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Temporal stabilizability and compensatability of time-varying linear discrete-time systems with white stochastic parameters

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ABSTRACT

This paper reveals that apart from changes of system structure vital system properties such as stabilizability and compensatability may be lost *temporarily* due to the stochastic nature of system parameters. To that end *new system properties* called temporal mean-square stabilizability (tms-stabilizability) and temporal mean-square compensatability (tms-compensatability) for time-varying linear discrete-time systems with white stochastic parameters (multiplicative white noise) are developed. When controlling such systems by means of (optimal) state feedback, tms-stabilizability identifies intervals where *mean-square stability* (*ms-stability*) *is lost temporarily*. This is vital knowledge to both control engineers and system scientists. Similarly, tms-compensatability identifies intervals where ms-stability is lost temporarily in case of full-order (optimal) output feedback. Tests explicit in the system matrices are provided to determine each temporal system property. These tests compute *measures* of the associated temporal system properties. Relations among the new system properties as well as relations with associated existing system properties are investigated and established. Examples illustrating principal applications and practical importance are provided.

1. Introduction

Research performed during the last decade showed that the *structure* of time-varying linear systems may change or almost change [28–32]. These changes of structure cause and explain differences between reachability and controllability and dually observability and reconstructability. They lead naturally to the definition of *temporal linear system structure* and associated *temporal properties* like temporal controllability and reconstructability. These temporal properties reveal *time intervals* where the associated ordinary system properties are *lost temporarily*. Obviously this is vital knowledge to control engineers and system scientists.

In continuous-time the intervals and associated changes of structure are detected by the differential Kalman decomposition [28,29]. In discrete-time they are detected by the *j*-step, *k*-step Kalman decomposition [30]. If controllability is lost temporarily over an interval, it is important to check whether or not stabilizability is lost temporarily over that interval. Developing and verifying temporal stabilizability was done in [31,32]. Dually temporal reconstructability and detectability were also developed and verified in these papers. These important, practical developments have sometimes been criticized because of their dependence on the selected state-vector norm. Recently it was explained in [33] that this dependence is inevitable when analysis is restricted to finite time-intervals. Also sensible choices of the state-vector norm were presented and discussed in [33].

This paper reveals that apart from changes of system structure, the *white stochastic nature of system parameters* can cause temporal loss of vital system properties, notably stabilizability and compensatability i.e. the ability to stabilize a system by means of state and output feedback respectively. To that end this paper *extends* temporal properties that have been introduced for time-varying linear discrete-time systems with deterministic parameters to systems having white stochastic parameters. Also a *new temporal system property* is introduced called *temporal compensatability*. Ordinary compensatability was introduced because separability and duality between estimation and control are lost if system parameters become stochastic [9]. If system parameters are deterministic, compensatability is equivalent with stabilizability plus detectability, otherwise it is stronger. This paper reveals that, even if the parameters are deterministic, temporal compensatability is still stronger than temporal stabilizability plus temporal detectability. Roughly speaking this is because the time needed for LQG compensators to start convergence is about the *sum* of the times needed by the state estimator and controller to start convergence.

Discrete-time linear systems with white stochastic parameters offer a way to design *non-conservative robust digital feedback controllers* [6,7,38,39]. These controllers may be *perturbation feedback controllers* based on *linearized dynamics* about possibly optimal state trajectories associated with a non-linear system. The perturbation

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feedback controllers as well as the possibly optimal state trajectories may be computed *off-line*. This enables the handling of a wide range of constraints and optimization criteria offering a wide range of application [5]. The linearized dynamics used for perturbation feedback controller design are generally *time-varying*. This is one important reason to study time-varying linear discrete-time systems with white stochastic parameters (multiplicative white noise). Other reasons are that discrete-time system parameters may be white due to stochastic sampling, randomly varying delays or Markovian jumps of system structure [4,11,17–19,22,24–26]. Optimal state and output feedback control of linear systems with white stochastic parameters has been addressed in [8,10,14–16,23,27].

The authors are aware of one other development that considers stability and stabilizability, but not compensatability, over finite time intervals. Two properties called finite-time stability and stabilizability have been introduced and investigated [1–3]. Like our temporal system properties they apply to finite time-intervals. Finite-time stability and stabilizability consider a state-vector norm over the full finite time-interval whereas temporal stability and stabilizability consider a state-vector norm at the initial and final time of the interval only. By shifting the initial time of the interval towards the final time however, a similar picture of closed loop stability over the full finite time interval is obtained [31,32]. On the other hand temporal stability and stabilizability consider arbitrary initial conditions whereas finite-time stability and stabilizability are defined for fixed initial conditions only. Moreover finite-time stability and stabilizability computations concern LMI's instead of standard LQ computations required by temporal stability and stabilizability when system parameters are deterministic.

To analyze the effect of stochastic parameters on stability, stabilizability and compensatability of systems the mean-square (ms) of the state must be considered. Temporal mean-square stabilizability (tms-stabilizability) identifies temporal loss of closed loop mean-square stability in case of (optimal) full state feedback. It is presented in Section 3. Temporal mean-square compensatability (tms-compensatability) does the same in case of (optimal) full-order output feedback and is presented in Section 4. In both sections important relations among these new system properties are established. Also relations with existing system properties, partly relating to linear systems with deterministic parameters, are established. Examples illustrating these relations are presented in Section 5. First however Section 2 presents a semi-industrial example to illustrate the main contribution and practical importance of the results developed in this paper. Conclusions are drawn in Section 6 an important one being that tms-stabilizability and tms-compensatability are most important for feedback control design based on time-varying linear dynamics with stochastic parameters.

2. Illustrative example

The new results and temporal system properties will be presented in the next two sections. In this section the main contribution of this paper and one of its major applications is illustrated and demonstrated first. This is done by means of a semi-industrial example.

Example 1. Consider the digital optimal perturbation feedback control of the "Goddard Rocket" around its optimal trajectory as presented in [5], Example 2. The example considered here is identical except for the parameters of the equivalent discrete time-varying linearized system (EDTVLS) used for digital optimal perturbation feedback design. These are turned into *stochastic parameters* using a possibly time-varying *parameter uncertainty measure* $\beta_i \geq 0$, where i denotes discrete-time. When $\beta_i = \beta = 0$ the parameters are deterministic at each time i and the results of Example 2 presented in [32] are obtained. With increasing β_i, parameter uncertainty at time i increases.

Fig. 1 presents values of the temporal mean-square stabilizability measure $\rho_{min}^{tms}(i,25)$, $i = 0, 1, .., 24$ of the closed loop system with full state feedback over time-interval $(i,25)$. If the value falls below one, the system is temporal mean-square stabilizable (tms-stabilizable) over time-interval $(i,25)$. For clarity the results are plotted using both a linear and logarithmic scale. Similarly Fig. 2 presents values of the temporal mean-square compensatability measure $\sigma_{min}^{tms}(i,25)$, $i = 0, 1, .., 24$ of the closed loop system with full-order output feedback over time-interval $(i,25)$. Again if the value falls below one, the system is temporal mean-square compensatable (tms-compensatable) over time-interval $(i,25)$. As expected, with increasing constant values of β, i.e. with increasing parameter uncertainty at each time i, tms-stabilizability and tms-compensatability become worse because their measures increase. Also observe that tms-stabilizability is far better than tms-compensatability. This represents the well-known fact that full state feedback is to be preferred over full-order output feedback. A time-varying uncertainty measure β_i may be used to indicate or describe time-varying levels of model parameter uncertainty. Observe from Figs. 1 and 2 that introducing stochastic parameters, e.g. to promote robustness, goes at the expense of tms-stabilizability and tms-compensatability.

3. Temporal stabilizability

The possibility of time-varying linear systems with deterministic parameters to change or almost change structure motivated the investigation into temporal properties of these systems [28–32]. A change of structure comes generally with a change of important

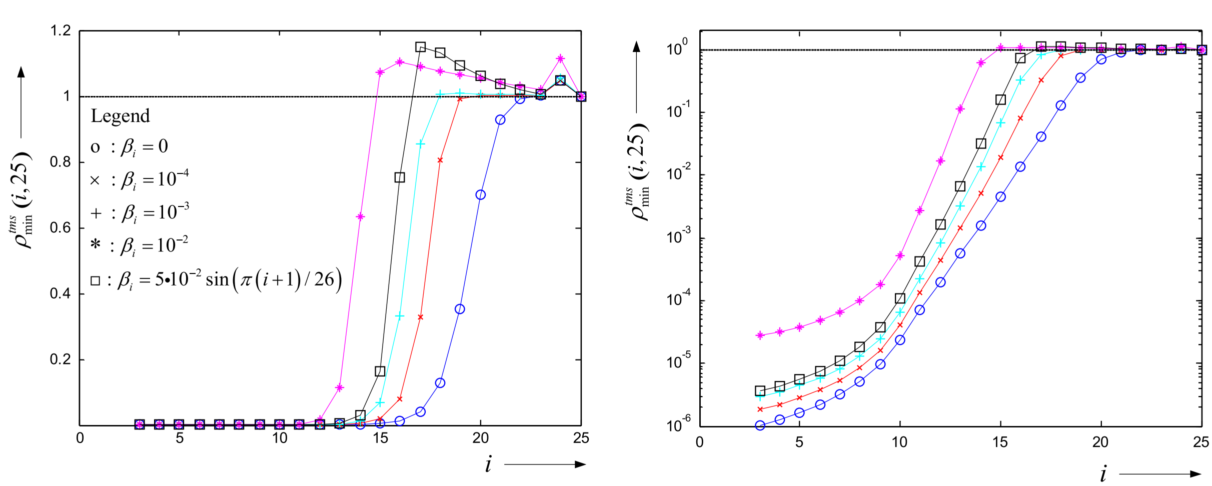

Fig. 1. tms-Stabilizability measures Example 1 for different values of β_i.

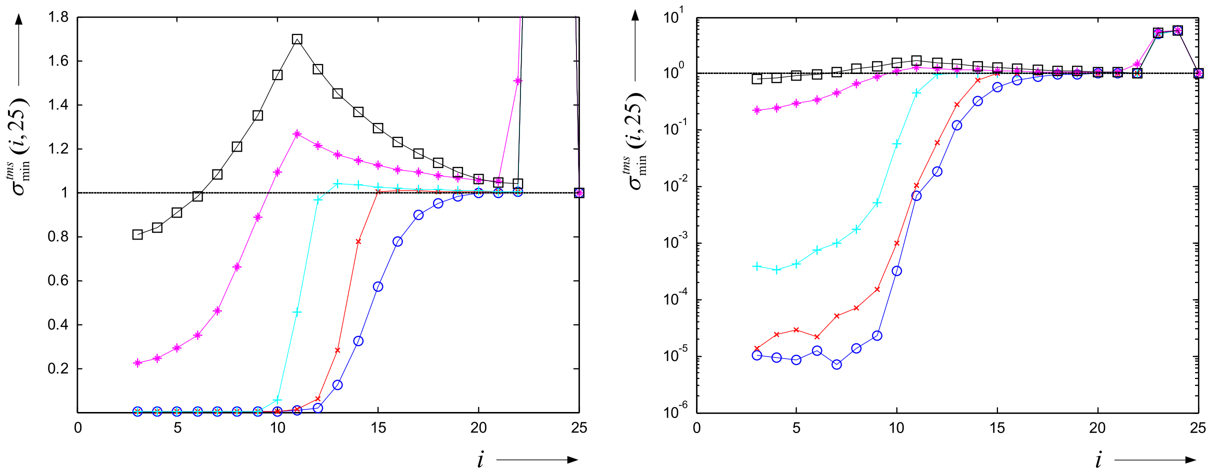

Fig. 2. tms-Compensatability measures Example 1 for different values of β_i.

system properties for control system design such as stability. Changes of structure may be accompanied by changes of the state dimension [34]. This for instance happens in asynchronously sampled digital control systems [35]. As we shall discover in this paper the stochastic nature of system parameters is another motivator to consider temporal system properties. Therefore in this paper we will consider time-varying linear discrete-time system with stochastic parameters and variable dimensions (VDD system) described by the following equation:

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \quad y_i = C_i x_i, \quad i \in I = \{i_0, i_0+1, ..., i_N\}, \quad i_0 < i_N \in \mathbb{Z}. \tag{1}$$

In Eq. (1) $x_i \in R^{n_i}$ represents the state, $u_i \in R^{m_i}$ the control and $y_i \in R^{l_i}$ the observations. $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are sequences of independent random matrices with compatible dimensions and known time-varying statistics that will be described later in this section. In addition x_i is independent of $\{\Phi_j,\ j=i,i+1,..,i_N\}$, $\{\Gamma_j,\ j=i,i+1,..,i_N\}$, $\{C_j,\ j=i,i+1,..,i_N\}$. Denote the system (1) by (Φ_i, Γ_i, C_i).

Stability analysis of linear systems with white stochastic parameters generally takes place at two levels. One level considers the stability of $\overline{x}_i$ i.e. stability of the state *mean* denoted by *m-stability*. Loosely speaking, to study m-stability conventional linear systems theory applies after replacing stochastic system parameters by their mean values. Doing so the system (Φ_i, Γ_i, C_i) turns into $\left(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i\right)$. The other level considers stability of $\overline{\|x_i\|^2} = \overline{x_i^T x_i}$ i.e. stability of the state *in the mean-square sense* denoted by *ms-stability*. Only at this level the effect of uncertainty caused by the stochastic nature of system parameters is accounted for. Therefore ms-stability is a stronger property than m-stability. Turning deterministic parameters into stochastic parameters having the same mean value may *destabilize* the system in the mean-square sense, as opposed to adding additive white system noise.

To define *temporal* stability, consider the autonomous system $(\Phi_i) = (\Phi_i, \theta, \theta)$ where θ indicates zero matrices with compatible dimensions. Throughout this paper let $\|\bullet\|$ denote the matrix 2 norm. For vectors this amount to the L2 norm. To define *temporal* m-stability (tm-stability) for the autonomous system (Φ_i), we introduce the following *measure of state-growth in the mean* over an arbitrary time interval $[i_s, i_f] \subseteq I$

$$\rho^{tm}(i_s, i_f) = \max_{x_{i_s} \neq 0} \left(\frac{\|\overline{x}_{i_f}\|^2}{\|\overline{x}_{i_s}\|^2} \right). \tag{2}$$

Observe that the measure (2) is equal to the one introduced in [31,32] with the state replaced by its mean value as noted above. Also note that, without loss of generality, the measure proposed in [31,32] already contained squares to enable compatibility with LQ computations in that paper. Next to define *temporal* ms-stability (tms-stability) consider a similar *measure of state growth in the mean-square sense*

$$\rho^{tms}(i_s, i_f) = \max_{x_{i_s} \neq 0} \left(\frac{\overline{\|x_{i_f}\|^2}}{\overline{\|x_{i_s}\|^2}} \right). \tag{3}$$

Remark 1. tm-Stability measure (2) equals the factor by which the state mean has grown over the interval $[i_s, i_f] \subseteq I$ whereas tms-stability measure (3) is the factor by which the mean-square of the state has grown.

Definition 1. The autonomous system (Φ_i) is called *temporal mean stable* (tm-stable) over $[i_s, i_f] \subseteq I$ if $\rho^{tm}(i_s, i_f) < 1$ and *temporal mean-square stable* (tms-stable) if $\rho^{tms}(i_s, i_f) < 1$.

To present the main results of this section we first have to fully represent the system (Φ_i, Γ_i, C_i). To describe the statistics of system (Φ_i, Γ_i, C_i) and their influence on the state x_i, the following relations are important:

$$\overline{\|x_i\|^2} = \overline{x_i^T x_i} = tr\left(\overline{x_i x_i^T}\right), \tag{4}$$

where *tr* denotes trace and

$$\overline{x_i x_i^T} \geq 0, \tag{5}$$

the *second moment* of x_i. Define

$$\tilde{x}_i = x_i - \overline{x}_i, \tag{6}$$

then

$$\overline{x_i x_i^T} = \overline{x}_i \overline{x}_i^T + \overline{\tilde{x}_i \tilde{x}_i^T}, \tag{7}$$

where $\overline{\tilde{x}_i \tilde{x}_i^T} \geq 0$ is the *covariance matrix* of x_i representing the state uncertainty. Similarly we may specify the time-varying statistics of system (Φ_i, Γ_i, C_i). To that end define

$$\tilde{\Phi}_i = \Phi_i - \overline{\Phi}_i, \quad \tilde{\Gamma}_i = \Gamma_i - \overline{\Gamma}_i, \quad \tilde{C}_i = C_i - \overline{C}_i, \tag{8}$$

where

$$\overline{\Phi}_i, \ \overline{\Gamma}_i, \ \overline{C}_i, \tag{9}$$

are the first moments of $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$. The second moments and covariances are described by Kronecker products, because instead of vectors, Φ_i, Γ_i, C_i are matrices. Similar to Eq. (7) they satisfy

$$\overline{\Phi_i \otimes \Phi_i} = \overline{\Phi}_i \otimes \overline{\Phi}_i + \overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i}. \tag{10}$$

Relations like (10) also hold for Γ_i, C_i. We will not allow Γ_i, C_i to be correlated. If Φ_i, Γ_i are correlated their covariance $\overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i}$ is non-zero and

$$\overline{\Phi_i \otimes \Gamma_i} = \overline{\Phi}_i \otimes \overline{\Gamma}_i + \overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i}. \tag{11}$$

Similarly when Φ_i, C_i are correlated their covariance $\overline{\tilde{\Phi}_i \otimes \tilde{C}_i}$ is non-zero. From (10) and (11) observe that the statistics of the processes $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ associated with system (Φ_i, Γ_i, C_i) are fully specified by the following equation:

$$\overline{\Phi}_i,\ \overline{\Gamma}_i,\ \overline{C}_i,\ \overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i},\ \overline{\tilde{\Gamma}_i \otimes \tilde{\Gamma}_i},\ \overline{\tilde{C}_i \otimes \tilde{C}_i},\ \overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i},\ \overline{\tilde{\Phi}_i \otimes \tilde{C}_i}, \tag{12}$$

i.e. the mean values and covariances.

Stabilizability is concerned with reducing state growth by means of control. Therefore it is associated with system $(\Phi_i, \Gamma_i) = (\Phi_i, \Gamma_i, \theta)$. Similar to [31,32] for (Φ_i, Γ_i) introduce the following *measure of minimal state growth in the mean achieved through full-state feedback control* over an arbitrary closed time interval $[i_s, i_f] \subseteq I$

$$\rho_{min}^{tm}(i_s, i_f) = \max_{x_{i_s} \neq 0} \left(\frac{\min\limits_{u_i | \{x_{i_s}, x_{i_s+1}, .., x_i\}} \left(\|\overline{x}_{i_f}\|^2 \right)}{\|\overline{x}_{i_s}\|^2} \right). \tag{13}$$

Similarly introduce the following *measure of minimal state growth in the mean-square sense achieved through full-state feedback control* over an arbitrary closed time interval $[i_s, i_f] \subseteq I$

$$\rho_{min}^{tms}(i_s, i_f) = \max_{x_{i_s} \neq 0} \left(\frac{\min\limits_{u_i | \{x_{i_s}, x_{i_s+1}, .., x_i\}} \left(\overline{\|x_{i_f}\|^2} \right)}{\overline{\|x_{i_s}\|^2}} \right). \tag{14}$$

Definition 2. The system (Φ_i, Γ_i) is called *temporal mean stabilizable* (tm-stabilizable) over $[i_s, i_f] \subseteq I$ if $\rho_{min}^{tm}(i_s, i_f) < 1$ and *temporal mean-square stabilizable* (tms-stabilizable) if $\rho_{min}^{tms}(i_s, i_f) < 1$

Loosely speaking tm-stabilizability and tms-stabilizability guarantee that control can always be selected such that the state does not grow in the mean respectively the mean-square sense. To obtain the main theorem in this section results concerning linear quadratic feedback control over a finite horizon have to be stated first. Let $+$ denote the Moore–Penrose pseudo inverse.

3.1. Finite horizon LQ feedback control

For the system (Φ_i, Γ_i) considered over $[i_s, i_f] \subseteq I$ consider the cost function

$$J_{LQ} = E\left\{ x_{i_f}^T H x_{i_f} + \sum_{i=i_s}^{i_f-1} x_i^T Q_i x_i + u_i^T R_i u_i \right\}, \tag{15}$$

where E also denotes mean or expected value and where $H \geq 0$, $Q_i \geq \theta$, $R_i \geq 0$. Let u_i be a deterministic function of $\{x_{i_s}, x_{i_s+1}, .., x_i\}$. From [4], that considers a similar problem, observe that with

$$u_i = -L_i x_i, \tag{16}$$

we obtain

$$J_{LQ} = \overline{x_{i_s}^T S_{i_s} x_{i_s}} = tr\left(S_{i_s} \overline{x_{i_s} x_{i_s}^T} \right), \tag{17}$$

where S_i, $i = i_s, i_{s+1}, ., i_f$ satisfy the generalized Riccati equation

$$S_i = \overline{\Phi_i^T S_{i+1} \Phi_i} - L_i^T \left(\overline{\Gamma_i^T S_{i+1} \Gamma_i} + R_i \right) L_i + Q_i,$$
$$i = i_s, i_s+1, .., i_f-1, \quad S_{i_f} = H. \tag{18}$$

The optimal solution denoted by the superscript * that minimizes the cost function is given by the following equation:

$$u_i^* = -L_i^* x_i, \ \ L_i^* = \left(\overline{\Gamma_i^T S_{i+1}^* \Gamma_i} + R_i \right)^+ \overline{\Gamma_i^T S_{i+1}^* \Phi_i}. \tag{19}$$

The main theorem in this section also requires the following technical lemma. To state this lemma let S^n denote the set of nonnegative real symmetric matrices with dimension, n.

Lemma 1. *Let $A, B \in S^n$. Then $\max\limits_{B \neq \theta}(tr(AB)/tr(B)) = \|A\|$. The maximum is achieved for $B = \alpha^2 ee^T$ where e is the eigenvector associated with the largest eigenvalue of A and $\alpha \neq 0$ an arbitrary scaling factor.*

Proof. If $A, B \in S^n$, from [13] Theorem 2 we obtain

$$\lambda_{min}(A) tr(B) \leq tr(AB) \leq \lambda_{max}(A) tr(B) = \|A\| tr(B), \tag{20}$$

where $\lambda_{min}(A)$, $\lambda_{max}(A)$ denote the smallest and largest eigenvalue of A. From the latter inequality we obtain $\max\limits_{B \neq \theta}(tr(AB)/tr(B)) = \lambda_{max}(A) = \|A\|$. The construction of B satisfying $tr(AB) = \lambda_{max}(A) tr(B)$ is provided in the proof of Theorem 2 in [13] by means of a singular value decomposition. For nonnegative matrices the positive singular values are identical to the positive eigenvalues while their associated vectors are identical up to a scalar factor. Therefore B given in [13] is equivalent with $B = \alpha^2 ee^T$, with e the eigenvector of A associated with $\lambda_{max}(A)$.

Let I_n denote the identity matrix of dimension n. The next theorem states how to compute both the tm and tms-stability and stabilizability measures from the system matrices using the results of the LQ problem above.

Theorem 1. *(a) Take $H = I_{n_{i_f}}, Q_i = \theta,\ R_i = \theta, L_i = \theta$, in (15)–(18). Then $\rho^{tms}(i_s, i_f) = \|S_{i_s}\|$. Furthermore $\rho^{tms}(i_r, i_f) = \|S_{i_r}\|$, $i_r \in [i_s, i_f]$.*

(b) Take $H = I_{n_{i_f}}, Q_i = \theta,\ R_i = \theta$ in (15)–(19). Then $\rho_{min}^{tms}(i_s, i_f) = \|S_{i_s}^*\|$. Furthermore $\rho_{min}^{tms}(i_r, i_f) = \|S_{i_r}^*\|$, $i_r \in [i_s, i_f]$.

(c) If in (15)–(19) we remove the overbar of all terms containing system matrices and next replace each system matrix by its mean, instead of $\rho^{tms}(i_r, i_f)$ we obtain $\rho^{tm}(i_r, i_f)$ and instead of $\rho_{min}^{tms}(i_r, i_f)$, $\rho_{min}^{tm}(i_r, i_f)$.

Proof. By setting to zero Q_i, R_i,L_i in (15)–(18) the system is effectively reduced to (Φ_i). With $H = I_{n_{i_f}}$ in (15)–(18) we have $J_{LQ} = \overline{x_{i_f}^T I_{n_{i_f}} x_{i_f}} = \overline{\|x_{i_f}\|^2}$ and from (4), (13) and Lemma 1, $\rho^{tms}(i_s, i_f) = \max\limits_{x_{i_s} \neq 0} \left(\frac{tr\left(S_{i_s} \overline{x_{i_s} x_{i_s}^T}\right)}{tr\left(\overline{x_{i_s} x_{i_s}^T}\right)} \right) = \|S_{i_s}\|$. Lemma 1 applies because from (18) S_{i_s} is independent of $\overline{x_{i_s} x_{i_s}^T}$. This proves (a). Taking $L_i = L_i^*$ in (18), with L_i^* given by (19), we obtain $J_{LQ}^* = tr\left(S_{i_s}^* \overline{x_{i_s} x_{i_s}^T} \right) = \min\limits_{u_i | x_{i_s}} \left(\overline{\|x_{i_f}\|^2} \right)$ and from (4), (13) and Lemma 1, $\rho_{min}^{tms}(i_s, i_f) = \max\limits_{x_{i_s} \neq 0} \left(\frac{tr\left(S_{i_s}^* \overline{x_{i_s} x_{i_s}^T}\right)}{tr\left(\overline{x_{i_s} x_{i_s}^T}\right)} \right) = \|S_{i_s}^*\|$. Finally from (18) we may replace i_s with any $i_r \in [i_s, i_f]$. This proves (b). The proof of (c) follows from [32].

Remark 2. From Lemma 1 and Theorem 1 we obtain all states for which optimal tms-feedback stabilization performs *worst*, i.e. states that grow *exactly* by the factor $\rho^{tms}(i_s, i_f)$ over interval $[i_s, i_f]$. These are all states satisfying the bound stated in Lemma 1. They have a second moment $\overline{x_{i_s} x_{i_s}^T} = \alpha^2 ee^T$ where e is now the eigenvector associated with the largest eigenvalue of $S_{i_s}^*$. Observe that $\overline{x_{i_s} x_{i_s}^T} = \alpha^2 ee^T$ has rank one and therefore describes states having just a single excited mode.

Definition 3. Denote a state x_{i_s} that solves $\max\limits_{x_{i_s} \neq 0} \left(\frac{tr\left(S_{i_s}^* \overline{x_{i_s} x_{i_s}^T}\right)}{tr\left(\overline{x_{i_s} x_{i_s}^T}\right)} \right)$ by $x_{i_s}^\rho$.

From the proof of Theorem 1(b) and Definition 3

$$\rho_{min}^{tms}(i_s, i_f) = \frac{tr\left(S_{i_s}^* \overline{x_{i_s}^\rho x_{i_s}^{\rho^T}} \right)}{tr\left(\overline{x_{i_s}^\rho x_{i_s}^{\rho^T}} \right)} = \frac{J_{LQ}^*\left(x_{i_s}^\rho \right)}{tr\left(\overline{x_{i_s}^\rho x_{i_s}^{\rho^T}} \right)} = \|S_{i_s}^*\| \tag{21}$$

Remark 3. Despite the fact that tms-stabilizability measure (13) represents a max–min computation, only a *single* optimal feedback computation is needed to determine it. This is due to the fact that the optimal feedback law (19) is *independent of the initial state* x_{i_s} and the fact that $\min_{u_i|x_{i_s}}\left(\overline{\|x_{i_f}\|^2}\right)$ is a *homogeneous function* of $\overline{\|x_{i_s}\|^2}$.

Remark 4. As to the computation of (15)–(19) from the system specification (12) note that

$$\overline{\Gamma_i^T S_{i+1}\Phi_i} = st^{-1}\left(\overline{\Phi_i \otimes \Gamma_i}^T st\left(S_{i+1}\right)\right), \quad (22)$$

where st is the stack operator that stacks the columns of a matrix into a column vector and st^{-1} its inverse that unstacks the column vector back into a matrix.

Remark 5. Observe from Eq. (18) that a single application of Theorem 1 requires computation of S_i, $i \in [i_s, i_f]$. This automatically provides all associated tm or tms-stabilizability measures $\|S_i\|$ associated with all sub-intervals $[i, i_f]$ of $[i_s, i_f]$, $i \in [i_s, i_f]$.

Remark 6. According to Definition 2 tm- and tms-stability and tm- and tms-stabilizability are associated with an interval $[i_s, i_f] \subseteq I$. To simplify statements, in the remainder of this paper we will no longer mention this interval, unless different intervals appear in one statement.

In [29] time intervals were detected over which time-varying discrete-time linear systems with *deterministic parameters* are temporal uncontrollable. These time intervals are the ones were temporal unstabilizability of these systems may occur. Next temporal stabilizability (t-stabilizability) of time-varying linear discrete-time systems with deterministic parameters can be checked using the results in [32]. Also from [32] temporal stabilizability is dual to temporal detectability (t-detectability).

Theorem 2. $\left(\overline{\Phi}_i^T, \overline{\Gamma}_i^T\right)$ *t-detectable* $\Leftrightarrow (\overline{\Phi}_i, \overline{\Gamma}_i)$ *t-stabilizable* $\Leftrightarrow (\Phi_i, \Gamma_i)$ *tm-stabilizable.*

Proof. The first equivalence follows from [32]. tm-Stabilizability of (Φ_i, Γ_i) considers temporal state behavior in the mean. This behavior is fully determined by $(\overline{\Phi}_i, \overline{\Gamma}_i)$ which proves the second equivalence.

Theorem 3. *(a)* (Φ_i) *tm-stable* $\Rightarrow (\Phi_i, \Gamma_i)$ *tm-stabilizable. (b)* (Φ_i) *tms-stable*$\Rightarrow (\Phi_i, \Gamma_i)$ *tms-stabilizable. (c)* (Φ_i, Γ_i) *tms-stabilizable*$\Rightarrow (\Phi_i, \Gamma_i)$ *tm-stabilizable.*

Proof. Parts (a) and (b) follow from Theorem 1 taking $L_i = \theta$ in (16) and (18). The proof of (c) is given in the Appendix.

Remark 7. At this stage one may wonder about relations between *temporal* (tms) and ordinary (ms) mean square stability and stabilizability properties. Ordinary ms-stability and ms-stabilizability only consider whether $\overline{\|x_i\|^2} \to 0$ as $i \to \infty$. This provides *no* information about the behavior of $\overline{\|x_i\|^2}$ over the specified finite interval $[i_s, i_f]$ that determines tms-stability and tms-stabilizability and vice-versa. Only when the interval $[i_s, i_f]$ over which tms-stability and tms-stabilizability is considered has the property $i_s = i_0,\ i_f \to \infty$ the properties become related. In that case, as $i \to \infty$, ms-stability and ms-stabilizability imply $\overline{\|x_i\|^2} \to 0$ whereas tms-stability and tms-stabilizability imply $\overline{\|x_i\|^2} < \overline{\|x_{i_s}\|^2}$ which is weaker.

Remark 8. According to (c) of Theorems 3 and 2 , $(\overline{\Phi}_i, \overline{\Gamma}_i)$ t-unstabilizable$\Rightarrow(\Phi_i, \Gamma_i)$ tms-unstabilizable. So $(\overline{\Phi}_i, \overline{\Gamma}_i)$ t-unstabilizable is *one cause* of tms-unstabilizability of (Φ_i, Γ_i) fully related to the first moments $\{\overline{\Phi}_i\}$, $\{\overline{\Gamma}_i\}$. It can be checked using the results from [5]. *The second cause* of tms-unstabilizability relates to the uncertainty in the system parameters. These do not affect state behavior in the mean but do affect state behavior in the mean-square sense. So besides intervals where t-unstabilizability of $(\overline{\Phi}_i, \overline{\Gamma}_i)$ occurs intervals where large parameter uncertainty exists are also of interest to check.

Remark 9. As with other mean-square (ms) properties of linear systems [10], tms-stability and tms-stabilizability become equal to their associated mean (m) properties tm-stability and tm-stabilizability, if all system matrices are replaced with their mean values. This may also be interpreted as replacing stochastic system parameters by associated deterministic ones having the same mean. This comes down to ignoring all covariances associated with system matrices being all terms with a tilde in (12).

When analyzing control systems, the state behavior over the entire interval $[i_s, i_f]$ is generally of interest, not just the behavior at initial time i_s and final time i_f. This behavior is partly considered by Theorem 1 that determines the stabilizability measure for each sub-interval $[i, i_f]$ of $[i_s, i_f]$, $i \in [i_s, i_f - 1]$ as noted in Remark 5. Similar to [5] the next theorem introduces a *one-step* mean-square stabilizability measure (osms-stabilizability measure) that applies to individual time instants.

Theorem 4. $\|S_i^*\| - \|S_{i+1}^*\|$ *is a one-step mean-square stabilizability measure (osms-stabilizability measure) that applies to individual time instants* $i \in [i_s, i_f - 1]$ *of time interval* $[i_s, i_f]$

Proof. $\rho_{min}^{tms}(i_s, i_f) = \|S_{i_s}^*\| = \|S_{i_f}^*\| + \sum_{i=1}^{i_f-1} \|S_i^*\| - \|S_{i+1}^*\|$ so $\|S_i^*\| - \|S_{i+1}^*\|$ is the *contribution at time i* to $\rho_{min}^{tms}(i_s, i_f)$. The smaller this contribution the better stabilizability.

Definition 4. Denote the one-step mean-square stabilizability measure mentioned in Theorem 4 by $\rho_{min}^{osms}(i, i_s, i_f) = \|S_i^*\| - \|S_{i+1}^*\|$. Then $(\Phi_i, \Gamma_i)$is called *one-step mean-square stabilizable* (osms-stabilizable) *at time* $i \in [i_s, i_f - 1]$ if $\rho_{min}^{osms}(i, i_s, i_f) < 0$.

Time instants where (Φ_i, Γ_i) is osms-stabilizable increase osms-stabilizability over time interval $[i_s, i_f]$ because $\rho_{min}^{osms}(i, i_s, i_f) < 0$ decreases $\rho_{min}^{tms}(i_s, i_f)$. Time instants where $\rho_{min}^{osms}(i, i_s, i_f) > 0$ decrease tms-stabilizability over time-interval $[i_s, i_f]$ because they increase $\rho_{min}^{tms}(i_s, i_f)$.

As a special case of $\rho_{min}^{osms}(i, i_s, i_f)$ we obtain a one-step mean stabilizability measure (osm-stabilizability measure) and an associated definition of one-step mean stabilizability.

Definition 5. $\rho_{min}^{osm}(i, i_s, i_f)$ is a *one-step mean stabilizability measure* (osm-stabilizability measure) of system (Φ_i, Γ_i) that equals $\rho_{min}^{osms}(i, i_s, i_f)$ associated with system $(\overline{\Phi}_i, \overline{\Gamma}_i)$. System (Φ_i, Γ_i) is called *one-step mean stabilizable* if $\rho_{min}^{osm}(i, i_s, i_f) < 0$.

For systems (Φ_i, Γ_i) with deterministic parameters *one-step stabilizability* (os-stabilizability) was defined in [32] and complies with the results in this paper as can be seen from the next definition.

Definition 6. (Φ_i, Γ_i) with deterministic parameters is called *one-step stabilizable* (os-stabilizable) if $\rho_{min}^{osms}(i, i_s, i_f) = \rho_{min}^{osm}(i, i_s, i_f) < 0$.

Remark 10. According to Definitions 4–6 os-, osm- and osms-stabilizability relate to time $i \in [i_s, i_f - 1] \subset I$. To simplify statements, in the remainder of this paper we will not specifically mention i, i_s, i_f unless different values are used within the same statement.

Theorem 5. *(a)* (Φ_i, Γ_i) *osm-stabilizable* $\Leftrightarrow (\overline{\Phi}_i, \overline{\Gamma}_i)$ *os-stabilizable. (b)* (Φ_i, Γ_i) *osms-stabilizable* $\Rightarrow (\Phi_i, \Gamma_i)$ *osm-stabilizable.*

Proof. Part (a) follows immediately from [32] and Definition 5. Part (b) is proved along the same lines as Theorem 3(c) that is given in the Appendix.

4. Temporal compensatability

Optimal full-state feedback computations were employed in the previous section to compute temporal mean and mean-square stabilizability measures over arbitrary intervals $[i_s, i_f] \subseteq I$. Since the full state is often unavailable for feedback output feedback is often the only means of stabilizing the system. This leads naturally to temporal stabilizability measures associated with dynamic output feedback called *compensation*. The associated measures are called *temporal compensatability measures*.

When the linear system has deterministic parameters the optimal compensation problem equals the LQG problem which is known to have the separation property [21]. This implies that the optimal compensator consists of a full-state estimator and full-state feedback law that can be computed separately. In addition their computations are dual and so are associated system properties. When the system has stochastic parameters separability and duality are lost. This requires considering another system property called mean-square compensatability (ms-compensatability) [9]. Of course here we are interested in temporal mean-square compensatability (tms-compensatability). Like in the previous section mean compensatability (m-compensatability) is obtained by replacing stochastic parameters by their mean values and applying standard LQG results. Associated with this, in this section we consider temporal mean compensatability (tm-compensatability). When the horizon is finite, the state estimator of the LQG compensator is known to *depend on initial conditions of the system*. Given Remark 3, this already shows that the computation of temporal compensatability measures will require more than a single compensator computation.

So instead of full-state feedback we now consider a linear full-order compensator for the system (Φ_i, Γ_i, C_i) over the interval $[i_s, i_f] \subseteq I$ given by the following equation:

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_i y_i, \quad u_i = -L_i \hat{x}_i, \quad i = i_s, i_s+1, ..., i_f-1, \tag{23}$$

where $\hat{x}_i \in R^{n_i}$ represents the compensator state with $\hat{x}_{i_s}$ deterministic and where F_i, K_i, L_i have dimensions compatible with (Φ_i, Γ_i, C_i). Compensator (23) is denoted by $(\hat{x}_{i_s}, F_i, K_i, L_i)$. At time i it uses observations $\{y_j, \ j = i_s, i_s+1, .., i-1\}$ to compute u_i. Also a-priori knowledge of $\bar{x}_{i_s}$, $\overline{x_{i_s} x_{i_s}^T}$ is presumed. Temporal mean and mean-square compensatability are concerned with the ability to limit state growth in the mean and mean-square sense over finite intervals $[i_s, i_f] \subseteq I$ by means of compensators $(\hat{x}_{i_s}, F_i, K_i, L_i)$. Similar to (13) introduce the following *temporal mean compensatability measure* (tm-compensatability measure):

$$\sigma_{min}^{tm}(i_s, i_f) = \max_{x_{i_s} \neq 0} \left(\frac{\min\limits_{(\hat{x}_{i_s}, F_i, K_i, L_i) | x_{i_s}} \left(\|\bar{x}_{i_f}\|^2 \right)}{\|\bar{x}_{i_s}\|^2} \right). \tag{24}$$

Similar to (14) introduce the following *temporal mean-square compensatability measure* (tms-compensatability measure):

$$\sigma_{min}^{tms}(i_s, i_f) = \max_{x_{i_s} \neq 0} \left(\frac{\min\limits_{(\hat{x}_{i_s}, F_i, K_i, L_i) | x_{i_s}} \left(\overline{\|x_{i_f}\|^2} \right)}{\overline{\|x_{i_s}\|^2}} \right). \tag{25}$$

In Eqs. (24) and (25) the state x_{i_s} is specified by its first moment $\bar{x}_{i_s}$ and second moment $\overline{x_{i_s} x_{i_s}^T}$.

Definition 7. (Φ_i, Γ_i, C_i) is called *temporal mean compensatable* (tm-compensatable) over $[i_s, i_f] \subseteq I$ if $\sigma_{min}^{tm}(i_s, i_f) < 1$ and *temporal mean-square compensatable* (tms-compensatable) if $\sigma_{min}^{tms}(i_s, i_f) < 1$.

Note that computation of tm and tms-compensatability measures $\sigma_{min}^{tm}(i_s, i_f)$, $\sigma_{min}^{tms}(i_s, i_f)$ in (25) can be done by algorithms that compute finite horizon optimal full-order compensators for time-varying linear discrete-time systems with white stochastic parameters minimizing the quadratic cost function (15). Such algorithms are presented in [36,37]. Denote such algorithms by the following equation:

$$A(x_{i_s}, \Phi_i, \Gamma_i, C_i, V_i, W_i, Q_i, R_i, H). \tag{26}$$

In (26) the arguments represent all problem data determining the finite-horizon optimal full-order compensator where V_i, W_i represent covariance matrices of additive white system and measurement noise [36,37]. Whenever a full-order compensator is used to minimize the costs J_{LQ} in (15) these costs will be denoted by J_C.

Theorem 6. *(a) A single run of* $A\left(x_{i_s}, \Phi_i, \Gamma_i, C_i, \theta, \theta, \theta, \theta, I_{n_{i_f}}\right)$ *provides* $\min\limits_{(\hat{x}_{i_s}, F_i, K_i, L_i) \,|\, x_{i_s}} \left(\overline{\|x_{i_f}\|^2} \right)$. *(b) A single run of* $A\left(x_{i_s}, \overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i, \theta, \theta, \theta, \theta, I_{n_{i_f}}\right)$ *provides* $\min\limits_{(\hat{x}_{i_s}, F_i, K_i, L_i) \,|\, x_{i_s}} \left(\|\bar{x}_{i_f}\|^2 \right)$.

Proof. With $H = I_{n_{i_f}}, Q_i = \theta$, $R_i = \theta$ in (15) we obtain $J_C = \overline{x_{i_f}^T I_{n_{i_f}} x_{i_f}} = \overline{\|x_{i_f}\|^2}$. $V_i = \theta$, $W_i = \theta$ describe that system (Φ_i, Γ_i, C_i) has no additive white system and measurement noise. This proves part (a). Part (b) follows from the fact that replacing the stochastic parameters by their mean values provides results in the mean which equal standard LQG results.

Remark 11. Note from its proof that case (b) in Theorem 6 is just a special case of (a) in Theorem 6. Similarly $\sigma_{min}^{tm}(i_s, i_f)$ is just a special case of $\sigma_{min}^{tms}(i_s, i_f)$.

The next theorem states how tms-compensatability measure (25) may be computed from a max–min optimization that is much less demanding than the original one stated by (25). To state this theorem we need to state three lemmas first. From now on let α denote any *positive* scalar. Recall that state x_{i_s} is the state of system (Φ_i, Γ_i, C_i) at time i_s fully specified by its first moment $\bar{x}_{i_s}$ and second moment $\overline{x_{i_s} x_{i_s}^T}$.

Lemma 2. $\max\limits_{x_{i_s} | \ \overline{\|x_{i_s}\|^2} = \alpha} \left(J_C^*(x_{i_s}) \right) = \max\limits_{x_{i_s} | \ \overline{\|x_{i_s}\|^2} = \alpha, \ \bar{x}_{i_s} = 0} \left(J_C^*(x_{i_s}) \right)$.

Proof. A fixed mean-square $\overline{\|x_{i_s}\|^2} = \alpha$ of initial state x_{i_s} still allows for different choices of the first moment $\bar{x}_{i_s}$ and covariance $\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}$ because $\overline{\|x_{i_s}\|^2} = tr\left(\overline{x_{i_s} x_{i_s}^T}\right)$ with $\overline{x_{i_s} x_{i_s}^T} = \bar{x}_{i_s} \bar{x}_{i_s}^T + \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}$. From the latter two relations it follows that for a given value $\overline{\|x_{i_s}\|^2} = \alpha$, the covariance matrix $\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}$ that represents the *uncertainty* in the initial state, is *maximal* if $\bar{x}_{i_s} = 0$. Maximal initial state uncertainty represents the *worst case* for optimal compensation because compensation involves state estimation. The worst case has *maximal* costs $\overline{\|x_{i_f}\|^2} = tr\left(\overline{x_{i_f} x_{i_f}^T}\right)$.

Related to state x_{i_s} of system (Φ_i, Γ_i, C_i) at time i_s, let $x_{i_s}^{\alpha}$ denote a state of system (Φ_i, Γ_i, C_i) at time i_s with mean $\sqrt{\alpha} \bar{x}_{i_s}$ and second moment $\alpha \overline{x_{i_s} x_{i_s}^T}$.

Lemma 3. $J_C^*\left(x_{i_s}^{\alpha}\right) = \alpha J_C^*(x_{i_s})$.

Proof. Given in the Appendix.

Lemma 4. $\max\limits_{x_{i_s} | \overline{\|x_{i_s}\|^2} = \alpha} \left(J_C^*(x_{i_s}) \right) = \alpha \max\limits_{x_{i_s} | \ \overline{\|x_{i_s}\|^2} = 1} \left(J_C^*(x_{i_s}) \right)$ *i.e.* $\max\limits_{x_i^s \neq 0} \left(J_{i_f}^*(x_{i_s}) \right)$ *is a homogeneous function of* $\overline{\|x_{i_s}\|^2}$.

Proof. Let $x_{i_s}^*$ denote x_{i_s} that solves $\max\limits_{x_{i_s} | \ \overline{\|x_{i_s}\|^2} = 1} \left(J_C^*(x_{i_s}) \right)$. So $\overline{\|x_{i_s}^*\|^2} = 1$ and $J_C^*\left(x_{i_s}^*\right)$ is the maximum. According to Lemma 3, $J_C^*\left(x_{i_s}^{*^{\alpha}}\right) = \alpha J_C^*\left(x_{i_s}^*\right)$ with $\overline{\|x_{i_s}^{*^{\alpha}}\|^2} = \alpha$. We need to prove $\max\limits_{x_{i_s} | \ \overline{\|x_{i_s}\|^2} = \alpha} \left(J_C^*(x_{i_s}) \right) = \alpha J_C^*\left(x_{i_s}^*\right)$ and do this by contradiction. Suppose x_{i_s}', $\overline{\|x_{i_s}'\|^2} = \alpha$ exists such that $J_C^*\left(x_{i_s}'\right) > \alpha J_C^*\left(x_{i_s}^*\right)$. Then by Lemma 3 $J_C^*\left(x_{i_s}' 1/\alpha\right) > \frac{1}{\alpha} \alpha J_C^*\left(x_{i_s}^*\right)$ with $\overline{\|x_{i_s}' 1/\alpha\|^2} = 1$ contradicting that $J_C^*\left(x_{i_s}^*\right)$ is maximal.

Theorem 7. $\sigma_{min}^{tms}(i_s, i_f) = \max\limits_{x_{i_s} | \ \overline{\|x_{i_s}\|^2} = 1, \ \bar{x}_{i_s} = 0} \left(J_C^*(x_{i_s}) \right)$

Proof. By Lemma 4 and Lemma 2 we may replace $\max_{x_{i_s} \neq 0}$ by $\max_{x_{i_s} \mid \|x_{i_s}\|^2 = 1,\, \bar{x}_{i_s} = 0}$ in Eq. (25).

Definition 8. Denote a state that solves $\max_{x_{i_s} \mid \|x_{i_s}\|^2 = 1,\, \bar{x}_{i_s} = 0} \left(J_C^*(x_{i_s})\right)$ by $x_{i_s}^{\sigma}$.

From Theorem 7 and Definition 8,

$$\sigma_{min}^{tms}(i_s, i_f) = \max_{x_{i_s} \mid \|x_{i_s}\|^2 = 1,\, \bar{x}_{i_s} = 0} \left(J_C^*(x_{i_s})\right) = J_C^*\left(x_{i_s}^{\sigma}\right) \tag{27}$$

Remark 12. From eqs. (4), (7) observe that Theorem 7 restricts the maximization to initial states x_{i_s} having zero mean and covariance satisfying $tr(\tilde{x}_{i_s}\tilde{x}_{i_s}^T) = 1$. The minimization is performed by a single run of optimal compensation algorithm $A(x_{i_s}, \Phi_i, \Gamma_i, C_i, \theta, \theta, \theta, \theta, I_{n_{i_f}})$ mentioned in Theorem 6. Moreover, since $\sigma_{min}^{tms}(i_s, i_f)$ depends on x_{i_s}, for each sub-interval $[i, i_f]$ of $[i_s, i_f]$, $i \in [i_s, i_f - 1]$, we must *recompute* $\sigma_{min}^{tms}(i, i_f)$. Compared with optimal state feedback, which required only *a single recursive computation* over the interval $[i_s, i_f]$ to produce $\rho_{min}^{tms}(i, i_f)$, $i \in [i_s, i_f - 1]$, this represents a significant increase of computational effort. However, since these computations concern control system analysis and design, they will generally be performed *off-line*.

To state and prove the next theorem we need the next lemma and remark.

Lemma 5. $J_C^*(x_{i_s}) \geq J_{LQ}^*(x_{i_s})$.

Proof. Given in the Appendix.

Remark 13. According to Definition 7 tm- and tms-compensatability are associated with an interval $[i_s, i_f] \subseteq I$. To simplify statements, in the remainder of this paper we will no longer mention this interval, unless different intervals appear in one statement.

Theorem 8. *(a) (Φ_i) tms-stable$\Rightarrow (\Phi_i, \Gamma_i, C_i)$ tms-compensatable.*

(b) (Φ_i) tm-stable$\Rightarrow (\Phi_i, \Gamma_i, C_i)$ tm-compensatable.

(c) (Φ_i, Γ_i, C_i) tms-compensatable $\Rightarrow (\Phi_i, \Gamma_i, C_i)$ tm-compensatable.

(d) $\rho_{min}^{tms}(i_s, i_f) \leq \sigma_{min}^{tms}(i_s, i_f)$, $\rho_{min}'^{tms}(i_s, i_f) \leq \sigma_{min}^{tms}(i_s, i_f)$ where $\rho_{min}'^{tms}(i_s, i_f)$ is the tms-stabilizability measure of system $\left(\Phi_i^T, C_i^T\right)$ with discrete-time reversed.

(e) (Φ_i, Γ_i, C_i) tms-compensatable $\Rightarrow (\Phi_i, \Gamma_i)$ tms-stabilizable, $\left(\Phi_i^T, C_i^T\right)$ tms-stabilizable.

(f) (Φ_i, Γ_i, C_i) tm-compensatable $\Rightarrow (\Phi_i, \Gamma_i)$ tm-stabilizable, $\left(\Phi_i^T, C_i^T\right)$ tm-stabilizable.

Proof. Taking $(\hat{x}_{i_s}, F_i, K_i, L_i) = (0, \theta, \theta, \theta)$ proves part (a) and (b). Part (c) is proved in the Appendix. To prove part (d) observe from Remark 2 and Definition 3 that without loss of generality we may take $tr\left(x_{i_s}^{\rho} x_{i_s}^{\rho T}\right) = \overline{\|x_{i_s}^{\rho}\|^2} = 1$ in Eq. (21). Then from (21), $\rho_{min}^{tms}(i_s, i_f) = J_{LQ}^*\left(x_{i_s}^{\rho}\right)$. Next from Definition 8, Lemma 5, Theorem 7 and (25)

$$\rho_{min}^{tms}(i_s, i_f) = J_{LQ}^*\left(x_{i_s}^{\rho}\right) \leq J_C^*\left(x_{i_s}^{\rho}\right) \leq J_C^*\left(x_{i_s}^{\sigma}\right) = \sigma_{min}^{tms}(i_s, i_f) \tag{28}$$

The proof of $\rho_{min}'^{tms}(i_s, i_f) \leq \sigma_{min}^{tms}(i_s, i_f)$ is dual as is further explained in the Appendix. Part (e) follows from (d) and Definition 2 and Definition 7. Finally (f) follows from (e) and Remark 9.

Remark 14. Temporal compensation involves both control and state estimation over a finite interval. One might imagine a growing transient of one, which is compensated for by a decreasing transient of the other, leading overall to temporal mean or mean-square compensatability. Associating control with temporal stabilizability of (Φ_i, Γ_i) and state estimation with temporal stabilizability of $\left(\Phi_i^T, C_i^T\right)$, Theorem 8(d) and (e) state that this type of compensation cannot happen. Finally Theorem 8(f) states that considering system parameters in the mean does not lead to equivalence of tm-compensatability with tm-stabilizability of both (Φ_i, Γ_i) and $\left(\Phi_i^T, C_i^T\right)$. Together with Theorem 2 this means that if (Φ_i, Γ_i, C_i) has deterministic parameters tm-compensatability, which similar to Theorem 2 may now be called t-compensatability, is still a *stronger* property than t-stabilizability of (Φ_i, Γ_i) plus t-detectability of (Φ_i, C_i). Because parameters are deterministic, $\Delta P_{i+1}^C = 0$ in (54) in the Appendix. Together with (38) and (53) in the Appendix we now find $J_{LQ}^*\left(x_{i_s}^{\rho}\right) + tr\left(\hat{P}_{i_f}^C\right) = J_C^*\left(x_{i_s}^{\rho}\right)$. So in Eq. (28) equality does not hold in general also since generally $x_{i_s}^{\rho} \neq x_{i_s}^{\sigma}$ in (28).

Similar to Theorem 4, (a) *one-step mean-square compensatability measure* (osms-compensatability measure) may be introduced and similar to Definition 4 *one-step mean-square compensatability* (osms-compensatability). Given Remark 12, instead of $\|S_i^*\|$ that could be used because $\rho_{min}^{tms}(i, i_f) = \|S_i^*\|$, $i \in [i_s, i_f]$, we now have to use $\sigma_{min}^{tms}(i, i_f)$ instead.

Theorem 9. *$\sigma_{min}^{tms}(i, i_f) - \sigma_{min}^{tms}(i+1, i_f)$ is a one-step mean-square compensatability measure (osms-compensatability measure) that applies to individual time instants $i \in [i_s, i_f - 1]$ of time interval $[i_s, i_f]$.*

Proof. $\sigma_{min}^{tms}(i_s, i_f) = 1 + \sum_{i=1}^{i_f - 1} \sigma_{min}^{tms}(i, i_f) - \sigma_{min}^{tms}(i+1, i_f)$ so $\sigma_{min}^{tms}(i, i_f) - \sigma_{min}^{tms}(i+1, i_f)$ is the *contribution at time i* to $\sigma_{min}^{tms}(i_s, i_f)$. The smaller this contribution the better compensatability.

Definition 9. Denote the one-step mean-square compensatability measure mentioned in Theorem 9 by $\sigma_{min}^{osms}(i, i_s, i_f) = \sigma_{min}^{tms}(i, i_f) - \sigma_{min}^{tms}(i+1, i_f)$. Then (Φ_i, Γ_i, C_i) is called *one-step mean-square compensatable* (osms-compensatable) *at time* $i \in [i_s, i_f - 1]$ if $\sigma_{min}^{osms}(i, i_s, i_f) < 0$.

Time instants where (Φ_i, Γ_i, C_i) is osms-compensatable increase osms-compensatability over time interval $[i_s, i_f]$ because $\sigma_{min}^{osms}(i, i_s, i_f) < 0$ decreases $\sigma_{min}^{tms}(i_s, i_f)$. Time instants where $\sigma_{min}^{osms}(i, i_s, i_f) > 0$ decrease osms-compensatability over time-interval $[i_s, i_f]$ because they increase $\sigma_{min}^{tms}(i_s, i_f)$.

Following Remark 9, as a special case of $\sigma_{min}^{osms}(i, i_s, i_f)$ we obtain a one-step mean compensatability measure (osm-compensatability measure) and an associated definition of one-step mean compensatability.

Definition 10. $\sigma_{min}^{osm}(i, i_s, i_f)$ is a *one-step mean compensatability measure* (osm-compensatability measure) of system (Φ_i, Γ_i, C_i) that equals $\sigma_{min}^{osms}(i, i_s, i_f)$ associated with system $\left(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i\right)$. System (Φ_i, Γ_i, C_i) is called *one-step mean compensatable* if $\sigma_{min}^{osm}(i, i_s, i_f) < 0$.

For systems (Φ_i, Γ_i, C_i) with deterministic parameters *one-step compensatability* was *not* defined in [32]. Similar to Theorem 5 (a) the following definition is obtained.

Definition 11. (Φ_i, Γ_i, C_i) with deterministic parameters is called *one-step compensatable* (os-compensatable) if $\sigma_{min}^{osm}(i, i_s, i_f) < 0$.

Remark 15. According to Definitions 11, 10 and 4, os-, osm- and osms-compensatability relate to time $i \in [i_s, i_f - 1] \subset I$. To simplify statements, in the remainder of this paper we will not specifically mention i, i_s, i_f unless different values are used within the same statement.

Theorem 10. *(a) (Φ_i, Γ_i, C_i) osm-compensatable $\Leftrightarrow \left(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i\right)$ os-compensatable. (b) (Φ_i, Γ_i, C_i) osms-compensatable $\Rightarrow (\Phi_i, \Gamma_i, C_i)$ osm-compensatable.*

Table 1
tms-Stabilizability and tms-compensatability measures Example 1, $\overline{\Phi}_i^{11} = 0.9$.

$\overline{\Phi}_i^{22}$	0.95	0.95	0.95	0.95	0.95	0.95	1.05	1.05	1.05	1.05	1.05	1.05
β	0	0.001	0.003	0.01	0.03	0.1	0	0.001	0.003	0.01	0.03	0.1
$\rho_{min}^{tms}(0,10)$	0.3585	0.3621	0.3694	0.3960	0.4818	0.9298	2.6533	2.6800	2.7340	2.9309	3.5658	6.8820
$\rho'_{min}tms(0,10)$	0.1216	0.1228	0.1253	0.1343	0.1634	0.3153	0.1216	0.1228	0.1253	0.1343	0.1634	0.3154
$\sigma_{min}^{tms}(0,10)$	0.3585	0.3892	0.4536	0.7107	1.7668	11.8250	2.6530	2.7588	2.9790	3.8469	7.3057	38.5872

Proof. Part (a) follows from Remark 9, Definitions 10 and 11. Part (b) is proved along the same lines as Theorem 8(c) that is given in the Appendix.

Remark 16. Comparing Theorem 10 with Theorem 8, one may wonder why equivalents of Theorem 8(d)–(f) do not appear in Theorem 10. The reason is that Theorem 8 only considers the state at times i_s and i_f. These states determine a tms-compensatability measure that upper bounds two types of tms-stabilizability measures. Theorem 10 considers the individual contributions at each time $i \in [i_s, i_f]$ to these measures. These individual contributions do not necessarily obey the same upper bounding relations even though their summations do. Other important reason for the failure of the individual bounding is that $x_{i_s}^{\rho}$ in Eq. (21) is generally different from $x_{i_s}^{\sigma}$ in Eq. (27). Also $\sigma_{min}^{tms}(i_s, i_f)$ depends on x_{i_s} as opposed to $\rho_{min}^{tms}(i_s, i_f)$

5. Applications and examples

In this section we will present a series of simple examples illustrating several important results from theorems in this paper. For illustrative purposes piecewise time-invariant VDD systems with constant dimensions are especially useful because their behavior and piecewise constant structure over finite time intervals are easily constructed and recognized [31,32].

Example 2. $\overline{\Phi}_i = \begin{bmatrix} \overline{\Phi}_i^{11} & 2 \\ 0 & \overline{\Phi}_i^{22} \end{bmatrix}$, $\overline{\Gamma}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\overline{C}_i = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $i = 0, 1, .., 10$

$\overline{\Phi}_i = \begin{bmatrix} \overline{\Phi}_i^{11} & 2 \\ 0.1 & \overline{\Phi}_i^{22} \end{bmatrix}$, $\overline{\Gamma}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\overline{C}_i = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $i = 11, 12, 13, 14$

$$\overline{\Phi_i \otimes \Phi_i} = \beta(\overline{\Phi}_i \otimes \overline{\Phi}_i), \quad \overline{\Gamma_i \otimes \Gamma_i} = \beta(\overline{\Gamma}_i \otimes \overline{\Gamma}_i), \quad \overline{C_i \otimes C_i} = \beta(\overline{C}_i \otimes \overline{C}_i)$$

First note that $\beta \geq 0$ in Example 2 is a parameter uncertainty measure of VDD system (Φ_i, Γ_i, C_i). If $\beta = 0$ the system has deterministic parameters. Observe that VDD system $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ in Example 2, i.e. system (Φ_i, Γ_i, C_i) in the mean, changes structure around $i = 11$ because then $\overline{\Phi}_i^{21}$, i.e. element 2,1 of $\overline{\Phi}_i$, becomes non-zero while $\overline{\Gamma}_i^{21}$ and $\overline{C}_i^{11}$ remain zero. The exact structure of VDD system $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ is obtained from the j-step, k-step Kalman decomposition presented in [3]. Taking $j = k = n = 2$, from Definition 7 in [3] we find that over interval [0, 10] system $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ is 2-step uncontrollable/unreachable. From Definition 9 in [30] system $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ is also 2-step unreconstructable/unobservable over interval [0, 10]. So from Remark 8, interval [0, 10] is important to check for tms-stabilizability and tms-compensatability of both $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ and (Φ_i, Γ_i, C_i). tms-Stabilizability and tms-compensatability measures over interval [0, 10] for different values of $\beta \geq 0$ and $\overline{\Phi}_i^{22}$ are listed in Table 1. These are obtained from Eqs. (18) and (19), Theorems 1(b), 8(d) and finally Eq. (26) and Theorem 6(a) and (b).

Computation of tms-compensatability measures according to Theorem 7 requires maximization over initial conditions having zero mean $\overline{x}_{i_s} = 0$ and unit mean-square $\overline{\|x_{i_s}\|^2} = tr\left(\overline{x_{i_s} x_{i_s}^T}\right) = 1$. Because $\overline{x}_{i_s} = 0$, from (4)–(7) $\overline{x_{i_s} x_{i_s}^T} = \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}$ and the maximization can be implemented by varying the U–D factors of the nonnegative covariance matrix $\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}$ with the diagonal elements constrained to be nonnegative. This minimizes the number of optimization parameters to $\frac{1}{2} n_0(n_0 - 1)$. From (4) observe that the unit mean-square can be realized by normalization: $\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T} \to \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T} / tr(\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T})$. This is achieved by dividing the nonnegative diagonal elements of the U–D factors by $tr(\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T})$. The constrained maximization was performed using Matlab function fminsearch with default settings.

From Remark 9, for $\beta = 0$ we find $\rho_{min}^{tm}(0, 10) = \rho_{min}^{tms}(0, 10)$ and $\sigma_{min}^{tm}(0, 10) = \sigma_{min}^{tms}(0, 10)$. When $\beta = 0$ the system has deterministic parameters. Only the second state of $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ in Example 2 is t-unreachable and autonomous over interval [0, 10]. Therefore this state fully determines both $\rho_{min}^{tm}(0, 10) = \rho_{min}^{tms}(0, 10)$. Alternatively these can be calculated as $\left(\left(\overline{\Phi}_i^{22}\right)^{10}\right)^2 = \left(\overline{\Phi}_i^{22}\right)^{20}$ [33]. Observe that the first state is t-unobservable over interval [0, 10] [30]. Then for $\beta = 0$ we obtain $\rho'^{tm}_{min}(0, 10) = \rho'^{tms}_{min}(0, 10) = \left(\overline{\Phi}_i^{11}\right)^{20} = (0.9)^{20} = 0.1216$.

As β increases, system parameter uncertainty increases and with it $\rho_{min}^{tms}(0, 10)$, $\rho'_{min}tms(0, 10)$ and $\sigma_{min}^{tms}(0, 10)$. For $\overline{\Phi}_i^{22} = 0.95$ this reveals that parameter uncertainty *alone*, i.e. an increase of β, can cause the system to become tms-unstabilizable ($\beta = 0.15$) and tms-uncompensatable ($\beta = 0.03,\ 0.15$). Finally observe that for each value β, $\rho_{min}^{tms}(0, 10) \leq \sigma_{min}^{tms}(0, 10)$, $\rho'_{min}tms(0, 10) \leq \sigma_{min}^{tms}(0, 10)$ as stated by Theorem 8(d). This condition appears to be very slightly violated for small values β in Table 1. This however is due to small numerical errors. For small parameter uncertainties and $\overline{\Phi}_i^{22} = 0.95$ in Table 1, the system is both tms-stabilizable as well as tms-compensatable, despite the t-unreachability and t-unobservability of the associated mean system. According to Theorem 8(d), (e) and Theorem 2 this implies that the mean system $(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i)$ is both t-stabilizable and t-detectable. This is due to $\left|\overline{\Phi}_i^{22}\right| < 1$ and $\left|\overline{\Phi}_i^{11}\right| < 1$ respectively [5]. When element $\left|\overline{\Phi}_i^{22}\right| \geq 1$ in Table 1, the mean system $(\overline{\Phi}_i, \overline{\Gamma}_i)$ becomes t-unstabilizable which from Theorem 2 is equivalent with (Φ_i, Γ_i) tm-unstabilizable. Then from Theorem 3 (c) and Theorem 8(d), (e) it follows that VDD system (Φ_i, Γ_i, C_i) is both tms-unstabilizable and tms-uncompensatable for all $\beta \geq 0$. This is confirmed by Table 1. Dually when $\left|\overline{\Phi}_i^{11}\right| \geq 1$ the mean system $(\overline{\Phi}_i, \overline{C}_i)$ becomes t-undetectable which from Theorem 2 is equivalent with $\left(\Phi_i^T, C_i^T\right)$ tm-unstabilizable implying (Φ_i, Γ_i, C_i) tms-unstabilizable, tm-uncompensatable and tms-uncompensatable.

Observe from Definitions 4 and 9 that to compute the one-step mean-square stabilizability and compensatability measures $\rho_{min}^{osms}(i, 0, 10)$, $\sigma_{min}^{osms}(i, 0, 10)$, $i = 0, 1, .., 9$ we need to compute $\rho_{min}^{tms}(i, 10)$, $i = 0, 1, .., 9$, $\sigma_{min}^{tms}(i, 10)$, $i = 0, 1, .., 9$. These are tms-stabilizability and compensatability measures of subintervals. From Theorem 1(b) and Remark 5, $\rho_{min}^{tms}(i, 10)$, $i = 0, 1, .., 9$ are

Table 2
tms-Stabilizability and compensatability measures of sub-intervals.

i	0	1	2	3	4	5	6	7	8	9	10
$\rho_{min}^{tms}(i,10)$	0.3960	0.4344	0.4766	0.5229	0.5736	0.6293	0.6904	0.7576	0.8363	0.9924	1
$\sigma_{min}^{tms}(i,10)$	5.6369	1.3862	1.3923	1.2265	1.0965	0.9971	0.9139	0.8402	0.7729	0.7107	1

Table 3
One-step stabilizability and compensatability measures associated with Table 2.

i	0	1	2	3	4	5	6	7	8	9
$\rho_{min}^{osms}(i)$	−0.0384	−0.0422	−0.0463	−0.0507	−0.0557	−0.0611	−0.0672	−0.0787	−0.1561	−0.0076
$\sigma_{min}^{osms}(i)$	4.2507	−0.0061	0.1658	0.1300	0.0994	0.0832	0.0737	0.0673	0.0622	−0.2893

Table 4
t-Stabilizability, t-detectability and t-compensatability over [i,15], i=11, 14.

i	11	12	13	14
$\rho_{min}^{tms}(i,15)$	2.7100E-08	4.1205E-08	8.4266E-08	9.1250E-01
$\rho_{min}^{\prime tms}(i,15)$	2.7100E-08	4.1205E-08	8.4266E-08	9.1250E-01
$\sigma_{min}^{tms}(i,15)$	1.6079E-07	7.6583E-02	1.2497E+00	5.6461E+00

directly obtained from the computation of $\rho_{min}^{tms}(0,10)$. They are recorded in Table 2. A similar result unfortunately does not hold for $\sigma_{min}^{tms}(i,10)$, $i=0,1,..,9$. This follows from Remark 3 and the fact that optimal compensators depend on initial conditions of the system, as mentioned at the start of this section. So $\sigma_{min}^{tms}(i,10)$, $i=0,1,..,9$ in Table 2 are obtained from separate computations. Table 2 records values for $\overline{\Phi}_i^{11}=0.9$, $\overline{\Phi}_i^{22}=0.95$ and $\beta=0.01$ in Example 2.

From Table 2 we now easily obtain the one-step mean-square stabilizability and compensatability measures $\rho_{min}^{osms}(i,0,10)$, $\sigma_{min}^{osms}(i,0,10)$, $i=0,1,..,9$ as differences of consecutive values in Table 2 with an appropriate sign. These are recorded in Table 3. From Table 3, Definitions 4 and 9 observe that (Φ_i,Γ_i,C_i) is osms-stabilizable at each time $i\in[0,9]$ while most of the time it is not osms-compensatable.

Consider the system in Example 2 with $\beta=0$, i.e. with deterministic parameters. Then $\rho_{min}^{tms}(\bullet,\bullet)$ equals the temporal stabilizability (t-stabilizability) measure developed in [32] for systems with deterministic parameters. Over time-interval [22,38] the system is 2-step controllable because $\overline{\Phi}_i^{21}\neq 0$. Therefore $\rho_{min}^{tms}(i,15)=\rho_{min}^{tm}(i,15)=0$, $11\le i\le 13$ and their computation concerns the *dead beat feedback control to zero* of the perfectly measured state. Dually, the system is also 2-step reconstructable over [22,38] so $\rho_{min}^{\prime tms}(i,15)=\rho_{min}^{\prime tm}(i,15)=0$, $11\le i\le 13$ and equal to the temporal detectability (t-detectability) measure developed in [32]. Since compensation involves both state estimation and state feedback, 2+2=4 time steps are needed for the temporal compensatability (t-compensatability) measure to become zero. This implies $\sigma_{min}^{tms}(11,15)=0$. All this is confirmed by Table 4 taking into account Remark 18 at the end of this section. Also for $i=12,13,14$, Table 4 shows that t-compensatability is *stronger* than t-stabilizability plus t-detectability since $\sigma_{min}^{tms}(i,15)>\rho_{min}^{tms}(i,15)$, $\sigma_{min}^{tms}(i,15)>\rho_{min}^{\prime tms}(i,15)$. In particular over [13,15] the system is not t-compensatable but is t-stabilizable plus t-detectable.

Remark 17. According to Theorem 6, the tms-compensatability measure may be calculated from repeated application of $A(x_{i_s},\Phi_i,\Gamma_i,C_i,V_i,W_i,Q_i,R_i,H)$ with $V_i=\theta,\ W_i=\theta,\ Q_i=\theta,\ R_i=\theta$. Setting $R_i=\theta$ and $W_i=\theta$ requires the use of the Moore–Penrose pseudo inverse in the algorithms instead of the standard inverse because the to be inverted matrices may become singular [9,36,37]. Numerically this may be problematic if the to be inverted matrices are close to singular because this may cause switching between singularity and non-singularity preventing convergence. Other reasons for deliberately choosing a small control penalty are mentioned in [31], Remark 3. We selected $R_i=\varepsilon I_{m_i}$ and $W_i=\varepsilon I_{l_i}$, $\varepsilon=10^{-6}$ in Example 2 as well as Example 1 in Section 2. As to Example 1, the same choice was made in the corresponding Example 2 presented in [32]. This choice results in slightly conservative tms-stabilizability and compensatability measures because a very small control penalty is introduced. Dually also a very small amount of measurement uncertainty is introduced.

Remark 18. Another numerical issue arises if the tms-compensatability measure becomes very close to the minimum zero. Although theoretically $H=I_{n_N}$ causes all modes of the system to contribute to the costs at each discrete-time instant, these contributions become very small if the compensatability measure becomes very close to zero. When they become zero numerically, this violates conditions for the optimal compensation algorithm to produce proper results [9,36,37]. This is prevented by selecting $V_i=\varepsilon I_{n_i}$ and $Q_i=\varepsilon I_{n_i}$, $\varepsilon=10^{-9}$ in Example 2. This causes slight additional conservatism in the tms-compensatability measures of Example 2.

6. Conclusions

Temporal loss of stability of closed loop control systems is of major concern to control engineers because it may cause the system state to grow beyond practical bounds while not being detected by conventional stability analysis. *New temporal system properties* to detect this, called tms-stability, tms-stabilizability and tms-compensatability and associated *measures* were introduced Algorithms to calculate the properties and associated measures from the system matrices were also presented. They apply to systems with white stochastic parameters. Because the measures were selected to be quadratic, the algorithms to compute them are algorithms to compute optimal full-order state and output feedback controllers for time-varying discrete-time linear systems with white stochastic parameters. Examples were presented illustrating major applications and causes of temporal loss of closed loop mean-square stability. One major application concerns digital optimal perturbation feedback control of non-linear systems. When system parameters instead of deterministic are stochastic, their uncertainty is *another* cause of loss of tms-stabilizability or tms-compensatability.

Computation of tms-compensatability and tms-stabilizability measures constitute max–min problems. The maximization concerns initial conditions, the minimization concerns control. As demonstrated in this paper, solving the max–min problem associated with a tms-stabilizability measure comes down to solving just a *single* LQ control problem. Moreover the solution of this LQ problem also provides tms-stabilizability measures associated with all subintervals having the same terminal time. The max–min problem associated with a tms-compensatability measure is more involved. The maximization can be restricted to initial conditions having zero-mean and a norm equal to one. The minimization of controls is performed by an algorithm that computes the optimal full-order output feedback controller. For each subinterval a separate computation is required. Because they concern analysis of control system designs, computations of tms-stabilizability and tms-compensatability measures will generally be performed *off-line*. In that case computational efficiency is not critical.

During the development and computation of tms-stabilizability and tms-compensatability we found, rather unexpectedly (see [37], remark 13), that some *finite-horizon* optimal full-order compensation problems appeared to possess local minima. Given the fact that solutions are unique in the infinite horizon time-invariant full-order case [9], this phenomenon must be due to the time-varying nature of the linear system and/or the influence of boundary conditions. When calculating tms-compensatability measures we require the global minimum. We tried to implement this by repeating computations that were initialized randomly and picking the best solution. Clearly this issue requires further investigation. Temporal *reduced-order* compensatability may be introduced and investigated along the lines of this paper and [12]. As in [37], by exploiting the delta domain we may carry over the results of this paper from discrete-time to continuous-time. Finally the use of stochastic parameters to enhance robustness of controller designs for linear or linearized systems that are time-varying is another topic that is by no means fully researched.

Appendix

Proof of Theorem 3. (c) Eq. (18), concerning state-feedback of system (Φ_i, Γ_i), may also be written as follows [9]:

$$S_i = \overline{(\Phi_i - \Gamma_i L_i)^T S_{i+1} (\Phi_i - \Gamma_i L_i)}, \quad S_{i_f} = I_{n_{i_f}}. \tag{29}$$

When system (Φ_i, Γ_i) is replaced with $(\overline{\Phi}_i, \overline{\Gamma}_i)$ Eq. (29) becomes

$$S_i = \left(\overline{\Phi}_i - \overline{\Gamma}_i L_i\right)^T S_{i+1} \left(\overline{\Phi}_i - \overline{\Gamma}_i L_i\right), \quad S_{i_f} = I_{n_{i_f}}. \tag{30}$$

Recall that replacing (Φ_i, Γ_i) by $(\overline{\Phi}_i, \overline{\Gamma}_i)$ produces results concerning the state mean instead of the state mean square. Let superscript m denote results concerning the state mean. If in (29) we substitute the optimal feedback gains L_i^* for system (Φ_i, Γ_i) we obtain S_i^*. If we substitute the *same* feedback gains in (30) we obtain S_i^m which are not optimal in general. From this observation and (29), (30) themselves, it follows immediately that $S_i^{m*} \le S_i^m \le S_i^*$. Using this and Theorem 1, $\rho^{tm}(i_s, i_f) = \|S_{i_s}^{m*}\| \le \|S_{i_s}^m\| \le \|S_{i_s}^*\| = \rho^{tms}(i_s, i_f)$ is obtained. From $\rho^{tm}(i_s, i_f) \le \rho^{tms}(i_s, i_f)$ and Definition 1, tms-stabilizability implies tm-stabilizability.

Algorithm properties. Algorithm $A(x_{i_s}, \Phi_i, \Gamma_i, C_i, \theta, \theta, \theta, \theta, I_{n_{i_f}})$, i.e. algorithm (26) presented in [36] with $Q_i = \theta$, $R_i = \theta$, $V_i = \theta$, $W_i = \theta$ is used to determine tms-compensatability measure $\sigma_{min}^{tms}(i_s, i_f)$. To prove results related to tms-compensatability we will need properties of this algorithm. Since in the current paper full-order compensators are considered $G_i = H_i = \tau_i = I_{n_i}$ applies in [36]. Also the ordinary matrix inverse may be replaced with the Moore–Penrose pseudo inverse [9] denoted by $+$. Then the following equations determine the optimal full-order compensator over interval $[i_s, i_f]$ computed by $A\left(x_{i_s}, \Phi_i, \Gamma_i, C_i, \theta, \theta, \theta, \theta, I_{n_{i_f}}\right)$

$$\begin{aligned} P_{i+1} = {} & \overline{\Phi_i P_i \Phi_i^T} - K_i \left(\overline{C_i P_i C_i^T} + \overline{\tilde{C}_i \hat{P}_i \tilde{C}_i^T}\right) K_i^T \\ & + \overline{\tilde{\Phi}_i \hat{P}_i \tilde{\Phi}_i^T} - \overline{\tilde{\Phi}_i \hat{P}_i L_i^T \tilde{\Gamma}_i^T} - \overline{\tilde{\Gamma}_i L_i \hat{P}_i \tilde{\Phi}_i^T} + \overline{\tilde{\Gamma}_i L_i \hat{P}_i L_i^T \tilde{\Gamma}_i^T}, \\ & i = i_s, i_s+1, .., i_f-1, \quad P_{i_s} = \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}, \end{aligned} \tag{31}$$

$$\begin{aligned} S_i = {} & \overline{\Phi_i^T S_{i+1} \Phi_i} - L_i^T \left(\overline{\Gamma_i^T S_{i+1} \Gamma_i} + \overline{\tilde{\Gamma}_i^T \hat{S}_{i+1} \tilde{\Gamma}_i + R_i}\right) L_i \\ & + \overline{\tilde{\Phi}_i^T \hat{S}_{i+1} \tilde{\Phi}_i} - \overline{\tilde{\Phi}_i^T \hat{S}_{i+1} K_i \tilde{C}_i} - \overline{\tilde{C}_i^T K_i^T \hat{S}_{i+1} \tilde{\Phi}_i} + \overline{\tilde{C}_i^T K_i^T \hat{S}_{i+1} K_i \tilde{C}_i}, \\ & i = i_s, i_s+1, .., i_f-1, \quad S_{i_f} = I_{i_f}, \end{aligned} \tag{32}$$

$$\begin{aligned} \hat{P}_{i+1} = {} & \left(\overline{\Phi}_i - \overline{\Gamma}_i L_i\right) \hat{P}_i \left(\overline{\Phi}_i - \overline{\Gamma}_i L_i\right)^T \\ & + K_i \left(\overline{C_i P_i C_i^T} + \overline{\tilde{C}_i \hat{P}_i \tilde{C}_i^T}\right) K_i^T, \quad i = i_s, i_s+1, .., i_f-1, \quad \hat{P}_{i_s} = \overline{x}_{i_s} \overline{x}_{i_s}^T, \end{aligned} \tag{33}$$

$$\begin{aligned} \hat{S}_i = {} & \left(\overline{\Phi}_i - K_i \overline{C}_i\right)^T \hat{S}_{i+1} \left(\overline{\Phi}_i - K_i \overline{C}_i\right) \\ & + L_i^T \left(\overline{\Gamma_i^T S_{i+1} \Gamma_i} + \overline{\tilde{\Gamma}_i^T \hat{S}_{i+1} \tilde{\Gamma}_i}\right) L_i, \quad i = i_s, i_s+1, .., i_f-1, \quad \hat{S}_{i_f} = \theta, \end{aligned} \tag{34}$$

$$K_i = \left(\overline{\Phi_i P_i C_i^T} + \overline{\tilde{\Phi}_i \hat{P}_i \tilde{C}_i^T}\right) \left(\overline{C_i P_i C_i^T} + \overline{\tilde{C}_i \hat{P}_i \tilde{C}_i^T}\right)^+, \quad i = i_s, i_s+1, .., i_f-1, \tag{35}$$

$$L_i = \left(\overline{\Gamma_i^T S_{i+1} \Gamma_i} + \overline{\tilde{\Gamma}_i^T \hat{S}_{i+1} \tilde{\Gamma}_i}\right)^+ \left(\overline{\Gamma_i^T S_{i+1} \Phi_i} + \overline{\tilde{\Gamma}_i^T \hat{S}_{i+1} \tilde{\Phi}_i}\right), \quad i = i_s, i_s+1, .., i_f-1, \tag{36}$$

$$F_i = \overline{\Phi}_i - K_i \overline{C}_i - \overline{\Gamma}_i L_i, \quad i = i_s, i_s+1, .., i_f-1. \tag{37}$$

The optimal costs $J^*(x_{i_s})$ associated with the optimal full-order compensator satisfying (31)–(37) are given by the following equation [36]:

$$J^*(x_{i_s}) = tr\left(P_{i_f} + \hat{P}_{i_f}\right) = tr\left(\overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}\left(S_{i_s} + \hat{S}_{i_s}\right) + \overline{x}_{i_s} \overline{x}_{i_s}^T S_{i_s}\right). \tag{38}$$

Matrices $P_i, \hat{P}_i$ in (31), (33) are related to the second moment of closed loop system (23), (1)

$$\begin{bmatrix} x_{i+1} \\ \hat{x}_{i+1} \end{bmatrix} = \begin{bmatrix} \Phi_i & -\Gamma_i L_i \\ K_i C_i & F_i \end{bmatrix} \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, \quad i = i_s, i_s+1, ..., i_f-1. \tag{39}$$

Let $x_i' = \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}$, $\Phi_i' = \begin{bmatrix} \Phi_i & -\Gamma_i L_i \\ K_i C_i & F_i \end{bmatrix}$. Denote the second moment of the closed loop system [39] by $P_i' = \overline{x_i' x_i'^T}$. Then the second moment P_i' propagates according to the following equation:

$$P_{i+1}' = \overline{\Phi_i' P_i' \Phi_i'^T}, \tag{40}$$

and is partitioned according to the following equation [39]:

$$P_i' = \begin{bmatrix} P_i^1 & P_i^{12} \\ P_i^{12^T} & P_i^2 \end{bmatrix}. \tag{41}$$

The costs $J(x_{i_s})$ obtained with compensator (F_i, K_i, L_i) is given by the following equation:

$$J(x_{i_s}) = tr\left(P_{i_f}^1\right). \tag{42}$$

If closed loop system [39] is *optimal*, i.e. when (31)-(37) are satisfied then [36,9]

$$P_i = P_i^1 - P_i^2, \tag{43}$$

$$\hat{P}_i = P_i^{12} = P_i^2. \tag{44}$$

Matrices $S_i, \hat{S}_i$ in (32), (34) are related to the dual of closed loop system [39] fully specified by $\Phi_i'^T$. Its second moment S_i' propagates according to the following equation:

$$S_i' = \overline{\Phi'^T S_{i+1}' \Phi_i'}, \quad i = i_s, i_s+1, ..., i_f-1, \tag{45}$$

and is partitioned dual to (41)

$$S_i' = \begin{bmatrix} S_i^1 & S_i^{12} \\ S_i^{12^T} & S_i^2 \end{bmatrix}. \tag{46}$$

$$P_{i_s}^1 = \overline{x_{i_s} x_{i_s}^T}, \; P_{i_s}^2 = \overline{\overline{x}_{i_s} \overline{x}_{i_s}^T} = \hat{x}_{i_s} \hat{x}_{i_s}^T \Rightarrow P_{i_s} = \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}, \; \hat{P}_{i_s} = \overline{x}_{i_s} \overline{x}_{i_s}^T = \hat{x}_{i_s} \hat{x}_{i_s}^T, \tag{47}$$

If closed loop system [39] is *optimal*, i.e. when (31)-(37) are satisfied then [36,9]

$$S_i = S_i^1 - S_i^2, \tag{48}$$

$$\hat{S}_i = S_i^2 = -S_i^{12}, i = i_s, i_s+1, .., i_f, \tag{49}$$

$$S_{i_f}^1 = I_{n_{i_f}}, \; S_{i_f}^2 = \theta \Rightarrow S_{i_f} = I_{n_{i_f}}, \hat{S}_{i_f} = \theta. \tag{50}$$

Proof of Lemma 3. Assume closed loop system (39) is optimal in the sense of Theorem 6, i.e. Eqs. (31)–(37) are satisfied as well as (39)–(50). Then from (47), (41), $P_{i_s}' = \begin{bmatrix} \overline{x_{i_s} x_{i_s}^T} & \overline{\overline{x}_{i_s} \overline{x}_{i_s}^T} \\ \overline{\overline{x}_{i_s} \overline{x}_{i_s}^T} & \overline{\overline{x}_{i_s} \overline{x}_{i_s}^T} \end{bmatrix}$. If $\overline{x_{i_s} x_{i_s}^T} \to \alpha \overline{x_{i_s} x_{i_s}^T}$, $\overline{x}_{i_s} \to \sqrt{\alpha} \overline{x}_{i_s}$ then from (40), (41), and (47),$P_{i_s}' \to \alpha P_{i_s}'$. Using (31)–(37), (39)–(50) one may verify that $P_i' \to \alpha P_i'$, $i \in [i_s, i_f]$, while S_i', $i \in [i_s, i_f]$ remain unchanged. Also the optimal compensator matrices F_i, K_i, L_i, $i \in [i_s, i_f-1]$ remain unchanged. Then from (38), (42), $J^*(x_{i_s}) = tr\left(P_{i_f} + \hat{P}_{i_f}\right) = tr\left(P_{i_f}^1\right) \to J^*\left(x_{i_s}^\alpha\right) = tr\left(\alpha P_{i_f}^1\right) = \alpha tr\left(P_{i_f}^1\right) = \alpha J^*(x_{i_s})$.

Proof of Lemma 5. Recall the propagation of S_i associated with state-feedback given by Eq. (29) that is another representation of Eq. (18) [7]. For convenience Eq. (29) is restated with S_i replaced with S_i^{LQ}

$$S_i^{LQ} = \overline{(\Phi_i - \Gamma_i L_i)^T S_{i+1}^{LQ} (\Phi_i - \Gamma_i L_i)}, \; S_{i_f}^{LQ} = I_{n_{i_f}}. \tag{51}$$

Next consider the propagation of S_i in Eq. (32) associated with full-order optimal compensation. S_i is denoted below by S_i^C. Its propagation can also be written as follows [9]:

$$S_i^C = \overline{(\Phi_i - \Gamma_i L_i)^T S_{i+1}^C (\Phi_i - \Gamma_i L_i)} + \Delta S_i^C, \; S_{i_f}^C = I_{n_{i_f}}, \tag{52}$$

where $\Delta S_i^C \geq 0 \in S^{n_i}$ captures the remaining terms in (32) not captured by the first term in Eq. (52). From (38) the minimal costs denoted by $J_C^*(x_{i_s})$ associated with the optimal compensator are $J_C^*(x_{i_s}) = tr\left(\overline{x_{i_s} x_{i_s}^T} S_{i_s}^C + \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T} \hat{S}_{i_s}^C\right)$. Applying the optimal compensator feedback gains L_i^* in (52) as state feedback gains in (51) leads to associated costs $J_{LQ}(x_{i_s}) = tr\left(\overline{x_{i_s} x_{i_s}^T} S_{i_s}^{LQ}\right)$ given by (17) that are no longer optimal. Because $\Delta S_i^C \geq 0$, from (51) and (52), $S_{i_s}^C \geq S_{i_s}^{LQ}$. Then $J_C^*(x_{i_s}) = tr\left(\overline{x_{i_s} x_{i_s}^T} S_{i_s}^C + \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T} \hat{S}_{i_s}^C\right) \geq J_{LQ}(x_{i_s}) = tr\left(\overline{x_{i_s} x_{i_s}^T} S_{i_s}^{LQ}\right) \geq J_{LQ}^*(x_{i_s})$.

Proof remaining parts of Theorem 8. Proof of (c). Eqs. (40), (42) represent the state propagation and costs associated with closed loop system [39] when an *arbitrary* compensator $(\hat{x}_{i_s}, F_i, K_i, L_i)$ is applied to system (Φ_i, Γ_i, C_i). If we replace (Φ_i, Γ_i, C_i) with $\left(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i\right)$ results are obtained concerning the state mean instead of the state mean square. Let the superscript m denote results concerning the state mean. Application of this replacement changes Φ_i' into $\Phi_i'^m = \overline{\Phi'}_i$. Suppose closed loop system [39] consists of (Φ_i, Γ_i, C_i) and the associated optimal compensator. Then (40) provides $P_i'*$. Next consider closed loop system [39] with the *same* compensator applied to $\left(\overline{\Phi}_i, \overline{\Gamma}_i, \overline{C}_i\right)$. This provides $P_i'^m$ which are not optimal in general. From (40) observe that the change of Φ_i' into $\Phi_i'^m = \overline{\Phi'}_i \Rightarrow P_i^m \leq P_i^*$. From [20], Eq. (42) and Definition 3 applied to the state mean, it follows that $P_i^m \leq P_i^* \Rightarrow P_i^{1^m} \leq P_i^{1^*} \Rightarrow J^m\left(x_{i_s}^{\rho^m}\right) \leq J^*\left(x_{i_s}^{\rho^m}\right)$. Then $\sigma^{tm}(i_s, i_f) = J^{*^m}\left(x_{i_s}^{\rho^m}\right) \leq J^m\left(x_{i_s}^{\rho^m}\right) \leq J^*\left(x_{i_s}^{\rho^m}\right) \leq J^*\left(x_{i_s}^{\sigma}\right) = \sigma^{tms}(i_s, i_f)$. Using Definition 7 from $\sigma^{tm}(i_s, i_f) \leq \sigma^{tms}(i_s, i_f)$ it follows that tms-compensatability implies tm-compensatability.

Finally we prove the inequality $\rho_{min}'^{tms}(i_s, i_f) \leq \sigma_{min}^{tms}(i_s, i_f)$. Reconsider the proof of Lemma 5 with Φ_i replaced by Φ_i^T, Γ_i by C_i^T, L_i by K_i^T and S by P in (51), (52). Also reverse time and apply the boundary conditions (47) to obtain

$$P_{i+1}^{LQ} = \overline{(\Phi_i - K_i C_i) P_i^{LQ} (\Phi_i - K_i C_i)}, \; P_{i_s}^{LQ} = \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}, \tag{53}$$

$$P_{i+1}^C = \overline{(\Phi_i - K_i C_i) P_i^C (\Phi_i - K_i C_i)} + \Delta P_{i+1}^C, \; P_{i_s}^C = \overline{\tilde{x}_{i_s} \tilde{x}_{i_s}^T}. \tag{54}$$

Now Eq. (54) equals Eq. (31) when $\Delta P_{i+1}^C \geq 0 \in S^{n_{i+1}}$ captures the remaining terms in (31) not captured by the first term in (54). From (38) the costs of the associated optimal compensator are now represented by $J_C^*(x_{i_s}) = tr\left(P_{i_f}^C + \hat{P}_{i_f}^C\right)$. Because $\Delta P_{i+1}^C \geq 0$ in (54), from (53), (54) $P_{i_f}^C \geq P_{i_f}^{LQ}$. Then $J_{LQ}^*\left(x_{i_s}^\rho\right) = tr\left(P_{i_f}^{LQ}\right) \leq tr\left(P_{i_f}^C + \hat{P}_{i_f}^C\right) = J_C^*\left(x_{i_s}^\rho\right) \leq J_C^*\left(x_{i_s}^\sigma\right)$ from which we obtain (28) again. But now $J_{LQ}^*\left(x_{i_s}^\rho\right) = tr\left(P_{i_f}^{LQ}\right)$ is associated with $\left(\Phi_i^T, C_i^T\right)$ instead of (Φ_i, Γ_i). Therefore in (28) $\rho_{min}^{tms}(i_s, i_f)$ changes into $\rho_{min}'^{tms}(i_s, i_f)$.

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