{n_i}$ is the state, $u_i \in R^{m_i}$ is the input and $y_i \in R^{l_i}$ is the output. The dimensions n_i, m_i, l_i of the state, input and output may all vary over time. Therefore from now on we will call the system (1) a variable dimension discrete-time linear system (a VDD system). The matrices $\Phi_i \in R^{n_{i+1}\times n_i}$, $\Gamma_i \in R^{n_{i+1}\times m_i}$ and $C_i \in R^{l_i \times n_i}$, are real valued and have compatible dimensions. Note Φ_i need not be square. Observe from (1) that the time-domain I of VDD systems satisfies

$$I = \{i_0, i_0+1, \dots, i_N\} \tag{2}$$

In [23], different types of well-known system properties like reachability, observability and minimality for different types of linear systems have been reviewed and their equivalences and differences investigated. Based on these results, 'temporal' versions of these properties were established in continuous-time and found to be of practical importance [3–5]. Similar results were established in discrete-time [10]. In this section, we state the most important ones from [10] needed to obtain the main results of this paper.

Definition 1: The 'reachability/controllability grammian' $W_{i,j}$, $j \ge i$, of a VDD system (1) is given by

$$\begin{aligned} W_{i,j} &= \Phi_j W_{i,j-1}\Phi_j^{\mathrm{T}} + \Gamma_j\Gamma_j^{\mathrm{T}} \in R^{n_j\times n_j}, \quad j > i \\ W_{i,i} &= 0 \in R^{n_i\times n_i} \end{aligned} \tag{3}$$

The 'observability/reconstructability grammian' $M_{i,j}$, $j > i$, of the VDD system (1) is given by

$$\begin{aligned} M_{i,j} &= \Phi_i^{\mathrm{T}} M_{i-1,j}\Phi_i + C_i^{\mathrm{T}} C_i \in R^{n_i\times n_i}, \quad i < j \\ M_{j,j} &= 0 \in R^{n_j\times n_j} \end{aligned} \tag{4}$$

Definition 2: A VDD system is called '*j*-step reachable at time' i as well as '*j*-step controllable from time' $i-j$ if $j \ge 0$, i, $i-j \in \mathrm{I}$ and if any state $x_{i-j} \in R^{n_{i-j}}$ can be transferred to any state $x_i \in R^{n_i}$ through an appropriate choice of the input sequence $U_{i-j,\,i} = \{u_{i-j}, u_{i-j+1}, \dots, u_{i-1}\}$.

Definition 3: A VDD system is called '*k*-step observable at time' i as well as '*k*-step reconstructable from time' $i+k$, if $k \ge 0$, i, $i+k \in I$ and if from the output sequence $Y_{i,\,i+k} = \{y_i, y_{i+1}, \dots, y_{i+k-1}\}$ the state $x_i \in R^{n_i}$ can be determined.
From Definitions 2, 3 above and Lemmas 2.2, 2.3 in [10], we have the following lemma

Lemma 1: A VDD system is *j*-step reachable at time i and therefore *j*-step controllable from time $i-j$ if and only if i, $i-j \in I$ and $W_{i-j,\,i} > 0$. Dually, a VDD system is *k*-step observable at time i and therefore *k*-step reconstructable from time $i+k$ if and only if i, $i+k \in I$ and $M_{i,\,i+k} > 0$.

Definition 4: For VDD systems the Kalman decomposition that uses gramians $W_{i-j,i} \in R^{n_i\times n_i}$, $j \ge 0$ and $M_{i,i+k} \in R^{n_i\times n_i}$, $k \ge 0$ as inputs to compute a Kalman decomposition at every time i is called the '*j*-step and *k*-step Kalman decomposition'.

Theorem 1: 1) For VDD systems, after application of the *j*-step, *k*-step Kalman decomposition at every time i the states x_i' are decomposed: $x_i'^{\mathrm{T}} = \left[x_i'^{a\mathrm{T}} x_i'^{b\mathrm{T}} x_i'^{c\mathrm{T}} x_i'^{d\mathrm{T}}\right]^{\mathrm{T}}$, $x_i'^a \in R^{n_i^a}$, $x_i'^b \in R^{n_i^b}$, $x_i'^c \in R^{n_i^c}$, $x_i'^d \in R^{n_i^d}$. The states $x_i'^a$ are *j*-step reachable at time *i*/*j*-step controllable from time $i-j$ and *k*-step unobservable at time *i*/*k*-step unreconstructable from time $i+k$. The states $x_i'^b$ are *j*-step reachable at time *i*/*j*-step controllable from time $i-j$ and *k*-step observable at time *i*/*k*-step reconstructable from time $i+k$. The states $x_i'^c$ are *j*-step unreachable at time *i*/*j*-step uncontrollable from time $i-j$ and *k*-step unobservable at time *i*/*k*-step unreconstructable from time $i+k$. Finally, the states $x_i'^d$ are *j*-step unreachable at time *i*/*j*-step uncontrollable from time $i-j$ and *k*-step observable at time *i*/*k*-step reconstructable from time $i+k$. The system matrices Γ_i', C_i' decompose as follows: $\Gamma_i'^{\mathrm{T}} = \left[\Gamma_i'^{a\mathrm{T}} \quad \Gamma_i'^{b\mathrm{T}} \quad 0 \quad 0\right]^{\mathrm{T}}$, $\Gamma_i'^a \in R^{n_{i+1}^a\times m_i}$,

doi: 10.1049/iet-cta.2012.0678

$C_i'^b \in R^{l_i \times n_i^b}$, $C_i'^d \in R^{l_i \times n_i^d}$, $C_i' = \begin{bmatrix} 0 & C_i'^b & 0 & C_i'^d \end{bmatrix}$, $C_i'^b \in R^{l_i \times n_i^b}$, $C_i'^d \in R^{l_i \times n_i^d}$.

Remark 1: As opposed to the conventional Kalman decomposition, the system matrices Φ_i' obtained after the Kalman decomposition in Theorem 1, do not generally decompose into matrices containing zero parts, as explained in [10].

Definition 5: The '*j*-step, *k*-step structure' of a VDD system at time $i \in I$ is determined by the dimensions $n_i^a, n_i^b, n_i^c, n_i^d$ obtained from the *j*-step, *k*-step Kalman decomposition at time *i*. If $n_h^a = n_i^a, n_h^b = n_i^b, n_h^b = n_i^b, n_h^c = n_i^c$ and $n_h^d = n_i^d$, then a VDD system has the same *j*-step, *k*-step structure at times $h, i \in I$. Otherwise, the *j*-step, *k*-step structure is different.

Remark 2: At each discrete-time instant $i \in I$ the values *j*, *k* associated with the *j*-step, *k*-step Kalman decomposition may be different, that is, $j(i)$, $k(i)$. But only if $j(i)$ is selected such that $\sum_{i'=i-j(i)}^{i-1} m_{i'} \geq n_i$ can the system be *j*-step reachable at *i*/*j*-step controllable from $i - j$. Dually only if $k(i)$ is selected such that $\sum_{i'=i}^{i+k-1} l_{i'} \geq n_i$ can the system be *k*-step observable at *i*/*k*-step reconstructable from $i + k$. Presuming $\forall i \in \{0, 1, \ldots, N-1\}$: $m_i \geq 1$, $l_i \geq 1$, then $j(i) = k(i) = n_i$ are the 'smallest' values of $j(i)$, $k(i)$ that are guaranteed to satisfy these constraints. Since our aim is to consider 'temporal' controllability and reconstructability we do not want to take $j(i)$, $k(i)$ very large. Therefore a natural choice that satisfies the conditions, except near the time boundaries i_0, i_N where the conditions cannot be satisfied, is

$$j(i) = \min\left(2n_i, i - i_0\right), \quad k(i) = \min\left(2n_i, i_N - i\right) \quad (5)$$

Remark 3: As opposed to continuous-time, where the temporal linear system structure remains constant over open time-intervals almost covering the systems time-domain [3, 4], in discrete time the temporal *j*-step, *k*-step structure may be different at each time $i \in I$, even if $j(i)$, $k(i)$ and n_i, m_i, l_i do not depend on *i* [10].

3 Temporal and one-step stabilisability and detectability

The aim to investigate temporal uncontrollability/unreachability and temporal unreconstructability/unobservability is to detect intervals where stability of the closed-loop system may be lost temporarily when designing static state or dynamic output feedback controllers. Definition 2 is used to formalise the notion of temporal uncontrollability/unreachability.

Definition 6: A VVD system is called *j*-step unreachable over the interval $[i_s, i_f + j]$/ *j*-step uncontrollable over the interval $[i_s - j, i_f]$, $i_0 + j \leq i_s < i_f \leq i_N - j$ if $\forall i \in [i_s, i_f + j]$ the system is not *j*-step reachable at time *i*/ not *j*-step controllable from time $i - j$.

Lemma 2: If i_s, i_f satisfy the conditions in Definition 6 then over the interval $[i_s, i_f]$, the VDD system is (i) not *j*-step reachable at each time and (ii) not *j*-step controllable each time.

Proof: Follows immediately from Definition 2 and Definition 6. □

Definition 7: A VDD system that satisfies the conditions in Definition 6 is called *j*-step uncontrollable/unreachable over the interval $[i_s, i_f]$.

Definition 8 (dual to Definition 6): A VVD system is called *k*-step unobservable over the interval $[i_s - k, i_f]$/ *k*-step unreconstructable over the interval $[i_s, i_f + k]$, $i_0 + k \leq i_s < i_f \leq i_N - k$ if $\forall i \in [i_s - k, i_f]$ the system is not *k*-step observable at time *i*/not *k*-step reconstructable from time $i + k$.

Lemma 3 (dual to Lemma 2): If i_s, i_f satisfy the conditions in Definition 8 then over the interval $[i_s, i_f]$, the VDD system is (i) not *k*-step observable at each time and (ii) not *k*-step reconstructable each time.

Definition 9 (dual to Definition 7): A VDD system that satisfies the conditions in Definition 8 is called *k*-step unreconstructable/unobservable over the interval $[i_s, i_f]$.

Application of the *j*-step *k*-step Kalman decomposition [10, 24] at each time $i \in [i_0, i_N]$, reveals all closed intervals (i.e. consisting of at least two consecutive discrete-time instants) where the system is *j*-step uncontrollable/unreachable and dually all closed intervals where the system is *k*-step unreconstructable/unobservable. As in Definitions 7 and 9, such intervals will be denoted by $[i_s, i_f]$. These closed intervals are precisely the intervals where stability of the closed-loop system may be lost temporarily when designing static state and dynamic output feedback controllers.

Stabilisability is a property that relates entirely to the uncontrollable part of a system. A general approach to determine stabilisability is to extract this uncontrollable part, that is autonomous, by means of a Kalman decomposition, and to determine its stability. It will become clear in this section that application of a state basis transformation changes temporal stability and stabilisability properties. To recover them, we therefore need to transform back to the original state basis. As opposed to this general approach, the stabilisability analysis presented in this section is much more straightforward and simple. It does not require transformation of the state basis because it relies fully on a well-established standard LQ theory applied to the original system representation. Therefore the associated numerical computations are also very efficient.

The stabilisability analysis in this section is 'unconventional' in the sense that stability, stabilisability and detectability over 'finite-time intervals' is required. Stability over an interval relates to growth of the magnitude of the state over this interval. Throughout this paper $\|\bullet\|$ denotes the matrix 2 norm. For vectors this amounts to the L2 norm. In the next section we will demonstrate how to compute numerically temporal and one-step stabilisability and detectability measures presented in this section, using only evaluations of the system matrices.

Definition 10: An autonomous VVD system is called 'temporal stable over the interval' $[i_s, i_f]$, if for any $x_{i_s} \neq 0$, $\|x_{i_f}\| / \|x_{i_s}\| < 1$.

Loosely speaking, according to Definition 10 an autonomous VDD system is called temporal stable over $[i_s, i_f]$, if for any initial state the magnitude of the associated terminal state is smaller than that of the initial state. An important difference between our definition and other finite-time stability concepts [12, 13] is that ours does not impose any restrictions on the magnitude of the state inside the interval. The advantage of Definition 10 is that it matches LQ control design as opposed to finite-time stability that relates to control system design using matrix inequalities [12] that are generally much more complicated.

Definition 11: Associate to Definition 10 the following ‘temporal stability measure’

$$\rho(i_{\mathrm{s}}, i_{\mathrm{f}}) = \max_{x_{i_{\mathrm{s}}} \neq 0} \left(\frac{\|x_{i_{\mathrm{f}}}\|^2}{\|x_{i_{\mathrm{s}}}\|^2} \right) \geq 0 \tag{6}$$

Observe that $\rho(i_{\mathrm{s}}, i_{\mathrm{f}})$ in Definition 11 is the largest possible ratio $\|x_{i_{\mathrm{f}}}\|^2/\|x_{i_{\mathrm{s}}}\|^2$. This ratio matches the largest possible ratio $\|x_{i_{\mathrm{f}}}\|/\|x_{i_{\mathrm{s}}}\|$ in Definition 10. Therefore $\rho(i_{\mathrm{s}}, i_{\mathrm{f}})$ is indeed a measure of temporal stability associated with Definition 10. If $\rho(i_{\mathrm{s}}, i_{\mathrm{f}})$ is smaller, temporal stability is larger. It will become clear that the squares in (6) are needed to achieve compatibility with LQ control computations. Let $\Phi_{i_{\mathrm{s}},i_{\mathrm{f}}}$ denote the state transition matrix of the associated autonomous system from time i_{s} to i_{f} given by

$$\Phi_{i_{\mathrm{s}},i_{\mathrm{f}}} = \Phi_{i_{\mathrm{f}}-1}\Phi_{i_{\mathrm{f}}-2}\cdots\Phi_{i_{\mathrm{s}}+1}\Phi_{i_{\mathrm{s}}} \tag{7}$$

Theorem 2: An autonomous VDD system is temporal stable over the time interval $[i_{\mathrm{s}}, i_{\mathrm{f}}]$ if and only if

$$\rho(i_{\mathrm{s}}, i_{\mathrm{f}}) = \left\| \Phi^{\mathrm{T}}_{i_{\mathrm{s}},i_{\mathrm{f}}} \Phi_{i_{\mathrm{s}},i_{\mathrm{f}}} \right\| < 1 \tag{8}$$

Proof: Since Theorem 2 applies to autonomous systems

$$x_{i_{\mathrm{f}}} = \Phi_{i_{\mathrm{s}},i_{\mathrm{f}}} x_{i_{\mathrm{s}}} \tag{9}$$

Using (9) the temporal stability measure (6) becomes

$$\begin{aligned}\rho(i_{\mathrm{s}}, i_{\mathrm{f}}) &= \max_{x_{i_{\mathrm{s}}} \neq 0} \left(\frac{\|\Phi_{i_{\mathrm{s}},i_{\mathrm{f}}} x_{i_{\mathrm{s}}}\|^2}{\|x_{i_{\mathrm{s}}}\|^2} \right) \\ &= \max_{x_{i_{\mathrm{s}}} \neq 0} \left(\frac{x^{\mathrm{T}}_{i_{\mathrm{s}}} \Phi^{\mathrm{T}}_{i_{\mathrm{s}},i_{\mathrm{f}}} \Phi_{i_{\mathrm{s}},i_{\mathrm{f}}} x_{i_{\mathrm{s}}}}{x^{\mathrm{T}}_{i_{\mathrm{s}}} x_{i_{\mathrm{s}}}} \right) = \left\| \Phi^{\mathrm{T}}_{i_{\mathrm{s}},i_{\mathrm{f}}} \Phi_{i_{\mathrm{s}},i_{\mathrm{f}}} \right\|\end{aligned} \tag{10}$$

The last equality in (10) holds because $\Phi^{\mathrm{T}}_{i_{\mathrm{s}},i_{\mathrm{f}}} \Phi_{i_{\mathrm{s}},i_{\mathrm{f}}}$ is non-negative symmetrically. Theorem 2 follows from (10), Definitions 10 and 11 and

$$\|x_{i_{\mathrm{f}}}\|/\|x_{i_{\mathrm{s}}}\| < 1 \Leftrightarrow \|x_{i_{\mathrm{f}}}\|^2/\|x_{i_{\mathrm{s}}}\|^2 < 1 \tag{11}$$

□

Stabilisability over a finite-time interval relates to the ability to stabilise the system over that interval by means of control.

Definition 12: Associate to Definitions 10 and 11, the following ‘temporal stabilisability measure’ that applies to VDD systems considered over the interval $i_{\mathrm{s}}, i_{\mathrm{f}}$

$$\rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) = \max_{x_{i_{\mathrm{s}}} \neq 0} \left(\frac{\min_{u_i|x_{i_{\mathrm{s}}}} \left\| x_{i_{\mathrm{f}}} \right\|^2}{\left\| x_{i_{\mathrm{s}}} \right\|^2} \right) \geq 0 \tag{12}$$

where $u_i|x_{i_{\mathrm{s}}}$ indicates a control law dependent on $x_{i_{\mathrm{s}}}$.

Definition 13: A VDD system is called ‘temporal stabilisable over’ $[i_{\mathrm{s}}, i_{\mathrm{f}}]$ if $\rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) < 1$.

Theorem 3: A VDD system is temporal controllable over $[i_{\mathrm{s}}, i_{\mathrm{f}}] \Rightarrow \rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) = 0 \Rightarrow$ the VDD system is temporal stabilisable over $[i_{\mathrm{s}}, i_{\mathrm{f}}]$.

Proof: If a VDD system is temporal controllable over $[i_{\mathrm{s}}, i_{\mathrm{f}}]$, then according to Definitions 6 and 2, any state $x_{i_{\mathrm{s}}}$ can be controlled to $x_{i_{\mathrm{f}}} = 0$. This implies $\rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) = 0$ and, according to Definition 13, temporal stabilisability over $[i_{\mathrm{s}}, i_{\mathrm{f}}]$. □

Remark 4: As with ordinary controllability and stabilisability, temporal controllability is a stronger property than temporal stabilisability.

To state the main theorem in this section consider the following parameterised discrete-time LQ problem. Given the system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \quad i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1] \tag{13}$$

with initial state

$$x_{i_{\mathrm{s}}} \tag{14}$$

find the control u_i, $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$ that minimises the cost function

$$J_{\mathrm{LQ}}(\varepsilon) = x^{\mathrm{T}}_{i_{\mathrm{f}}} H x_{i_{\mathrm{f}}} + \sum_{i=i_{\mathrm{s}}}^{i_{\mathrm{f}}-1} \left[x^{\mathrm{T}}_i Q_i x_i + u^{\mathrm{T}}_i R^{\varepsilon}_i u_i \right] \tag{15}$$

with

$$H = I_n, \quad Q_i = 0, \quad R^{\varepsilon}_i = \varepsilon I_m, \quad 0 \leq \varepsilon \ll 1 \tag{16}$$

If $\varepsilon > 0$, the linear quadratic control problem (13), (14)–(16) satisfies $H \geq 0$, $Q_i \geq 0$, $R^{\varepsilon}_i > 0$. In this standard case it is well known that the optimal control is given by

$$u_i = -L^{\varepsilon}_i x_i, \quad L^{\varepsilon}_i = \left(\Gamma^{\mathrm{T}}_i S^{\varepsilon}_{i+1} \Gamma_i + R^{\varepsilon}_i \right)^{-1} \Gamma^{\mathrm{T}}_i S^{\varepsilon}_{i+1} \Phi_i \tag{17}$$

and the minimum cost by

$$J^*_{\mathrm{LQ}}(\varepsilon) = x^{\mathrm{T}}_{i_{\mathrm{s}}} S^{\varepsilon}_{i_{\mathrm{s}}} x_{i_{\mathrm{s}}} \tag{18}$$

where S^{ε}_i, $i \in [i_{\mathrm{s}}, i_{\mathrm{f}}]$ is the solution of the matrix Riccati difference equation

$$S^{\varepsilon}_i = \Phi^{\mathrm{T}}_i S^{\varepsilon}_{i+1} \Phi_i - L^{\mathrm{T}}_i \left(\Gamma^{\mathrm{T}}_i S^{\varepsilon}_{i+1} \Gamma_i + R^{\varepsilon}_i \right) L_i + Q_i, \quad S^{\varepsilon}_{i_{\mathrm{f}}} = H \tag{19}$$

doi: 10.1049/iet-cta.2012.0678

Theorem 4:

$$S_i^* = \lim_{\varepsilon \downarrow 0} S_i^\varepsilon \tag{20}$$

exists, where $S_i^\varepsilon,\ \varepsilon \downarrow 0$ satisfies the matrix Riccati difference (19) with data as specified by (16). Furthermore

$$\rho_{\min}(i, i_{\mathrm{f}}) = \left\| S_i^* \right\|, \quad i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1] \tag{21}$$

As a special case of (21)

$$\rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) = \left\| S_{i_{\mathrm{s}}}^* \right\| \tag{22}$$

Proof: First observe that in the parameterised LQ problem (13)–(16) we may replace the initial time i_{s} by $i' \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$. This also holds for the stabilisability measure $\rho_{\min}$. Next from (15) and (16) observe that

$$\min_{u_i|x_{i'}} J_{\mathrm{LQ}}(0) = \min_{u_i|x_{i'}} \left(x_{i_{\mathrm{f}}}^{\mathrm{T}} x_{i_{\mathrm{f}}} \right) = \min_{u_i|x_{i'}} \left\| x_{i_{\mathrm{f}}} \right\|^2 \tag{23}$$

Now the key to proving (20), (21) is to prove that

$$\min_{u_i|x_{i'}} J_{\mathrm{LQ}}(0) = \lim_{\varepsilon \downarrow 0} J_{\mathrm{LQ}}^*(\varepsilon) = x_{i'}^{\mathrm{T}} S_{i'}^* x_{i'} \tag{24}$$

Suppose (24) holds. Then from (11), (23) and (24)

$$\begin{aligned} \rho_{\min}\left(i', i_{\mathrm{f}}\right) &= \max_{x_{i'} \neq 0} \left(\frac{\min_{u_i|x_{i'}} \left\| x_{i_{\mathrm{f}}} \right\|^2}{\left\| x_{i'} \right\|^2} \right) \\ &= \max_{x_{i'}} \left(\frac{x_{i'}^{\mathrm{T}} S_{i'}^* x_{i'}}{x_{i'}^T x_{i'}} \right) = \left\| S_{i'}^* \right\| \end{aligned} \tag{25}$$

The last equality in (25) holds because $S_{i'}^*$ is non-negative symmetrical. Hence, we are left to prove (24). Consider the j-step, k-step Kalman decomposition at time i_f with $j = i_f - i'$. According to this decomposition the linear system (13) can be decomposed into a part that is j-step controllable from time i' and a part that is autonomous. The contribution of the j-step controllable part to $\min_{u_i|x_{i'}} J_{\mathrm{LQ}}(0)$ is zero. The contribution to $J_{\mathrm{LQ}}^*(\varepsilon)$ tends to zero as $\varepsilon \downarrow 0$. The contribution of the autonomous part to both $\min_{u_i|x_{i'}} J_{\mathrm{LQ}}(0)$ and $J_{\mathrm{LQ}}^*(\varepsilon)$ is fixed and independent of ε. Because the system matrices are bounded, this contribution is also finite. This proves the existence of the limit (20) and the equality (24). □

Remark 5: There are three reasons for considering $0 < \varepsilon \ll 1$ in (16), instead of $\varepsilon = 0$. Taking $0 < \varepsilon \ll 1$, ε may be used to (i) keep the control within certain bounds that apply in practice and (ii) as a numerical tolerance to prevent ill-conditioning of the computation of (17) when $\Gamma_i^{\mathrm{T}} S_{i+1}^\varepsilon \Gamma_i$ is not full rank and $L_i \to \infty$ as $\varepsilon \downarrow 0$. In practice the selection of $0 < \varepsilon \ll 1$ will be a compromise and S_i^ε will 'approximate' S_i^*, $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$. As a result, all computations in this paper involving S_i^ε will be approximations, although generally very good ones. Thirdly, $\varepsilon = 0$ leads to a singular LQ problem that is generally much more difficult to solve and whose solution need not be unique.

When analysing control systems, the state behaviour over the entire interval $[i_{\mathrm{s}}, i_{\mathrm{f}}]$ is generally of interest, not just the behaviour at initial time i_{s} and final time i_{f}. This behaviour is partly considered by (21) of Theorem 4 that determines the stabilisability measure for each sub-interval $[i, i_{\mathrm{f}}]$, $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$. The following theorem introduces a one-step stabilisability measure that applies to individual time instants.

Theorem 5: $\left\| S_i^* \right\| - \left\| S_{i+1}^* \right\|$ is a one-step stabilisability measure (os-stabilisability measure) at time $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$.

Proof: Using (21)

$$\rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) = \left\| S_{i_{\mathrm{f}}}^* \right\| + \sum_{i=i_{\mathrm{s}}}^{i_{\mathrm{f}}-1} \left(\left\| S_i^* \right\| - \left\| S_{i+1}^* \right\| \right) \tag{26}$$

so $\left\| S_i^* \right\| - \left\| S_{i+1}^* \right\|$, $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$ is the 'one-step contribution at time' i to the temporal stabilisability measure $\rho_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}})$. □

Definition 14: A VDD system is called one-step stabilisable (os-stabilisable) at time $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$ if $\left\| S_i^* \right\| - \left\| S_{i+1}^* \right\| < 0$. Because for the VDD system (1), temporal and one-step detectability are dual to temporal and one-step stabilisability, the following definitions and theorems are stated without further explanation and proof.

Theorem 6 (dual to Theorem 4):

$$P_i^* = \lim_{\varepsilon \downarrow 0} P_i^\varepsilon \tag{27}$$

exists, where P_i^ε, $\varepsilon \downarrow 0$ satisfies the matrix Riccati difference equation that is dual to (19)

$$\begin{aligned} P_{i+1}^\varepsilon &= \Phi_i P_i^\varepsilon \Phi_i^{\mathrm{T}} - L_i^\varepsilon \left(C_i P_i^\varepsilon C_i^{\mathrm{T}} + R_i \right) L_i^{\varepsilon \mathrm{T}} + Q_i, P_{i_{\mathrm{s}}}^\varepsilon \\ &= H \end{aligned} \tag{28}$$

with

$$L_i^\varepsilon = \Phi_i P_i^\varepsilon C_i^{\mathrm{T}} \left(C_i P_i^\varepsilon C_i^{\mathrm{T}} + R_i^\varepsilon \right)^{-1} \tag{29}$$

with data as specified by (16). Furthermore

$$\sigma_{\min}(i_{\mathrm{s}}, i) = \left\| P_i^* \right\|, \quad i \in [i_{\mathrm{s}} + 1, i_{\mathrm{f}}] \tag{30}$$

where $\sigma_{\min}(i, i_{\mathrm{f}})$ is a temporal detectability measure over the interval $[i, i_{\mathrm{f}}]$. As a special case

$$\sigma_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) = \left\| P_{i_{\mathrm{f}}}^* \right\| \tag{31}$$

Definition 15 (dual to Definition 13): A VDD system is called 'temporal detectable over' $[i_{\mathrm{s}}, i_{\mathrm{f}}]$ if $\sigma_{\min}(i_{\mathrm{s}}, i_{\mathrm{f}}) < 1$.

Theorem 7 (dual to Theorem 5): $\left\| P_{i+1}^* \right\| - \left\| P_i^* \right\|$ is a one-step detectability measure (os-detectability measure) at time $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$.

Definition 16 (dual of Definition 14): A VDD system is called one-step detectable (os-detectable) at time $i \in [i_{\mathrm{s}}, i_{\mathrm{f}} - 1]$ if $\left\| P_{i+1}^* \right\| - \left\| P_i^* \right\| < 0$.

4 Two motivating examples

A major application of the results presented in this note concerns digital LQG output 'perturbation' feedback control for non-linear systems along open-loop digital optimal control and state trajectories. These open-loop optimal trajectories may be obtained from general, not necessarily quadratic, criteria [1]. The goal of digital LQG output perturbation feedback design is to stabilise the control system against disturbances, possibly in an optimal manner [2]. If the horizon is finite, LQG output perturbation feedback controllers can be designed irrespective of stabilisability and detectability properties of the linearised system. Only if the horizon is infinite the LQG controller design in general requires stabilisability and detectability. As explained in this paper, if the horizon is finite, closed-loop LQG output perturbation feedback control may be 'temporarily unstable' if the linearised system is not temporal stabilisable or not temporal detectable. The measures of temporal and one-step stabilisability and detectability developed in this note enable us 'to analyse if and when this happens'. They moreover enable us 'to quantify the temporal instability'. This provides highly valuable information concerning the 'practical applicability' of LQG output perturbation feedback control. The numerical examples in this section therefore concern digital LQG output perturbation feedback design along a digital optimal control and state trajectory. The first example is deliberately kept simple and is constructed such that it clearly illustrates temporal unstabilisability of the closed-loop system. The second example concerns digital LQG perturbation feedback control of a Rocket. It represents 'a real world engineering case'.

Example 1: Consider the following optimal control problem. Given the non-linear system

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$f\big(x(t), u(t)\big) = \begin{bmatrix} -x_1(t) - x_2(t) + u(t) + 0.05x_1(t)u(t) \\ x_1(t)x_2(t) \end{bmatrix} \tag{32}$$

with initial conditions

$$x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix} \tag{33}$$

find the bounded control

$$-10 \le u(t) \le 10 \tag{34}$$

that minimises the performance measure

$$J = \big(x_1(4) - 5\big)^2 + \big(x_2(4) - 5\big)^2 + \int_0^4 x_2^2(t)\,\mathrm{d}t \tag{35}$$

The performance measure (35) promotes $x_2(t)$, $t \in [0, 4]$ to be close to zero, whereas it also promotes x_1 and x_2 to be close to 5 at final time 4.

Example 1 is taken from [5], where it is used to demonstrate temporal stabilisability of LQG perturbation feedback control in continuous-time. In this paper, we consider digital control and temporal stabilisability of equivalent discrete-time LQG perturbation feedback control.

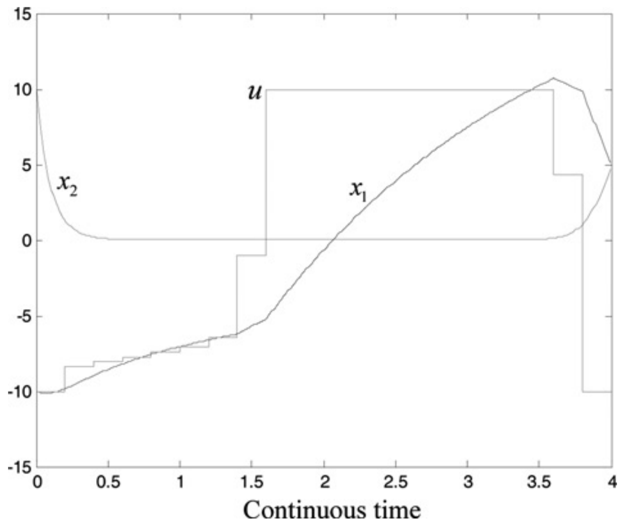

Fig. 1 *Optimal state trajectory and digital control Example 1*

So here, instead of continuously, all control inputs are updated and all outputs are sampled at the sampling instants

$$t_i = iT_s, \quad i = 0, 1, .., 20, \quad T_s = 0.2 \tag{36}$$

From (36) it is observed that the sampling is assumed to be periodic and synchronous (conventional sampling). The equivalent discrete-time (sampled-data) optimal control approach [8, 9, 25] provides the digital, piecewise constant, optimal open-loop control and the associated continuous state trajectory depicted in Fig. 1. The optimal trajectories are similar to those found in [5]. Equivalent discrete-time LQG design [8, 9] provides the digital LQG output perturbation feedback controller. To analyse the temporal stability of the latter control system, we need to analyse the temporal stabilisability and detectability properties of the equivalent discrete-time-varying linearised system (EDTVLS). Table 1 shows the temporal linear two-step, two-step structure of this EDTVLS computed from the two-step, two-step Kalman decomposition using an

Table 1 Temporal two-step, two-step structure of the equivalent discrete-time linearised system

I	$n_i^a + n_i^b$	$n_i^b + n_i^d$	n_i^a	n_i^b	n_i^c	n_i^d
0	0	2	0	0	0	2
1	1	2	0	1	0	1
2	2	2	0	2	0	0
3	2	2	0	2	0	0
4	1	2	0	1	0	1
5	1	1	0	1	1	0
6	1	1	0	1	1	0
7	1	1	0	1	1	0
8	1	1	0	1	1	0
9	1	1	0	1	1	0
10	1	1	0	1	1	0
11	1	1	0	1	1	0
12	1	1	0	1	1	0
13	1	1	0	1	1	0
14	1	1	0	1	1	0
15	1	1	0	1	1	0
16	2	1	1	1	0	0
17	2	2	0	2	0	0
18	2	2	0	2	0	0
19	2	1	1	1	0	0
20	2	0	2	0	0	0

IET Control Theory Appl., 2013, Vol. 7, Iss. 1, pp. 151–159
doi: 10.1049/iet-cta.2012.0678

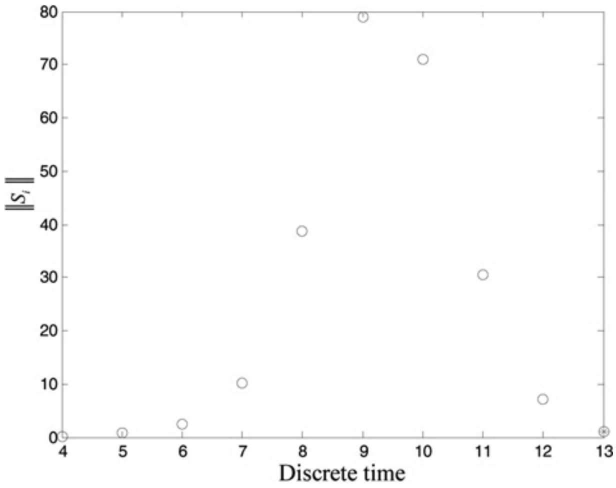

Fig. 2 *Temporal and one-step stabilisability measures Example 1*

algorithm proposed in [10]. According to Definitions 4 and 2, $n_i^a + n_i^b$ in Table 1 represents the rank of the two-step reachability gramian. Thus, from Definition 6, the EDTVLS is two-step unreachable over the interval [4, 15]. Then, according to Definition 7, the EDTVLS is two-step uncontrollable/unreachable over the interval [4, 13]. Fig. 2 shows plots of $\|S_i^{\varepsilon}\|$, $i \in [4, 13]$, $\varepsilon = 10^{-6}$, which according to Definition 12 and Theorem 4, are accurate approximations of the temporal stabilisability measures $\rho_{\min}(i, 13)$. From Fig. 2, $\|S_i^{\varepsilon}\| > 1$, $i = 6, 7, \ldots, 12$. Then, according to Definition 13, the EDTVLS is temporal unstabilisable over the intervals $[i, 13]$, $i = 6, 7, \ldots, 12$. According to Theorem 5, $\|S_i^{\varepsilon}\| - \|S_{i+1}^{\varepsilon}\|$, $i \in [4, 12]$ are accurate approximations of the one-step stabilisability measures $\|S_i^*\| - \|S_{i+1}^*\|$, $i \in [4, 12]$. Using Definition 14, it is observed from Fig. 2 that for $i = 9, 10, 11, 12$ the EDTVLS is one-step unstabilisable.

Dually $n_i^b + n_i^d$ in Table 1 represents the rank of the two-step observability gramian. Thus, from Table 1 and Definition 8, the EDTVLS is two-step unobservable over [5, 16]. Then, according to Definition 9, the EDTVLS is two-step unreconstructable/unobservable over the interval [7, 16]. Fig. 3 shows a plot of $\|P_i^{\varepsilon}\|$, c, $\varepsilon = 10^{-6}$. From this plot, Theorem 6 and Definition 15, the EDTVLS is temporal detectable over the intervals $[7, i]$, $i = 8, 9, \ldots, 16$.

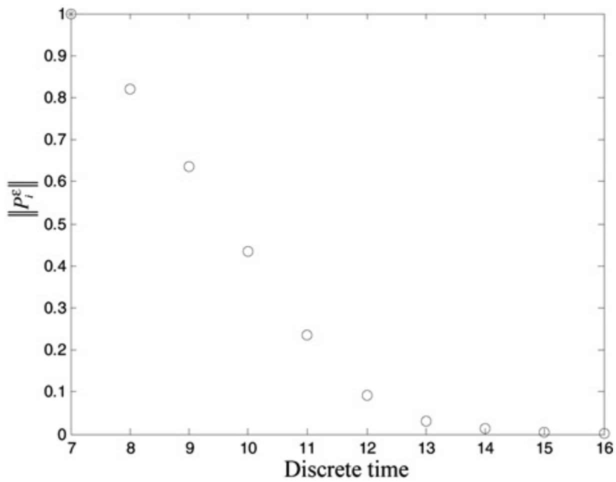

Fig. 3 *Temporal and one-step detectability measures Example 1*

Using Theorem 7 and Definition 16, the EDTVLS is one-step detectable at all times $i \in [7, 15]$.

In conclusion, the EDTVLS does not suffer from temporal and one-step undetectability. However, it does suffer from large one-step unstabilisability and associated large temporal unstabilisability. Neglecting second- and higher-order terms of the non-linear dynamics, the latter causes digital LQG output perturbation feedback control to be temporarily unstable. As expected, these results are similar to the ones obtained in continuous-time [5].

Example 2 (Goddard Rocket problem, see also [26]): Consider the following optimal control problem. Given the non-linear system

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$f\big(x(t), u(t)\big) = \begin{bmatrix} x_2(t) \\ p_1 u(t) - p_2 x_2^2(t)\exp\big(-p_3 x_1(t)\big) - p_4 \\ -u(t) \end{bmatrix},$$
$$p = \begin{bmatrix} 2060 \\ 1.27 \times 10^{-2} \\ 1.45 \times 10^{-4} \\ 9.81 \end{bmatrix} \tag{37}$$

with initial conditions

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 214.839 \end{bmatrix} \tag{38}$$

find the bounded control

$$0 \leq u(t) \leq 9.52551 \tag{39}$$

that minimises the performance measure

$$J = -x_1(t_f), \quad t_f = 56 \tag{40}$$

while satisfying the terminal state constraint

$$x_3(t_f) = 67.9833 \tag{41}$$

Example 2 is a well-known engineering problem in the optimal control literature [26]. Because this problem is singular, application of Theorem 4.1 in [27] implies that the linearised system around the singular part of the open-loop trajectory is differentially uncontrollable. From [4], this implies that in continuous-time the linearised system used for LQG perturbation feedback control is temporarily uncontrollable and, according to van Willigenburg and De Koning [5], may thus be temporarily unstabilisable. Therefore the closed-loop control system may be temporarily unstable. The digital version of Example 2 with a sampling period $T_s = 2$ is presented next. It shows that similar arguments apply in the case of digital control. The digital optimal open-loop control and state trajectory are shown in Fig. 4. The singular character of digital optimal control is clearly visible. The system has three states of which the first two (rocket height and velocity) are measured. The third

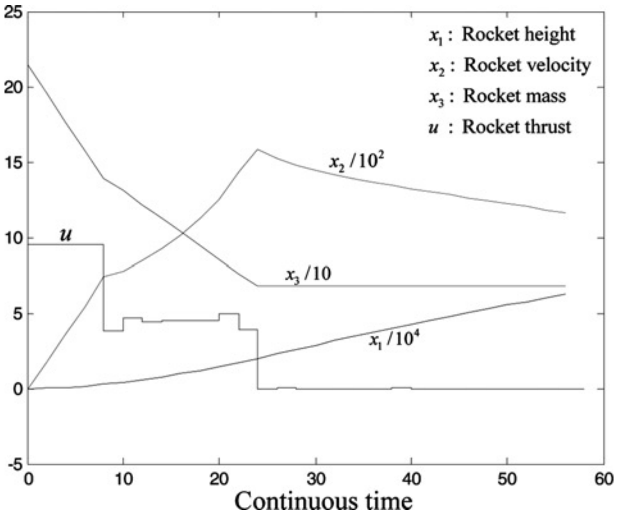

Fig. 4 *Optimal state trajectory and digital control Example 2*

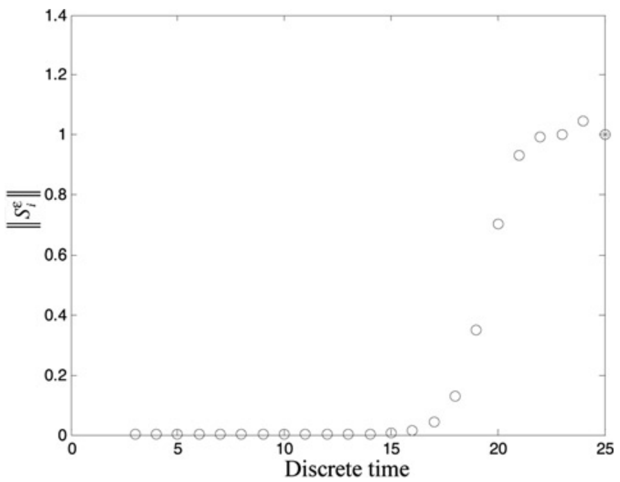

Fig. 5 *Temporal and one-step stabilisability measures Example 2*

state is the decreasing mass of the rocket because of burning fuel. Similar to Example 1, from the three-step, three-step Kalman decomposition applied to the EDTVLS, it follows that the EDTVLS is nowhere three-step unobservable, whereas it is three-step uncontrollable over the discrete-time interval [3, 25]. Our temporal stabilisability measure depicted in Fig. 5 shows that the EDTVLS is temporal stabilisable over any discrete-time interval $[i, 25]$, $i = 3, 4, \ldots, 23$, because $\|S_i^\varepsilon\| < 1$, $\varepsilon = 10^{-6}$. To reveal more clearly one-step stabilisability, we have plotted in Fig. 6 the 'relative' one-step stabilisability measure $\left(\|S_i^\varepsilon\| - \|S_{i+1}^\varepsilon\|\right)/\|S_{i+1}^\varepsilon\|$, $i = 3, 4, \ldots, 24$. Then from Theorem 5 at every discrete-time instant $i \in [3, 23]$, the EDTVLS is one-step stabilisable because the relative measure is less than zero.

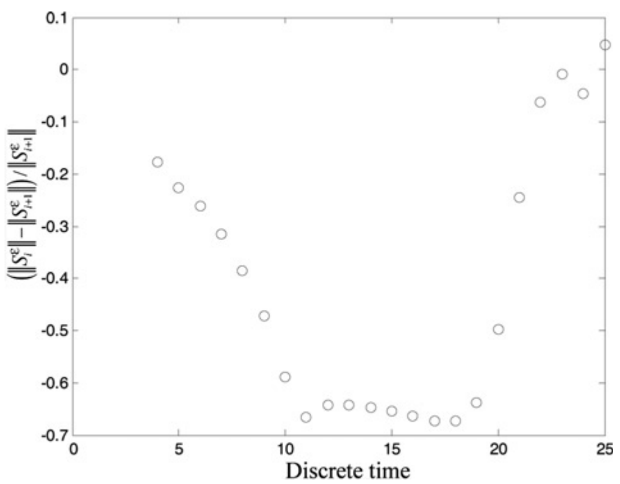

Fig. 6 *Relative one-step stabilisability measure Example 2*

In conclusion, the EDTVLS does not suffer from temporal and one-step undetectability, whereas it hardly suffers from temporal and one-step unstabilisability. Therefore digital LQG perturbation feedback control is expected to be successful in this case.

5 Conclusions

Control system stability is one of the major design objectives of almost any control system design. Remarkably, temporal loss of stability is generally ignored. The numerical examples in this paper considered temporal instability of a 'digital' optimal control system with LQG output perturbation feedback control. The paper revealed that temporal loss of stability is due to lack of temporal stabilisability and/or temporal detectability over certain time intervals of the linearised system about the optimal open-loop trajectory. When the optimal control system is a digital one, the linearised system is a time-varying discrete-time system. Therefore in this paper, we developed the properties of temporal stability, stabilisability and detectability for this type of system, as we did for continuous time-varying linear systems [5]. Standard LQ theory and algorithms are used to compute associated temporal and one-step stabilisability and detectability 'measures'. These measures quantify to what extent static or dynamic feedback control becomes temporarily unstable. Temporal stabilisability relates in a similar manner to temporal controllability [5, 10] as ordinary stabilisability relates to ordinary controllability. Dual arguments hold for temporal detectability.

As an alternative to the standard LQ algorithms, temporal stabilisability may be determined by extracting the temporal uncontrollable or temporal unreconstructable subsystems and analysing their temporal stability. In principle, the j-step, k-step Kalman decomposition introduced in [10] and also presented in this paper, is able to extract these subsystems. The extraction employs state basis transformations that generally change temporal stability properties. The approach presented in this paper is more simple and direct because it applies standard LQ theory to the original, untransformed system. The fact that state basis transformations change temporal stability properties appears at first to be unsatisfactory. In a future paper we plan to address this issue and intend to show that it is unavoidable.

Although the LQ problems in this paper are singular in principle, they can be very well approximated by non-singular LQ problems, as demonstrated in this paper. The interpretation of $\|S_i^*\|$ as a temporal stabilisability measure is new and highly interesting and valuable. The same applies to the interpretation of $\|S_i^*\| - \|S_{i+1}^*\|$ as a one-step stabilisability measure.

To solve the digital optimal control problems in this paper, the so called equivalent discrete-time optimal control approach [8, 9, 25] was used. This approach translates continuous-time transitions and performance into discrete-time equivalents and subsequently solves the equivalent discrete-time problem. Hence, the approach in this paper can also be applied to hybrid systems whenever equivalent discrete-time transitions and performances can be established.

Along the lines of this paper we are also currently exploring temporal properties of time-varying linear systems with white stochastic parameters [28]. Among others these enable robust digital optimal perturbation feedback design for non-linear systems.

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