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Including Errata:
Equations (2.16), (2.17), (2.20)-(2.23), (2.25), (4.2).
Mapping current paper onto
Van Willigenburg (1998), De Koning (1992):
$X_1 = P,\ X_2 = S,\ X_3 = \hat{P},\ X_4 = \hat{S}$.

U-D factorisation of the strengthened discrete-time optimal projection equations

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Algorithms for optimal reduced-order dynamic output feedback control of linear discrete-time systems with white stochastic parameters are U-D factored in this paper. U-D factorisation enhances computational accuracy, stability and possibly efficiency. Since U-D factorisation of algorithms for optimal full-order output feedback controller design was recently published by us, this paper focusses on the U-D factorisation of the optimal oblique projection matrix that becomes part of the solution as a result of order-reduction. The equations producing the solution are known as the optimal projection equations which for discrete-time systems have been strengthened in the past. The U-D factored strengthened discrete-time optimal projection equations are presented in this paper by means of a transformation that has to be applied recursively until convergence. The U-D factored and conventional algorithms are compared through a series of examples.

Keywords: systems with state and control-dependent noise; optimal reduced-order controller design; compensatability and optimal compensation; multiplicative white noise; stochastic parameters; UDU factorisation

1. Introduction

There are mainly three reasons why linear systems with white stochastic parameters (also referred to as systems with state and/or control dependent noise or systems with multiplicative white noise) are important. First, system parameters may be white by their very nature (Wagenaar & De Koning, 1988). Second, parameters may be *assumed* white to obtain non-conservative robust feedback controllers with respect to structured parameter uncertainty (Banning & De Koning, 1995; Bernstein, 1987; Willems & Willems, 1983; Yaz & Skelton, 1994). Finally systems with stochastic parameters may arise due to stochastic sampling, randomly varying delays or Markovian jumps of system structure (Antunes, Hespanha, & Silvestre, 2009; De Koning & Van Willigenburg, 2001; Immer, Yükselb, & Basar, 2006; Karimi, 2013; Kögel, Blind, Allgöwer, & Findeisen, 2011; Li, Zhoua, & Wub 2013; Matveev & Savkin, 2003; Shi & Yu, 2011; Tsai & Ray, 1999). Possible other applications of systems with stochastic parameters can for instance be found in Karimi (2013), Yaz and Skelton (1994) and references therein.

Design of controllers for linear systems with white stochastic parameters has received considerable attention in the control literature (De Koning, 1982, 1992; Gunckel & Franklin, 1963; Hyland, 1982; Joshi, 1976; Karimi, 2013; Kleinman, 1969; Li, Zhou, & Duzhi, 2013; McLane, 1971; Moore, Xun, Zhou, & Lim, 1999; Phillis, 1985; Van Willigenburg & De Koning, 2010; Willems & Willems, 1983; Yaz, 1988; Yaz & Skelton, 1994). The same applies to order reduction of controllers (Bernstein & Hyland, 1986; Jaimoukha, Haitham, Limebeer, & Shah, 2005; Kin & Rantzer, 2010; Liu & Anderson, 1989). Related but different from order-reduction, information structure constraints are also being considered (Rubió-Masseqú, Rossell, Karimi, & Palacios-Quiñonero, 2013). One major application of the algorithms in this paper concerns *perturbation* feedback control of non-linear systems using linearised models (Athans, 1971). Especially if the dimension of the linearised model is large, i.e. after spatial discretisation of an infinite dimensional system, order-reduction is vital. Order-reduction and dynamic output feedback control of linear systems with stochastic parameters are simultaneously and optimally addressed by the algorithms in this paper. Applications involving optimal output feedback controllers may also be found in Fujimoto, Ota, and Nakayama (2011), Hounkpevi and Yaz (2008), Meenakshi and Bhat (2006) and Boje (2005). Several recent developments in this area are described by Karimi (2013).

U-D factorisation applies to non-negative matrices and enhances the numerical accuracy, stability and possibly the efficiency of computations involving these matrices. It has, for example, been employed to improve computations related to Kalman filtering (Bierman, 1977), reduced-order LQG controller design (Van Willigenburg & De Koning, 2004) and adaptive set membership filtering (Zhou, Han, & Liu, 2008).

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For linear systems with white stochastic parameters, the current state of the art of algorithms producing optimal dynamic output feedback controllers, called *mean square compensators*, can be found in Van Willigenburg and De Koning (2010). Linear time-invariant and time-varying systems are considered in both continuous and discrete-time. In addition, both optimal full- and reduced-order mean square compensators are considered for both a finite and infinite horizon. Finally differences between the ordinary and strengthened optimal projection equations are addressed.

The algorithms that are U-D factored in this paper verify *reduced-order mean square compensatability* (De Koning & Van Willigenburg, 1998; Van Willigenburg & De Koning, 2010). This is a *system property* required for the existence of an *optimal reduced-order mean square compensator*, if the horizon is infinite. The algorithms also compute *optimal reduced-order mean square compensators* over an infinite horizon. The algorithms solve the strengthened discrete-time optimal projection equations (SDOPE) which are equivalent to first-order necessary optimality conditions (De Koning & Van Willigenburg, 1998, 2010). In the special full-order case they can be used to compute a unique globally optimal mean square compensator (De Koning, 1992). The U-D factorisation of the associated full-order algorithm was recently published by us (Van Willigenburg & De Koning, 2013). In this paper we focus on the extension caused by order-reduction. This causes the entry of an optimal oblique projection matrix into the SDOPE. This significantly complicates matters because this matrix is not symmetric and non-negative. To our best knowledge, U-D factorisation of the SDOPE is considered here for the first time.

One final remark as to the mathematical notation in this paper: we use Kronecker products to represent statistics of system matrices, instead of sums of matrices multiplied by scalar stochastic processes. Using Kronecker products is more general and concise as explained in Van Willigenburg and De Koning (2013).

2. Compensatability and the optimal compensation problem

Since our main research interest concerns digital optimal control system design, in this paper we will consider discrete-time linear systems with white stochastic parameters. In addition to the multiplicative white noise, the discrete-time linear systems are also corrupted by additive white system and measurement noise. These discrete-time systems are described by

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i,\ i = 0, 1, \dots, \tag{2.1}$$

$$y_i = C_i x_i + w_i,\ i = 0, 1, \dots. \tag{2.2}$$

In Equations (2.1) and (2.2) $x_i \in R^n$ represents the system state, $u_i \in R^m$ the control inputs and $y_i \in R^l$ the observations at time, $i = 0, 1, \dots$. Furthermore v_i represents discrete-time zero-mean additive white system noise and w_i discrete-time zero-mean additive white measurement noise. Because the discrete-time system has white stochastic parameters, at each discrete-time instant, $i = 0, 1, \dots$, the system matrices $\Phi_i,\ \Gamma_i,\ C_i$ have entries that instead of deterministic, are white stochastic variables. As a result the processes $\{\Phi_i,\ i = 0, 1, \dots\}$, $\{\Gamma_i,\ i = 0, 1, \dots\}$, $\{C_i,\ i = 0, 1, \dots\}$ are sequences of independent random matrices. They are assumed to have *constant* statistics like $\{v_i,\ i = 0, 1, \dots\}$, $\{w_i,\ i = 0, 1, \dots\}$ which are sequences of independent stochastic vectors. We assume $\Phi_i,\ \Gamma_i,\ C_i$, are independent of v_j and w_j, $i \neq j$ and uncorrelated with $v_i,\ w_i$. The processes $\{v_i,\ i = 0, 1, \dots\}$, $\{w_i,\ i = 0, 1, \dots\}$ are zero-mean with covariance matrices $V \geq 0$, $W \geq 0$ and cross-covariance matrix Y, $\begin{bmatrix} V & Y \\ Y^T & W \end{bmatrix} \geq 0$. To facilitate U-D factorisation the processes $\{\Phi_i,\ i = 0, 1, \dots\}$, $\{\Gamma_i,\ i = 0, 1, \dots\}$, $\{C_i,\ i = 0, 1, \dots\}$ are assumed mutually uncorrelated. For stochastic vectors, the mean or first moment as well as the covariance and second moment are well known. The mean or first moment of $\{\Gamma_i,\ i = 0, 1, \dots\}$ is denoted by

$$\bar{\Gamma} \in R^{n \times m}. \tag{2.3}$$

where the overbar denotes expectation and where the subscript i of Γ is deleted indicating the statistics are constant. Each element of $\bar{\Gamma}$ thus equals the average of the corresponding element in Γ_i. Next define,

$$\tilde{\Gamma}_i = \Gamma_i - \bar{\Gamma}. \tag{2.4}$$

Then the covariance of $\{\Gamma_i,\ i = 0, 1, \dots\}$ equals

$$\overline{\tilde{\Gamma} \otimes \tilde{\Gamma}} \in R^{n^2 \times m^2} \tag{2.5}$$

where the subscript i of $\tilde{\Gamma}$ is deleted again. The second moment of $\{\Gamma_i,\ i = 0, 1, \dots\}$ equals

$$\overline{\Gamma \otimes \Gamma} \in R^{n^2 \times m^2} \tag{2.6}$$

where the subscript i of Γ is deleted again. It satisfies

$$\overline{\Gamma \otimes \Gamma} = \bar{\Gamma} \otimes \bar{\Gamma} + \overline{\tilde{\Gamma} \otimes \tilde{\Gamma}} \tag{2.7}$$

Similar relations apply to the processes $\{\Phi_i,\ i = 0, 1, \dots\}$ and $\{C_i,\ i = 0, 1, \dots\}$. Finally consider the dynamic output feedback compensator,

$$\hat{x}_{i+1} = F\hat{x}_i + Ky_i, \tag{2.8}$$

$$u_i = -L\hat{x}_i,\ i = 0, 1, \dots, \tag{2.9}$$

where $\hat{x}_i \in R^{n_c}$, $i = 0, 1, \ldots$ is the compensator state having prescribed dimension $n_c \leq n$. Denote this compensator by (F, K, L). Call (F, K, L) *minimal* if the matrix pair (F, K) is controllable and the matrix pair (F, L) is observable. Associated to this compensator consider the closed loop system,

$$\begin{bmatrix} x_{i+1} \\ \hat{x}_{i+1} \end{bmatrix} = \begin{bmatrix} \Phi_i & -\Gamma_i L \\ KC_i & F \end{bmatrix} \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, i = 0, 1, \ldots. \tag{2.10}$$

Introduce

$$x_i' = \begin{bmatrix} x_{i+1} \\ \hat{x}_{i+1} \end{bmatrix}, \; \Phi_i' = \begin{bmatrix} \Phi_i & -\Gamma_i L \\ KC_i & F \end{bmatrix}. \tag{2.11}$$

Then the closed loop system is also represented by

$$x_{i+1}' = \Phi_i' x_i', \; i = 0, 1, \ldots. \tag{2.12}$$

Let ρ denote spectral radius. From De Koning (1992) the closed loop system (2.12) is *mean-square stable (ms-stable)* if

$$\rho\left(\overline{\Phi' \otimes \Phi'}\right) < 1. \tag{2.13}$$

Definition 1: (De Koning & Van Willigenburg, 1998)

If for the system (2.1), (2.2), there exists a compensator (2.8), (2.9) with state dimension n_c such that the closed loop system (2.12) is ms-stable the system (2.1), (2.2) is called *n_c-mean-square compensatable* (n_c-ms-compensatable). With respect to system (2.1), (2.2), such a compensator is called *mean-square stabilising* (ms-stabilising).

The optimal reduced-order compensation problem to which the U-D factored algorithms presented in this paper apply can now be stated. Within this problem statement E denotes expectation.

2.1. Optimal reduced-order compensation problem

For the system (2.1), (2.2), find the minimal ms-stabilising compensator (F^*, K^*, L^*) with given state dimension $n_c \leq n$ that minimises the infinite horizon quadratic sum criterion,

$$\sigma_\infty(F, K, L) = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{i=1}^{N} [\, x_i \; u_i \,] \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}\right\},$$
$$\begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \geq 0 \tag{2.14}$$

and find the associated minimum costs, $\sigma_\infty^*(F^*, K^*, L^*)$.

According to Equations (2.3)–(2.7), De Koning (1992) and De Koning & Van Willigenburg (1998) the following problem data entirely determine the solution of the optimal compensation problem,

$$n_c, \; \bar{\Phi}, \; \overline{\tilde{\Phi} \otimes \tilde{\Phi}} = V^{\Phi\Phi}, \; \bar{\Gamma}, \; \overline{\tilde{\Gamma} \otimes \tilde{\Gamma}} = V^{\Gamma\Gamma}, \; \bar{C}, \; \overline{\tilde{C} \otimes \tilde{C}} = V^{CC}, \; Q, \; R, \; M, \; V, \; W, \; Y. \tag{2.15}$$

2.2. Algorithms for compensatability and optimal compensation

The algorithm suitable for U-D factorization holds for the case $M = \theta$, $Y = \theta$ where θ denotes a zero matrix. The algorithm is similar to the one presented in Van Willigenburg and De Koning (2013). It solves four coupled matrix equations, two of them being generalised Riccati equations and two of them being generalised Lyapunov equations. In the reduced-order case these four equations involve the additional oblique projection matrix τ. For convenience these four coupled matrix equations are captured in a single transformation. Let S^n denote the set of symmetric real $n \times n$ matrices. Let $X = \{X_1, X_2, X_3, X_4\}$, $X_1, X_2, X_3, X_4 \in S^n$. Define,

$$K_X = \bar{\Phi} X_1 \bar{C}^T \left(\overline{CX_1C^T} + \overline{\tilde{C}X_3\tilde{C}^T} + W\right)^\dagger, \tag{2.16}$$

$$L_X = \left(\overline{\Gamma^T X_2 \Gamma} + \overline{\tilde{\Gamma}^T X_4 \tilde{\Gamma}} + R\right)^\dagger \bar{\Gamma}^T X_2 \bar{\Phi}, \tag{2.17}$$

$$F_X = \bar{\Phi} - \bar{\Gamma} L_X - K_X \bar{C}, \tag{2.18}$$

where $\dagger$ denotes a generalised inverse of a square matrix having the properties,

$$AA^\dagger A = A, \; A^\dagger A A^\dagger = A^\dagger. \tag{2.19}$$

Also define,

$$\Psi_1 = \left(\bar{\Phi} - K_X \bar{C}\right)^T X_3 \left(\bar{\Phi} - K_X \bar{C}\right) + L_X^T \left(\overline{\Gamma^T X_2 \Gamma} + \overline{\tilde{\Gamma}^T X_4 \tilde{\Gamma}} + R\right) L_X, \tag{2.20}$$

$$\Psi_2 = \left(\bar{\Phi} - \bar{\Gamma} L_X\right) X_4 \left(\bar{\Phi} - \bar{\Gamma} L_X\right)^T + K_X \left(\overline{CX_1C^T} + \overline{\tilde{C}X_3\tilde{C}^T} + W\right) K_X^T, \tag{2.21}$$

$$\tau = \left(X_3 X_4\right)^\# \in R^{n\times n}, \; \tau_\perp = I_n - \tau, \tag{2.22}$$

where $\#$ denotes the group generalised or Drazin inverse and I_n the identity matrix of dimension n. Then the transformation C: $S^n \times S^n \times S^n \times S^n \to S^n \times S^n \times S^n \times S^n$ that captures the necessary optimality conditions for optimal reduced-order compensation known as the SDOPE is given by (De Koning & Van

Willigenburg, 1998; Van Willigenburg & De Koning, 2013)

$$CX = \begin{pmatrix} \overline{(\Phi - K_X C) X_1 (\Phi - K_X C)^T} \\ \overline{+(\tilde{\Phi} - \tilde{\Gamma} L_X) X_3 (\tilde{\Phi} - \tilde{\Gamma} L_X)^T} + \\ K_X \left(\overline{\tilde{C} X_3 \tilde{C}^T} + W \right) K_X^T + V + \tau_\perp \Psi_1 \tau_\perp^T, \\ \overline{(\Phi - \Gamma L_X)^T X_2 (\Phi - \Gamma L_X)} \\ \overline{+(\tilde{\Phi} - K_X \tilde{C})^T X_4 (\tilde{\Phi} - K_X \tilde{C})} + \\ L_X^T \left(\overline{\tilde{\Gamma}^T X_4 \tilde{\Gamma}} + R \right) L_X + Q + \tau_\perp^T \Psi_2 \tau_\perp, \\ \frac{1}{2} \left(\tau \Psi_1 + \Psi_1 \tau^T \right), \\ \frac{1}{2} \left(\tau^T \Psi_2 + \Psi_2 \tau \right) \end{pmatrix} \quad (2.23)$$

Call $X = \{X_1, X_2, X_3, X_4\}$ non-negative if $X_1, X_2, X_3, X_4 \geq 0$. The two compensatability tests as well as the algorithm to compute the optimal reduced-order compensator seek non-negative solutions of the equation

$$X = CX, \quad (2.24)$$

by recursively applying the transformation C until convergence. A transformation similar to C is specified in De Koning and Van Willigenburg (1998) using a different, more convenient, representation. However, that representation is not suitable for U-D factorisation and is modified here according to Van Willigenburg and De Koning (2013) that is suitable for U-D factorisation. Together these results provide the following theorem.

Theorem 1: *Assume system* (2.1), (2.2) *is* n_c*-ms-compensatable and* $Q > 0$, $V > 0$. *If* $X^* = CX^* \geq 0$ *and* $rank\left(X_3^*\right) = rank\left(X_4^*\right) = rank\left(X_3^* X_4^*\right) = n_c$ *then* X^* *satisfies the necessary optimality conditions for reduced order compensation,* $(F^*, K^*, L^*) = \left(H F_{X^*} G^T, H K_{X^*}, L_{X^*} G^T\right)$ *where* $G, H \in R^{n_c \times n}$ *satisfy* $G^T H = \tau$, $G H^T = I_{n_c}$*and,*

$$\sigma_\infty^* = \mathrm{tr}\left[Q X_1^* + \left(Q + L_{X^*}^T R L_{X^*} \right) X_3^* \right]$$
$$= \mathrm{tr}\left[V X_2^* + \left(V + K_{X^*} V K_{X^*}^T \right) X_4^* \right]. \quad (2.25)$$

Remark 1: Observe from De Koning (1992), Theorem 3, that the condition $Q > 0$, $V > 0$ may be replaced with $R > 0$, $W > 0$, $\left(\Phi_i, V^{1/2}, Q^{1/2}\right)$ ms-detectable.

Remark 2: As opposed to the full-order case, Equation (2.25) may represent a *local* minimum that may exist in the reduced-order case because (2.24) may have multiple non-negative solutions. The full-order case is represented by Equation (2.23) setting $n_c = n$, $\tau = I_n$, $\tau_\perp = \theta_n$ where θ_n denotes a zero matrix of dimension n (De Koning & Van Willigenburg, 1998). Then Equation (2.23) becomes equal to Equation (6.14) in Van Willigenburg and De Koning (2013) after reordering the four elements of the transformation. In the full-order case, Equation (2.25) represents the global minimum (De Koning, 1992).

Remark 3: In the full-order case recursive application of transformation C starting from $X \geq 0$ guarantees both symmetry and non-negativeness of X during iteration (Van Willigenburg & De Koning, 2013). The situation is considerably more complicated in the reduced-order case. Although symmetry is still guaranteed, observe that non-negativeness is no longer guaranteed because of the third and fourth component of transformation C represented by Equation (2.23). To restore non-negativeness one may be tempted to replace $\frac{1}{2}\left(\tau \Psi_1 + \Psi_1 \tau^T\right)$ by $\tau \Psi_1 \tau^T$ and $\frac{1}{2}\left(\tau^T \Psi_2 + \Psi_2 \tau\right)$ by $\tau^T \Psi_2 \tau$. Doing so, the *conventional* discrete-time optimal projection equations (CDOPE) are implemented. The CDOPE are *weaker* than the SDOPE and provide erroneous solutions in general, because iterations of C leave τ unchanged. Although SDOPE may not satisfy $X \geq 0$ during parts of the iteration, if they converge they converge to solutions that do (De Koning & Van Willigenburg, 1998; Van Willigenburg & De Koning, 2000). U-D factorisation of X however demands $X \geq 0$ at all times. One major contribution of this paper is to present two modifications of C guaranteeing this.

Remark 4: Another complication in the reduced-order case is that $X^* = CX^*$ must satisfy the rank conditions $rank\left(X_3^*\right) = rank\left(X_4^*\right) = rank\left(X_3^* X_4^*\right) = n_c$ mentioned in Theorem 1. They can be realised by upper bounding the rank of τ to n_c, during iterations of C. To that end Equation (2.22) must be modified (De Koning & Van Willigenburg, 1998; Van Willigenburg & De Koning, 2000). Upper bounding the rank comes down to dropping a suitable part of the computation. If this part is not selected suitably convergence is not generally achieved. Another major contribution of this paper concerns the adaptation of Equation (2.22) to achieve this, directly from the U-D factors of X_3, X_4, instead of X_3, X_4 themselves. So what is called 'squaring up' of U-D factors to produce X_3, X_4 is prevented in Equation (2.22). This enhances computational accuracy and efficiency.

3. Modifications to ensure non-negativeness and facilitate U-D factorisation

The subtle but crucial difference between the SDOPE and CDOPE relates to the following equalities:

$$X_3^* = \tau \Psi_1 \tau^T = \tau \Psi_1 = \Psi_1 \tau^T,$$
$$X_4^* = \tau^T \Psi_2 \tau = \tau^T \Psi_2 = \Psi_2 \tau. \quad (3.1)$$

The CDOPE do not require the second and third equality for X_3^*, X_4^* in Equation (3.1) to hold, as opposed to the SDOPE. The second and third equality in Equation (3.1) need to hold however to ensure equivalence with first-order necessary optimality conditions (De Koning & Van Willigenburg, 1998; Van Willigenburg & De Koning, 2000). Therefore, if iterations of C are to converge to satisfy Equation (3.1), the third and fourth component of transformation C *must* involve $\tau\Psi_1$ and $\tau^T\Psi_2$ respectively, as in Equation (2.23). The terms $\tau\Psi_1$ and $\tau^T\Psi_2$ are neither symmetric nor non-negative by definition. According to Equation (3.1), they need to converge to become both symmetric and non-negative.

We first present two modifications that ensure the non-negativeness of X during iteration of C as mentioned in Remark 3 of the previous section. Assume Equation (3.1) is satisfied. Then the third and fourth component of C in Equation (2.23) may be replaced by

$$\Psi_1 - \tau_\perp \Psi_1 \tau_\perp^T,\ \ \Psi_2 - \tau_\perp^T \Psi_2 \tau_\perp, \tag{3.2}$$

which also include $\tau\Psi_1$ and $\tau^T\Psi_2$, as required. Although symmetric, the two terms in Equation (3.2) are not non-negative by definition. They also do not generally lead to convergence of C as mentioned in Van Willigenburg and De Koning (2000). Let $\lambda_1^{\min}$, $\lambda_2^{\min}$ denote the smallest, possibly negative, eigenvalues of the first and second term in Equation (3.2), respectively. One modification is to take the third and fourth component of C to be,

$$\Psi_1 - \tau_\perp \Psi_1 \tau_\perp^T + \max\left(-\lambda_1^{\min}, 0\right) I_n,\\ \Psi_2 - \tau_\perp^T \Psi_2 \tau_\perp + \max\left(-\lambda_2^{\min}, 0\right) I_n. \tag{3.3}$$

The terms in Equation (3.3) are guaranteed to be non-negative. Moreover, they are equal to Equation (3.2) whenever the terms in Equation (3.2) are non-negative. In Section 5 it is demonstrated that replacing the third and fourth component of C by Equation (3.3) does generally lead to convergence. Unfortunately, the negative signs appearing in Equation (3.3) prevent direct updating from U-D factors of Ψ_1, Ψ_2. The negative signs in Equation (3.3) occur precisely because of the need to involve $\tau\Psi_1$ and $\tau^T\Psi_2$.

Next we describe our second possible modification of the third and fourth component of C in Equation (2.23) guaranteeing non-negativeness of X during iteration of C. Consider the singular value decompositions,

$$\tau\Psi_1 = U_1 S_1 V_1^T,\ \ \tau^T\Psi_2 = U_2 S_2 V_2^T. \tag{3.4}$$

Then the third and fourth component of C in Equation (2.23) are taken to be,

$$U_1 S_1 U_1^T,\ \ U_2 S_2 U_2^T, \tag{3.5}$$

Because U, V are unitary real matrices and S diagonal matrices with real non-negative singular values on the diagonal, Equation (3.5) enforces both symmetry and non-negativeness. Moreover, they are equal to the terms in Equation (3.4) whenever these are non-negative and symmetric. Unfortunately, Equations (3.4) and (3.5) also require 'squaring up' Ψ_1, Ψ_2 from their U-D factors because $\tau\Psi_1$, $\tau^T\Psi_2$ are not symmetric in general.

Finally, in this section we present the computation of τ having maximal rank n_c directly from the U-D factors of X_3, X_4 as mentioned in Remark 4 in the previous section. To that end consider Cholesky decompositions,

$$X_3 = S_3 S_3^T,\ \ X_4 = S_4 S_4^T, \tag{3.6}$$

and the singular value decomposition,

$$S_3^T S_4 = U_{34} S_{34} V_{34}^T. \tag{3.7}$$

Assume the singular value decomposition (3.7) has the singular values in descending order on the diagonal of S_{34}. Then if $rank(X_3) = rank(X_4) = rank(X_3 X_4) = n_c$ Equation (3.7) equals

$$S_3^T S_4 = U_{34}(:, 1:n_c) S_{34}(1:n_c, 1:n_c)\, V_{34}(:, 1:n_c)^T \tag{3.8}$$

where the notation in between brackets in Equation (3.8) complies with Matlab notation, for example, $(:, 1:n_c)$ indicates all rows and the first n_c columns of U_{34}. Observe that $S_{34}(1:n_c, 1:n_c)$ in Equation (3.8) is a square diagonal invertible matrix. Then according to lemma 4 and the associated constructive algorithm presented by Zigic, Watson, and Beattie (1993),

$$\tau = S_3 U_{34}(:, 1:n_c)\, S_{34}^{-1}(1:n_c, 1:n_c)\, V_{34}(:, 1:n_c)^T\, S_4^T, \tag{3.9}$$

while G, H in Theorem 1 are given by

$$G = S_{34}^{-1/2} U_{34}(:, 1:n_c)^T\, S_3^T,\ \ H = S_{34}^{-1/2} V_{34}(:, 1:n_c)^T\, S_4^T \tag{3.10}$$

If $\operatorname{rank}(X_3 X_4) = n_r > n_c$ Equation (3.9) realises upper bounding of $\operatorname{rank}(\tau)$ to n_c. If $\operatorname{rank}(X_3 X_4) = n_r < n_c$ then n_c is replaced with n_r in Equations (3.9) and (3.10). The Cholesky decompositions (3.6) may be obtained directly from U-D factorisations of X_3, X_4 like $\operatorname{rank}(X_3)$ and $\operatorname{rank}(X_4)$, as shown in the next section.

4. U-D factorisation

The algorithm to compute optimal reduced-order compensators as well as the compensatability tests recursively applies the transformation C until convergence. The transformation C was specified by Equation (2.23) and modified in the previous section to ensure symmetry and non-negativeness of X at all times during iteration. Symmetry and non-negativeness are numerically attractive properties which are required to perform U-D factorisation. Moreover, they simplify the statements of the compensatability tests and main theorem that are stated first in this section. To state these, let Σ_n represent a square positive diagonal matrix of dimension n.

4.1. Compensatability test 1 (De Koning, 1992)

Choose $Q = V = I_n$, $R = \theta_n$, $W = \theta_n$. If $C^i(\theta_n, \Sigma_n, \theta_n, \Sigma_n)$ converges as $i \to \infty$ then system (2.1), (2.2) is n_c-ms-compensatable.

4.2. Compensatability test 2 (De Koning, 1992)

Choose $Q = V = \theta_n$, $R = \theta_n$, $W = \theta_n$. Let $(X_{1,i}, X_{2,i}, X_{3,i}, X_{4,i}) = C^i(\theta_n, \Sigma_n, \theta_n, \Sigma_n)$. If $\lim_{i\to\infty}\left[\frac{\mathrm{tr}(X_{1,i+1}+X_{3,i+1})}{\mathrm{tr}(X_{1,i}+X_{3,i})}\right] < 1$ then system (2.1), (2.2) is n_c-ms-compensatable.

Remark 5: As explained in De Koning (1992),

$$\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right) = \lim_{i\to\infty}\left[\frac{tr(X_{1,i+1} + X_{3,i+1})}{tr(X_{1,i} + X_{3,i})}\right] \tag{4.1}$$

where $\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right)$ denotes a *minimum* of the spectral radius of the closed loop system (2.12), achievable with a compensator. Therefore, Equation (4.1), computed by compensatability test 2, is actually a *measure* of n_c-ms-compensatability.

Remark 6: Notice that compensatability test 1 and 2 represent sufficient but not necessary conditions for n_c-ms-compensatability. This is due to the fact that in the reduced-order case $X = CX$ may have multiple non-negative solutions. As a result, convergence may sometimes not be obtained depending on Σ_n. Therefore Σ_n is mentioned in the compensatability tests instead of I_n that is mentioned in De Koning and Van Willigenburg (1998). In the full-order case a unique limit does exist, Σ_n may be replaced by I_n and the compensatability tests are necessary and sufficient. Then Equation (4.1) represents the global minimum (De Koning, 1992).

The main theorem below states a constructive solution of the optimal reduced-order compensation problem.

Theorem 2: *Assume system* (2.1), (2.2) *is* n_c*-ms-compensatable and* $Q > 0$, $V > 0$. *Then, if* $X^* = \lim_{i\to\infty} C^i(\theta_n, \Sigma_n, \theta_n, \Sigma_n)$ *exists,* X^* *is a non-negative solution of the equation* $X = CX$. *If, moreover,* $n_c = rank\left(X_3^* X_4^*\right)$ *then* $F^* = HF_{X^*}G^T$, $K^* = HK_{X^*}$, $L^* = L_{X^*}G^T$ *where* $G, H \in R^{n_c \times n}$ *satisfy* $G^T H = \tau$, $GH^T = I_{n_c}$ *and*

$$\begin{aligned}\sigma_\infty^* &= \mathrm{tr}\left[QX_1^* + \left(Q + L_{X^*}^T R L_{X^*}\right)X_3^*\right] \\ &= \mathrm{tr}\left[VX_2^* + \left(V + K_{X^*} V K_{X^*}^T\right)X_4^*\right].\end{aligned} \tag{4.2}$$

Proof: Theorem 2 follows from De Koning and Van Willigenburg (1998) and the modifications described in Section 3.

Remark 7: Generally, the condition $n_c = \mathrm{rank}\left(X_3^* X_4^*\right)$ in Theorem 2 is met because transformation C upper bounds $\mathrm{rank}\,(X_3 X_4)$ to n_c while it generally increases $\mathrm{rank}\,(X_3 X_4)$ if it falls below n_c. In exceptional cases, however, $n_c = \mathrm{rank}\left(X_3^* X_4^*\right)$ cannot be met. This happens if the full-order optimal compensator, that is globally optimal, has a minimal realisation with state dimension $n_m < n_c$. In that case $n_m = \mathrm{rank}\left(X_3^* X_4^*\right) < n_c$ and (F^*, K^*, L^*) is a minimal realisation of the full-order compensator that is globally optimal. Although globally optimal, (F^*, K^*, L^*) is not *formally* a solution of the reduced-order compensation problem because its state dimension is less than the prescribed one. Therefore, it is preferable to specify what is called a *max–min* prescribed compensator dimension $n_c \le n_m$ (Van Willigenburg & De Koning, 2002).

Finally, in this section we consider the U-D factorisation of the modified transformation C. The following two equations represent the basic computations used in Van Willigenburg and De Koning (2013) that are also used here to U-D factorise the algorithm,

$$P_1 = U_{P_1} D_{P_1} U_{P_1}^T, \quad P_2 = U_{P_2} D_{P_2} U_{P_2}^T, \tag{4.3}$$

$$P_3 = F P_1 F^T + P_2 = U_{P_3} D_{P_3} U_{P_3}^T. \tag{4.4}$$

In Equations (4.3) and (4.4) all matrices are square and have the same dimension. Equation (4.3) represents U-D factorisations of P_1, P_2 whereas Equation (4.4) specifies P_3 as well as its associated U-D factorisation. U represents unit upper triangular matrices and D non-negative diagonal matrices. Starting from P_1, P_2 one algorithm (A1) computes U_{P_1}, D_{P_1} and U_{P_2}, D_{P_2},

$$\mathrm{A1} : P \to U_P, D_P. \tag{4.5}$$

Another algorithm (A2) computes U_{P_3}, D_{P_3} in Equation (4.4) from $U_{P_1}, D_{P_1}, U_{P_2}, D_{P_2}$ and F,

$$\mathrm{A2} : U_{P_1}, D_{P_1}, U_{P_2}, D_{P_2}, F \to U_{P_3}, D_{P_3}. \tag{4.6}$$

Algorithms A1 and A2 are described in Bierman (1977, pp. 100–101, 131–133). The modification needed to apply them to non-negative instead of positive matrices is presented in Van Willigenburg and De Koning (2013).

The first two components of transformation C in Equation (2.23) are identical to the corresponding ones in Van Willigenburg and De Koning (2013) except for the final terms $\tau_\perp \Psi_1 \tau_\perp^T$ and $\tau_\perp^T \Psi_2 \tau_\perp$ that are added. Furthermore, $\Psi_1,\ \Psi_2$ in Equations (2.20) and (2.21) are identical to the other two components of C in Van Willigenburg and De Koning (2013). Their U-D factorisation, therefore, follows from Van Willigenburg and De Koning (2013). Having U-D factorisations of $\Psi_1,\ \Psi_2$ adding the terms $\tau_\perp \Psi_1 \tau_\perp^T$ and $\tau_\perp^T \Psi_2 \tau_\perp$ can be realised by algorithm A2 as follows. From $U_{\Psi_1}, D_{\Psi_1}, \tau_\perp$ we first compute $U_{\tau_\perp \Psi_1 \tau_\perp^T}, D_{\tau_\perp \Psi_1 \tau_\perp^T}$ using A2. Using A2 again the U-D factors of the first component of C are updated to include $\tau_\perp \Psi_1 \tau_\perp^T$. Similar arguments apply to the second component of C.

Consider Equation (3.3) that computes the final two components of C. The equation is less favourable because it requires subtracting $\tau_\perp \Psi_1 \tau_\perp^T$ and $\tau_\perp^T \Psi_2 \tau_\perp$ which cannot be performed by algorithm A2. Therefore, we are now forced to recover Ψ_1 from U_{Ψ_1}, D_{Ψ_1} and $\tau_\perp \Psi_1 \tau_\perp^T$ from $U_{\tau_\perp \Psi_1 \tau_\perp^T}, D_{\tau_\perp \Psi_1 \tau_\perp^T}$ to compute $\Psi_1 - \tau_\perp \Psi_1 \tau_\perp^T$ through ordinary matrix subtraction. Similar arguments apply to $\Psi_2 - \tau_\perp^T \Psi_2 \tau_\perp$. Next, eigenvalues of $\Psi_1 - \tau_\perp \Psi_1 \tau_\perp^T$ and $\Psi_2 - \tau_\perp^T \Psi_2 \tau_\perp$ have to be computed to determine $\lambda_1^{\min},\ \lambda_2^{\min}$ which are also needed to compute Equation (3.3). Finally, U-D factorisation of (3.3) has to be performed by algorithm A1. Alternatively, the last two components of C may be computed from Equations (3.4) and (3.5). As already mentioned in the previous section, they require recovering Ψ_1 from U_{Ψ_1}, D_{Ψ_1} and similarly for Ψ_2. After computation of (3.4), (3.5), U-D factorisation of (3.5) has to be performed by algorithm A1.

Finally, consider the computation of τ according to (3.6)–(3.9). As indicated in the previous section, S_3 and S_4 can be obtained from their U-D factors,

$$S_3 = U_{X_3} D_{X_3}^{1/2},\ \ S_4 = U_{X_4} D_{X_4}^{1/2} \tag{4.7}$$

where $D_{X_3}^{1/2}$ and $D_{X_4}^{1/2}$ are calculated taking scalar square roots of the non-zero diagonal elements of D_{X_3}, D_{X_4}. Also rank (X_3) is the number of non-zero diagonal elements of D_{X_3} and similarly for rank (X_4).

As indicated in De Koning and Van Willigenburg (1998) and Van Willigenburg and De Koning (2000) introduction of numerical damping may enhance convergence of the algorithms in critical cases. After each single iteration of transformation C numerical damping is realised by the following additional computation:

$$X_{j,i} := (1-a)\, X_{j,i} + a X_{j,i-1},\ \ j = 1,2,3,4,\quad i = 1,2,\ldots, \tag{4.8}$$

where lower index i indicates the result X_j after the ith iteration and $0 \le a < 1$ is the numerical damping factor. Let $U_{j,i},\ D_{j,i}$ denote the U-D factors of $X_{j,i}$. Then, the U-D factored implementation of Equation (4.8) multiplies the diagonal elements of $D_{j,i}$ and $D_{j,i-1}$ with the scalars $1-a$ and a, respectively and next uses a simplified version of A2 described by Equations (4.4) and (4.6), with $F = I_n$, to add the two terms on the right in Equation (4.8).

5. Numerical considerations and examples

The purpose of this paper is algorithm development for reduced-order control system design. To judge their overall performance, randomly generated examples up to a significant system order are most appropriate. These will be used in this section. We deliberately avoid specific industrial examples since their focus is different, namely towards a specific application. This does not mean to say that we consider industrial applications of less importance. On the contrary, control is an *applied science* and consideration of specific industrial applications we consider to be a *major*, *next* research step.

In Van Willigenburg and De Koning (2013) a similar algorithm development was presented for full-order control system design. It is highly interesting to see the effect of order-reduction on algorithm performance. Therefore, we use almost the same randomly generated examples in this paper. In the examples the system state dimension n varies from 2 to 70.

Example 1 in Van Willigenburg and De Koning (2013) turns out to be hardly affected by order-reduction from full-order $n_c = n = 2$ to reduced-order $n_c = 1$. To more clearly illustrate the effect of order reduction, element 2,1 of $\bar{\Phi}$ in Example 1 below is different.

Example 1:

$$\bar{\Phi} = \begin{bmatrix} 0.7092 & 0.3017 \\ 0.3017 & 0.9525 \end{bmatrix},\ \bar{\Gamma} = \begin{bmatrix} 0.7001 \\ 0.1593 \end{bmatrix},\quad \bar{C} = \begin{bmatrix} 0.3088 & 0.5735 \end{bmatrix} \tag{5.1}$$

$$\overline{\tilde{\Phi} \otimes \tilde{\Phi}} = V^{\Phi\Phi} = \beta_1 \bar{\Phi} \otimes \bar{\Phi},\ \ \overline{\tilde{\Gamma} \otimes \tilde{\Gamma}} = V^{\Gamma\Gamma} = \beta_2 \bar{\Gamma} \otimes \bar{\Gamma},\quad \overline{\tilde{C} \otimes \tilde{C}} = V^{CC} = \beta_3 \bar{C} \otimes \bar{C},\quad \beta_1, \beta_2, \beta_3 \ge 0 \tag{5.2}$$

$$V = diag\begin{pmatrix} 0.5627 & 0.7357 \end{pmatrix},\ W = 0.2588,\quad Q = diag\begin{pmatrix} 0.7350 & 0.9820 \end{pmatrix},\ R = 0.6644 \tag{5.3}$$

Observe that the problem data specification Equations (5.1)–(5.3) complies with Equation (2.15). From Van Willigenburg and De Koning (2013) and De Koning (1993) observe that $\beta_1,\ \beta_2,\ \beta_3 \ge 0$ in Equation (5.2) are measures of uncertainty of the system matrices $\Phi_i,\ \Gamma_i,\ C_i$, respectively. As these measures increase the minimal spectral

Table 1. Minimal spectral radius and costs against parameter uncertainty.

β_1	β_2	β_3	$\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right)\ n_c = n = 2$	$\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right)\ n_c = 1$	$\sigma^*_\infty\ n_c = n = 2$	$\sigma^*_\infty\ n_c = 1$
0.05	0.05	0.05	5.3353E−01	5.3361E−01	8.0727E+00	8.1204E+00
0.1	0.1	0.1	7.5000E−01	7.5024E−01	1.3257E+01	1.3416E+01
0.2	0.2	0.2	1.0610E+00	1.0611E+00	∞	∞
0.3	0.3	0.3	1.3051E+00	1.3051E+00	∞	∞
0	0.1	0.1	5.2619E−01	5.2652E−01	6.7386E+00	6.7605E+00
0.2	0.1	0.1	9.4759E−01	9.4771E−01	6.9151E+01	7.6070E+01
0.4	0.1	0.1	1.3056E+00	1.3057E+00	∞	∞
0.6	0.1	0.1	1.6370E+00	1.6370E+00	∞	∞
0.8	0.1	0.1	1.9531E+00	1.9531E+00	∞	∞
0.1	0	0.1	6.3981E−01	6.3992E−01	1.1139E+01	1.1254E+01
0.1	0.2	0.1	8.1879E−01	8.1895E−01	1.6204E+01	1.6435E+01
0.1	0.4	0.1	9.1120E−01	9.1127E−01	2.7939E+01	2.8589E+01
0.1	0.6	0.1	9.7466E−01	9.7469E−01	8.6663E+01	9.2848E+01
0.1	0.8	0.1	1.0226E+00	1.0226E+00	∞	∞
0.1	0.1	0	6.3981E−01	6.3992E−01	1.0899E+01	1.1016E+01
0.1	0.1	0.2	8.1879E−01	8.1895E−01	1.6502E+01	1.6730E+01
0.1	0.1	0.4	9.1120E−01	9.1127E−01	2.9340E+01	2.9990E+01
0.1	0.1	0.6	9.7466E−01	9.7469E−01	9.3510E+01	1.0002E+02
0.1	0.1	0.8	1.0226E+00	1.0226E+00	∞	∞

radius $\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right)$ of the closed loop system, achievable with a full and reduced-order compensator, increases (De Koning, 1993). This is confirmed by Table 1. The same applies to the minimum value σ^*_∞ of the cost function (2.14). If $\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right) \geq 1$ the system is not n_c-ms-compensatable and $\sigma^*_\infty = \infty$. The values of $\tilde{\rho}\left(\overline{\Phi' \otimes \Phi'}\right)$ and σ^*_∞ were computed with both the conventional and U-D factored algorithm. Both gave the same results within the specified convergence tolerance of 10^{-6}, indicating algebraic equivalence. The required number of algorithm iterations to obtain convergence was almost identical as well. Reducing the compensator state dimension from full-order $n_c = n = 2$ to reduced-order $n_c = 1$ increases the minimal costs. This increase is larger when parameter uncertainty is larger.

Like in the full-order case, within the U-D factored algorithms for reduced-order compensation, second moment computations play a dominant role. The efficiency of these second moment computations is discussed in Van Willigenburg and De Koning (2013). The number of floating point operations (multiply accumulate operations) required by them in the reduced-order case remains proportional to $(r_\Phi + 1)\, n^3$.

It is interesting to see how order reduction affects computational efficiency of the algorithms. To that end Table 2 records execution times of the U-D factored and conventional algorithms when the compensator order is both full and reduced. The compensation problems in Table 2 are randomly generated with $20 \leq n \leq 70$, $m = 3$, $l = 4$, $r_\Gamma = \min(nm, r_\Phi)$, $r_C = \min(nl, r_\Phi)$. The generation was such that they are all n_c-ms-compensatable for $n_c = n, 3$. Table 2 in this paper is similar to Table 3 in Van Willigenburg and De Koning (2013). As in that paper, when executed in MATLAB®, the U-D algorithms are computationally more efficient in cases of large n and small r_Φ, r_Γ, r_C. The numbers r_Φ, r_Γ, r_C relate to the minimal representation of $\overline{\Phi \otimes \Phi}$, $\overline{\Gamma \otimes \Gamma}$ and $\overline{C \otimes C}$ as explained in Willigenburg and De Koning (2013).

Table 2. Execution times (sec) of conventional and U-D factored algorithms.

n	r_Φ	Single iteration time $n_c = n$	Total iteration time $n_c = n$	Single iteration time $n_c = 3$	Total iteration time $n_c = 3$	U-D single iteration time $n_c = n$	U-D total iteration time $n_c = n$	U-D single iteration time $n_c = 3$	U-D total iteration time $n_c = 3$
20	1	4.42E−04	1.46E−02	1.02E−03	8.47E−02	1.12E−03	3.58E−02	1.30E−03	8.18E−02
50	1	6.88E−03	3.30E−01	1.12E−02	7.20E−01	4.85E−03	2.28E−01	7.27E−03	3.64E+01
70	1	2.53E−02	3.44E+00	3.19E−02	4.33E+00	1.21E−02	1.62E+00	1.78E−02	2.76E+00
20	5	3.40E−04	1.12E−02	1.01E−03	5.68E−02	1.25E−03	4.13E−02	1.77E−03	7.43E−02
50	5	6.58E−03	3.09E−01	9.93E−03	4.97E−01	7.71E−03	3.55E−01	1.15E−02	5.73E+01
70	5	2.51E−02	5.54E+00	3.21E−02	7.09E+00	2.19E−02	4.85E+00	2.78E−02	6.65E+00
20	20	5.34E−04	1.76E−02	1.20E−03	5.18E−02	3.93E−03	1.30E−01	3.83E−03	1.26E−01
50	50	7.15E−03	3.29E−01	1.06E−02	4.65E−01	4.91E−02	2.26E+00	5.19E−02	6.02E+00
70	70	2.53E−02	6.87E+00	3.23E−02	8.82E+00	1.67E−01	4.55E+01	1.75E−01	4.79E+01

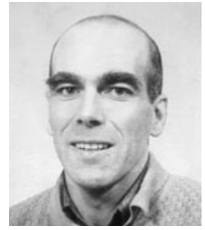
From Table 2 observe that order reduction increases the computational load of each single iteration for both the conventional algorithms (columns 3–6) and the U-D factored algorithms (columns 7–10). Also the number of iterations before convergence generally increases. U-D factorisation, however, has other advantages such as guaranteeing non-negativeness of the matrices during iterations of the algorithm that enhances numerical stability. Another potential advantage is doubling of precision (Bierman, 1977). Doubling of precision is achieved only if recovery of the U-D factored matrices into ordinary representation is avoided during the iterations. This type of recovery occurs in Equations (2.16) and (2.17) in the part that is not inverted. Furthermore, it occurs in Equation (3.3) that is absent in the full-order case. Further investigation into U-D factorisation is needed to see if this recovery can be avoided. As to the conventional algorithms programmed in MATLAB®, we conclude that they appear to be very efficient. Moreover, their efficiency is almost independent of r_Φ, r_Γ, r_C.

6. Conclusions

The U-D factorisation of algorithms solving the SDOPE for reduced-order compensation of systems with white parameters was presented. This extends the U-D factorisation of similar equations for full-order compensation recently presented by us. Since systems with deterministic parameters are a special case of systems with white parameters, reduced-order compensation of these more common type of systems is also realised by the algorithms. This special result is obtained by setting to zero all terms involving a tilde that relate to system matrices Φ_i, Γ_i, C_i. U-D factorisation enhances numerical accuracy and stability and possibly efficiency. As shown in this paper, the latter depends on the minimal representation of the stochastic system matrices.

To benefit from the possible doubling of precision obtained by U-D factorisation, the expressions of the compensator gains need further investigation. Presently their computation is not fully U-D factored. The same applies to the terms involving the optimal oblique projection matrix τ. Another topic for future research concerns the U-D factorisation of two Lyapunov equations that may be used instead of the SDOPE. This would extend the result of Van Willigenburg and De Koning (2004) that applies to systems with deterministic parameters. The two Lyapunov equations seem more suited for U-D factorisation because they do not involve the non-symmetric optimal oblique projection matrix. The modifications presented in this paper ensure non-negativeness and symmetry of matrices during iterations of the SDOPE. They may also be employed in previously published algorithms, where symmetry and non-negativeness were not guaranteed during iterations. This is expected to further improve numerical stability and convergence of these algorithms.

Notes on contributors



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