Proceedings of the Third International Conference on Advances in Control and Optimization of Dynamical Systems, Indian Institute of Technology Kanpur, India, March 13-15, 2014

Theoretical and numerical issues concerning temporal stabilisability and detectability

L. Gerard Van Willigenburg*, Willem L. De Koning**

**Biomass Refinery & Process Dynamics Group, Wageningen University, P.O. Box 17, 6700 AA Wageningen, The Netherlands, (e-mail: gerard.vanwilligenburg@wur.nl)*
***Retired from the Department of Mathematics of Delft University of Technology and lives at Kroeskarper 6, Leiden, The Netherlands (e-mail: wilros@planet.nl)*

Abstract. Motivated by the design of perturbation (output) feedback controllers for non-linear systems, *temporal* system properties have been developed by the authors, for time-varying linear continuous-time systems. In particular temporal controllability/reachability, temporal stabilisability, temporal reconstructability/observability and temporal detectability. As opposed to their ordinary counterparts, these temporal properties identify the *temporal loss* of stabilisability and detectability that may occur for time-varying linear continuous-time systems obtained e.g. by linearising around state and control trajectories of a non-linear system. One contribution of this paper is to show that *temporal* stabilisability and *temporal* detectability require *a measure of state decay* that *cannot* be independent of the state representation and state vector norm. This raises the issue of how to select the state representation or norm. The second contribution of this paper is to deal with this selection. Thirdly this paper presents two new, alternative ways to compute temporal stabilisability and detectability and their associated *measures.* They are compared with computations proposed earlier that rely on LQ control. Finally, through simple illustrative examples, numerical aspects concerning computation of temporal stabilisability and detectability and their associated measures are investigated.

Keywords: Linear Systems, Time-varying Systems, Optimal Control, Temporal Stability, Temporal Stabilisability, Temporal Detectability.

1. INTRODUCTION

Perturbation feedback design and stability analysis of non-linear systems along trajectories is often performed using the linearised dynamics about the trajectory [1]-[4]. If the trajectory is time-varying the linearised model is *time-varying*. If in addition the non-linear dynamics or the controls are non-smooth, i.e. in the case of bang-bang or digital control, the *structure* of the time-varying linearised system may no longer be constant. In other cases the structure may *almost* change. For control system design this is vital information since this structure reveals the *temporal loss* of controllability/reachability, stabilisability, reconstructability/ observability and detectability of the linearised system. They in turn may lead to *temporal instability* of a closed-loop control system [1], [4].

In [1] *measures* of *temporal stability, temporal stabilisability and temporal detectability* of time-varying linear continuous-time systems over arbitrary finite time intervals were developed and calculated by solving standard LQ problems. Temporal properties apply to time intervals that are generally finite. To define stability and stabilisability over a finite interval requires *a measure of state decay*. As demonstrated and argued in section 3 such a measure *cannot* be invariant under changes of the state representation or state vector norm. Therefore *temporal versions* of stability, stabilisability and detectability *cannot* be invariant under state transformations and changes of the state vector norm, as opposed to the conventional properties. This raises the issue of how to select the state representation or norm. In section 4 this issue is considered and discussed.

If the linear system is time-invariant the conventional stabilisability property is generally calculated by extracting the uncontrollable sub-system, that is autonomous, by means of a Kalman decomposition. Next this autonomous uncontrollable sub-system is checked for its stability. A similar procedure may be applied if the linear system is time-varying to determine temporal stabilisability. In section 5 we present two numerical procedures that rely on the latter idea. We show that these procedures are not generally applicable however, as opposed to the LQ approach presented in [1]. Next in section 6, through simple illustrative examples, numerical aspects concerning computation of temporal stabilisability and detectability and the associated measures are investigated. Finally section 7 presents conclusions some of which state the importance and fundamental difference of temporal stabilisability and detectability versus ordinary stabilisability and detectability.

2. TEMPORAL STABILISABILITY AND DETECTABILITY

As introduced and explained in [1] temporal stability and stabilisability properties are defined over *open* time-intervals,

$$\left(t_i, t_{i+1}\right),\ i = 0,1,..,N-1,\ t_{i+1} > t_i \tag{1}$$

that belong to the time domain of a time-varying linear system described by,

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t),\\ y(t) &= C(t)x(t),\ t \in \left(t_0, t_N\right),\ -\infty \le t_0 < t_N \le +\infty\end{aligned} \tag{2}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the bounded input and $y(t) \in R^l$ is the output of the system. $A(t)$, $B(t)$ and $C(t)$ are real bounded matrices of appropriate dimension. Denote the system (2) by $\left(A(t), B(t), C(t)\right)$. The time intervals (1) are obtained after application of the *differential Kalman decomposition* (d-Kalman decomposition) to the system $\left(A(t), B(t), C(t)\right)$. This Kalman decomposition is based on the differential controllability (d-controllability) gramian $W_t \in R^{n\times n}$ and the differential reconstructability (d-reconstructability) gramian $M_t \in R^{n\times n}$ in the same manner as the conventional Kalman decomposition is based on the conventional controllability and reconstructability gramian [5], [6]. Over each separate open time interval $\left(t_i, t_{i+1}\right)$, the d-Kalman decomposition provides a canonical decomposition of the system $\left(A(t), B(t), C(t)\right)$ into four parts denoted by a,b,c,d having constant dimensions denoted by n_a, n_b, n_c, n_d given by,

$$\begin{aligned}&n_a + n_b = rank\left(W_t\right),\ n_b + n_d = rank\left(M_t\right),\\ &n_b = rank\left(W_t M_t\right),\ n_a + n_b + n_c + n_d = n\end{aligned}. \tag{3}$$

The dimensions n_a, n_b, n_c, n_d define the *temporal linear system structure* that is *constant* over each open time interval $\left(t_i, t_{i+1}\right)$. So the time-varying system $\left(A(t), B(t), C(t)\right)$ is called a *piecewise constant rank system* (PCR system) [1], [5], [6]. Among others this implies that temporal controllability/reachability, differential controllability/ reachability and ordinary complete controllability/ reachability are all equivalent over each separate open time-interval $\left(t_i, t_{i+1}\right)$. Dually temporal reconstructability/ observability, differential reconstructability/observability and complete reconstructability/observability are also all equivalent [1]-[7]. This makes the temporal stability and stabilisability analysis over each separate open time-interval $\left(t_i, t_{i+1}\right)$ particularly easy and conventional. The analysis is only unconventional due to the fact that each separate open time interval $\left(t_i, t_{i+1}\right)$ has *finite length* in general. This requires a stability and stabilisability concept that applies over *finite time*. Detectability requires a concept dual to the one required for stabilisability. In [1] both were introduced.

Definition 1 (see Definitions 6,7 and Theorem 3 in [1])

An autonomous system $\left(A(t), 0, 0\right)$ is called *temporal stable over* $\left(t_i, t_{i+1}\right)$ if $\rho\left(t_i, t_{i+1}\right) = \max\limits_{x\left(t_i^+\right)\neq 0}\left(\left\|x\left(t_{i+1}^-\right)\right\|^2 \Big/ \left\|x\left(t_i^+\right)\right\|^2\right) < 1$ □

Definition 2 (see Definitions 8,9 in [1])

$\left(A(t), B(t), C(t)\right)$ is called *temporal stabilisable over* $\left(t_i, t_{i+1}\right)$ if $\rho_{\min}\left(t_i, t_{i+1}\right) = \max\limits_{x\left(t_i^+\right)\neq 0}\left(\dfrac{\min\limits_{u(t)|x\left(t_i^+\right)}\left\|x\left(t_{i+1}^-\right)\right\|^2}{\left\|x\left(t_i^+\right)\right\|^2}\right) < 1$ □

3. INFLUENCE OF SIMILARITY TRANSFORMATIONS AND STATE VECTOR NORMS

Conventional stabilisability considers decay *to zero* of *all states*, when time tends to *infinity*. When all states tend to zero so does any state vector norm and any state obtained after a similarity transformation. Therefore conventional stabilisability is invariant under associated changes of state.

When time is finite, decay of all states to exactly zero is not generally achieved and we must find a *measure* for *state decay* as used in Definition 1 and Definition 2. The next lemma and its proof might be considered a standard result. But they are fundamental to this paper and stated for clarity.

Lemma 1

Let $T \in R^{n\times n}$ be a non-singular real square matrix and $x \in R^n$ a real vector. Then $\max\limits_{x\neq 0}\left(\|Tx\|^2 / \|x\|^2\right) = \|T\|^2$. The maximum $\|T\|^2$ equals the largest eigenvalue of the nonnegative symmetric matrix $T^T T$ and is achieved for the associated eigenvector □

Proof

The 2-norm $\|T\|$ of matrix $T \in R^{n\times n}$ may be defined as the operator norm induced by vectors $x \in R^n$: $\|T\| \overset{def}{=} \max\limits_{x\neq 0}\left(\|Tx\| / \|x\|\right)$. This norm is also known as the spectral norm. The maximum $\|T\|$ is the square root of the largest eigenvalue of the nonnegative symmetric matrix $T^T T \in R^{n\times n}$ and is achieved by the associated eigenvector [9]. From this definition the first part of Lemma 1 is obtained by taking the square. Doing so the maximum value is squared while the value of $x \in R^n$ that achieves the maximum is unchanged. □

Definition 3

Let $T(t) \in R^{n\times n}$ represent a time-varying non-singular matrix that is defined and continuously differentiable over the open time intervals (1) associated with system $\left(A(t),B(t),C(t)\right)$. Then $T(t)$ defines a *similarity transformation*,

$$\begin{aligned}&x'(t)=T(t)x(t),\\&A'(t)=T(t)A(t)T^{-1}(t)+\dot{T}(t)T^{-1}(t),\\&B'(t)=T(t)B(t),\ C'(t)=C(t)T^{-1}(t),\\&t\in(t_i,t_{i+1}),\ i=0,1,..,N-1\end{aligned}\tag{4}$$

where the prime denotes system quantities obtained after the transformation □

Theorem 1

1. The temporal stability measure $\rho(t_i,t_{i+1})$ associated with the autonomous system $\left(A(t),0,0\right)$ in Definition 1 is invariant under similarity transformations of the state *if and only if* $\left\|T\left(t_{i+1}^-\right)\Phi\left(t_i^+,t_{i+1}^-\right)T^{-1}\left(t_i^+\right)\right\|=\left\|\Phi\left(t_i^+,t_{i+1}^-\right)\right\|$ where $\Phi\left(t_i^+,t_{i+1}^-\right)$ represents the state transition matrix from time t_i^+ to t_{i+1}^- of system $\left(A(t),B(t),C(t)\right)$.
2. The temporal stabilisability measure $\rho_{\min}(t_i,t_{i+1})$ associated with system $\left(A(t),B(t),C(t)\right)$ in Definition 2 is invariant under similarity transformations of the state *if and only if* $\left\|T\left(t_{i+1}^-\right)S^*\left(t_i^+\right)T^{-1}\left(t_i^+\right)\right\|=\left\|S^*\left(t_i^+\right)\right\|$ where $S^*\left(t_i^+\right)$ is the initial value of the solution $S(t),\ t\in(t_i,t_{i+1})$ to the LQ problem associated with Theorem 5 in [1] □

Proof

1. Since $\Phi\left(t_i^+,t_{i+1}^-\right)$ is non-singular from Lemma **1** we have,

$$\rho(t_i,t_{i+1})=\max_{x(t_i^+)\neq 0}\left(\left\|x\left(t_{i+1}^-\right)\right\|^2/\left\|x\left(t_i^+\right)\right\|^2\right)=\left\|\Phi\left(t_i^+,t_{i+1}^-\right)\right\|^2.$$

Applying the similarity transformation (4) gives $x'\left(t_{i+1}^-\right)=T\left(t_{i+1}^-\right)\Phi\left(t_i^+,t_{i+1}^-\right)T^{-1}\left(t_i^+\right)x'\left(t_i^+\right)$. Then from Lemma **1**: $\rho'(t_i,t_{i+1})=\max_{x(t_i^+)\neq 0}\left(\left\|x'\left(t_{i+1}^-\right)\right\|^2/\left\|x'\left(t_i^+\right)\right\|^2\right)=$ $\left\|T\left(t_{i+1}^-\right)\Phi\left(t_i^+,t_{i+1}^-\right)T^{-1}\left(t_i^+\right)\right\|^2$.

2. The proof is identical to the one above with $\Phi\left(t_i^+,t_{i+1}^-\right)\in R^{n\times n}$ replaced by $S^*\left(t_i^+\right)\in R^{n\times n}$ □

Example 1

Suppose $T(t)=T$ and randomly generate $\Phi\left(t_i^+,t_{i+1}^-\right)=\begin{bmatrix}-1.3830 & -4.4270\\ -2.0396 & 1.7494\end{bmatrix}$. Then by *minimizing* $\left(\left\|T\Phi\left(t_i^+,t_{i+1}^-\right)T^{-1}\right\|-\left\|\Phi\left(t_i^+,t_{i+1}^-\right)\right\|\right)^2$ as a function of the 4 elements of matrix $T\in R^{2\times 2}$ the matrix $T=\begin{bmatrix}1.0274 & 0.1740\\ -0.9086 & 0.8735\end{bmatrix}$ is found to satisfy the invariance condition $\left\|T\Phi\left(t_i^+,t_{i+1}^-\right)T^{-1}\right\|=\left\|\Phi\left(t_i^+,t_{i+1}^-\right)\right\|$ in Theorem 1. □

Remark 1

The minimization in Example 1 was performed using the Matlab optimization toolbox function fminsearch. Repeating the minimization with different initial guesses for T reveals many different solutions for T. Among these solutions are all similarity transformations represented by *orthogonal matrices* T since these are precisely the ones that preserve the 2-norm of vectors [9] and the stabilisability measure $\rho_{\min}(t_i,t_{i+1})$ in Definition 1. Still *generically* the conditions in Theorem 1 are not satisfied □

Well-known *alternative* vector norms are those induced by positive square matrices. Denote such a matrix by $P\in R^{n\times n}$. Then the associated vector norm $\|x\|_P$ of $x\in R^n$ equals,

$$\|x\|_P=\sqrt{x^TPx}\tag{5}$$

Although not necessarily, P is generally taken to be a symmetric matrix because,

$$\|x\|_P=\sqrt{x^TPx}=\sqrt{x^TP_sx},\ P_s=\frac{P+P^T}{2}\tag{6}$$

where P_s denotes the symmetric part of P that fully determines the norm. Now any positive matrix $P\in R^{n\times n}$ has positive square roots $P_r\in R^{n\times n}$,

$$P=P_r^TP_r\tag{7}$$

From equations (5), (7), observe that changing the vector norm from $\|x\|=\|x\|_I$ to $\|x\|_P$ is equivalent to performing a similarity transformation,

$$x'=P_rx,\tag{8}$$

because $\|x'\|=\sqrt{x^TP_r^TP_rx}=\sqrt{x^TPx}=\|x\|_P$. This indicates that changing the vector norm to one of type (5) will not fundamentally alter the invariance conditions. They will generically not be satisfied. Similar arguments apply to norms induced by time-varying positive square matrices $P(t)$ because $\|x'(t)\|=\sqrt{x^T(t)P_r^T(t)P_r(t)x(t)}=$ $\sqrt{x^T(t)P(t)x(t)}=\|x(t)\|_{P(t)}$.

Corollary 1

It appears that arbitrary similarity transformations or changes of the state vector norm offer too much freedom in manipulating measures of state decay for invariance conditions of temporal stability and stabilisability over finite time-intervals (t_i, t_{i+1}) to hold generically. Another indication is that finite-time stability [10], which is another type of stability defined for finite time-intervals, is also not invariant under arbitrary similarity transformations of the state. This can be seen from Definitions 1 and 2 in [10] because these use another *specified* (time-varying) squared matrix 2-norm $x^T(t)\Gamma(t)x(t),\ \Gamma(t)>0$ to define finite-time stability □

Corollary 2

Temporal as well as finite-time stability are *fundamentally different* stability concepts as compared to the conventional ones. The first include *transient state behaviour* and exclude behaviour when time tends to infinity whereas for conventional stability concepts it is precisely the other way around. Only if the length $t_{i+1}-t_i$ of the finite time interval tends to infinity results obtained from both stability concepts become comparable. Because temporal detectability is dual to temporal stabilisability the influence of similarity transformations and state vector norm selection on temporal detectability is similar □

4. SELECTION OF APPROPRIATE VECTOR NORMS TO MEASURE TEMPORAL STABILITY, STABILISABILITY AND DETECTABILITY

Roughly speaking, from an engineering perspective, stability and stabilisability are important properties guaranteeing the system state vector norm to remain within limits or to decay sufficiently or ultimately. Selecting a time-varying state vector norm of type (5) comes down to selecting the positive square matrices $P(t)>0$. This is comparable to selecting positive square state weighing matrices $Q(t)>0$ for an LQ control problem because,

$$x^T(t)Q(t)x(t)=\|x(t)\|^2_{Q(t)},\ Q(t)>0. \tag{9}$$

Since LQ control problems are very well-known, so are arguments for selecting $Q(t)$. Very often $Q(t)$ is taken to be diagonal because this results in a *straightforward weighing* of the magnitude of the individual states,

$$x^T(t)Q(t)x(t)=\sum_{j=1}^{n}Q_{j,j}(t)x_j^2(t),\ Q_{j,j}(t)>0. \tag{10}$$

where $Q_{j,j}(t)$ denotes the j^{th} diagonal element of $Q(t)$ and $x_j(t)$ the j^{th} element of state vector $x(t)$. Also very often, $Q(t)$ is taken to be time-invariant. Obviously in selecting $Q(t)>0$ it is vital for the individual states $x_j(t),\ j=1,2,..,n$ to have a clear meaning and interpretation. This provides another argument in favour of *first principles modelling*. Due to duality between temporal stabilisability and detectability similar arguments apply to temporal detectability.

5. COMPUTATION OF TEMPORAL STABILISABILITY AND DETECTABILITY

Although numerical methods to compute and quantify temporal stabilisability and detectability have been provided, alternatives will be presented and discussed in this section. The methods provided in [1], [4] use solutions of standard finite horizon time-varying LQ problems. Theorem 5 and Remark 3 in [1] indicate a technical difficulty because only an approximate version $S^{\varepsilon}(t),\ 0<\varepsilon\ll 1$ of the desired solution $S^*(t)$, $t\in(t_i,t_{i+1})$ of the LQ problem is obtained. Remark 3 in [1] also indicates that this technical difficulty may be considered a practical advantage.

In this section alternative computations are presented that can be made *after* application of the d-Kalman decomposition to the system $(A(t),B(t),C(t))$. These computations are similar to computations of conventional stabilisability. The d-Kalman decomposition not only detects the time instants t_i, $i=1,2,..,N-1$, that determine the open time-intervals (1), but also extracts the temporal uncontrollable, autonomous system part, consisting of parts c,d in equation (3), over each open time-interval (t_i,t_{i+1}), $i=0,1,..,N-1$. The alternative computations check the temporal stability of this temporal uncontrollable, autonomous system part that fully determines temporal stabilisability (see [1], proof of Theorem 5). However, to extract the temporal uncontrollable autonomous system part, the d-Kalman decomposition applies a *similarity transformation*. As demonstrated in the previous section, this similarity transformation generally *changes* temporal stability and stabilisability properties. By applying the inverse transformation however, this change can be undone. Dual arguments apply to the temporal unreconstructable system part and temporal detectability.

Consider the system $(A(t),B(t),C(t))$. Over each open time-interval (t_i,t_{i+1}), $i=0,1,..,N-1$, the d-Kalman provides a similarity transformation (4) specified by $T(t)$, $t\in(t_i,t_{i+1})$ that transforms $(A(t),B(t),C(t))$ into $(A'(t),B'(t),C'(t))$ where $A'(t),B'(t)$ decompose as follows [5], [6],

$$\begin{aligned}
&A'(t)=\begin{bmatrix}A'_{11}(t) & A'_{12}(t)\\ 0 & A'_{22}(t)\end{bmatrix},\ B'(t)=\begin{bmatrix}B'_1(t)\\ 0\end{bmatrix},\\
&A'_{11}(t)\in R^{(n-n_u)\times(n-n_u)},\ A'_{12}(t)\in R^{(n-n_u)\times n_u},\\
&A'_{22}(t)\in R^{n_u\times n_u},\ B'_1(t)\in R^{(n-n_u)\times m},\\
&n_u=n_c+n_d=rank(W_t)\le n.
\end{aligned} \tag{11}$$

In equation (11) $A'_{22}(t)$ represents the temporal uncontrollable, autonomous system part consisting of c,d in equation (3). Associated to the decomposition (11) consider the decomposition of the state,

$$x'(t)=\begin{bmatrix} x'_c(t) \\ x'_u(t) \end{bmatrix},\ x'_c \in R^{(n-n_u)},\ x'_u \in R^{n_u} \tag{12}$$

where $x'_u(t)$ represents temporal uncontrollable states and $x'_c(t)$ temporal controllable states. Let $T'(t)=T^{-1}(t)$. Associated with decomposition (11), (12) consider decompositions of $T(t)$, $T'(t)$,

$$\begin{aligned} &T(t)=\begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix},\ T_1(t)\in R^{(n-n_u)\times n},\ T_2(t)\in R^{n_u\times n} \\ &T'(t)=\begin{bmatrix} T'_1(t) & T'_2(t) \end{bmatrix},\ T'_1(t)\in R^{n\times(n-n_u)},\ T'_2(t)\in R^{n\times n_u} \end{aligned} \tag{13}$$

Finally consider the following relation,

$$x(t)=T'(t)x'(t)=\begin{bmatrix} T'_1(t) & T'_2(t) \end{bmatrix}\begin{bmatrix} x'_c(t) \\ x'_u(t) \end{bmatrix} \tag{14}$$

As to temporal stabilisability, of the original system state $x(t)$ in equation (14), only the states $x'_u(t)$ are of interest since the states $x'_c(t)$ are temporal controllable and therefore can be controlled to zero arbitrarily fast within (t_i,t_{i+1}). Therefore only $x'_u(t)$ in equation (14) contributes to the stabilisability measure of $x(t)$ [5], [6]. Discarding the contribution of $x'_c(t)$, from equation (14) we obtain,

$$x(t)=T'_2(t)x'_u(t),\ x(t)\in R^n,\ T'_2(t)\in R^{n\times n_u}. \tag{15}$$

Furthermore from (13),

$$x'_u(t)=T_2(t)x(t),\ x'_u(t)\in R^{n_u} \tag{16}$$

Using (15), (16),

$$x(t)=T'_2(t)T_2(t)\Phi(t_i^+,t)T'_2(t_i^+)T_2(t_i^+)x(t_i^+) \tag{17}$$

where $\Phi(t_i^+,t^-)$ represents the state transition matrix of the original system $(A(t),B(t))$ over (t_i,t), $t_i<t\le t_{i+1}$. In equation (17) note that in general $T_2(t)\in R^{n_u\times n}$ and $T'_2(t)\in R^{n\times n_u}$ are non-square matrices that are not each other's inverse. From equations (17) and Definition 1 we obtain,

$$\rho_{\min}(t_i,t_{i+1})=\left\|\Phi_r^T(t_i^+,t_{i+1}^-)\Phi_r(t_i^+,t_{i+1}^-)\right\|, \tag{18}$$

where,

$$\begin{aligned} &\Phi_r(t_i^+,t_{i+1}^-)= \\ &T'_2(t_{i+1}^-)T_2(t_{i+1}^-)\Phi(t_i^+,t_{i+1}^-)T'_2(t_i^+)T_2(t_i^+)\in R^{n_u\times n}. \end{aligned} \tag{19}$$

Corollary 3

Temporal stabilisability of system $(A(t),B(t))$ over (t_i,t_{i+1}) may be determined from $\Phi(t_i^+,t_{i+1}^-)$, $T_2(t_i^+)$ and $T'_2(t_{i+1}^-)$. Through equation (13) they determine (18), (19) i.e. the temporal stabilisability measure $\rho_{\min}(t_i,t_{i+1})$ □

Let $\Phi_{22}(t_i^+,t^-)$ denote the state transition matrix over (t_i,t), $t_i<t\le t_{i+1}$ associated with the autonomous system part $A'_{22}(t)$ in equation (11) related to temporal uncontrollable part $x'_u(t)$ of the state. Discarding again the contribution of the controllable part $x'_c(t)$ of the state, from (13)-(16) observe that $\Phi_r(t_i^+,t_{i+1}^-)\in R^{n_u\times n}$ in equation (18) is also given by,

$$\Phi_r(t_i^+,t_{i+1}^-)=T'_2(t_{i+1}^-)\Phi_{22}(t_i^+,t_{i+1}^-)T_2(t_i^+). \tag{20}$$

Corollary 4

Temporal stabilisability of system $(A(t),B(t))$ over (t_i,t_{i+1}) may be determined from $\Phi_{22}(t_i^+,t_{i+1}^-)$, $T_2(t_i^+)$ and $T'_2(t_{i+1}^-)$. Through equation (13) they determine (18), (20) i.e. the temporal stabilisability measure $\rho_{\min}(t_i,t_{i+1})$ □

Remark 2

Corollary 3 uses the state transition matrix $\Phi(t_i^+,t_{i+1}^-)$ associated with the original time-varying linear system (2) while Corollary 4 uses $\Phi_{22}(t_i^+,t_{i+1}^-)$ associated with $A'_{22}(t)$ obtained after the similarity transformation (11). Although theoretically they produce the same result, computation according to Corollary 4 uses only the lower right block $A'_{22}(t)$ of $A'(t)$. Corollary 3 on the other hand uses all parts of $A'(t)$ in equation (11). When small numerical errors occur in $T_2(t_i^+)$, $T'_2(t_i^+)$, $T_2(t_{i+1}^-)$, $T'_2(t_{i+1}^-)$, $\Phi(t_i^+,t_{i+1}^-)$ this causes the lower left part of $A'(t)$ to differ slightly from zero. Then the temporal controllable part slightly influences the temporal uncontrollable part in equation (11) and thereby the computation of temporal stabilisability measure $\rho_{\min}(t_i,t_{i+1})$ according to (19) □

Remark 3

Starting from appropriate gramians W_t, M_t of the original untransformed system, the transformation $T(t)$, $t\in(t_i,t_{i+1})$ is computed from the d-Kalman decomposition. The numerical procedure proposed in [6] to compute $T(t)$ from W_t, M_t performs decompositions at *isolated times* t. These computations therefore do *not* ensure $T(t)$, $t\in(t_i,t_{i+1})$ to be continuously differentiability as required by Definition 3. To establish the system structure (3), $T(t)$, $t\in(t_i,t_{i+1})$ need not be continuous and the numerical procedure proposed in [6] applies. Symbolic computation of the d-Kalman decomposition can fulfil the demand of continuous differentiability of $T(t)$, $t\in(t_i,t_{i+1})$. But it requires analytic expressions for the gramians and system matrices and only works for small enough dimensions. System matrices of time-varying linearised models about trajectories are generally computed numerically, at isolated times, not providing analytic expressions. Then Corollary 3 and Corollary 4 and the numerical procedure proposed in [6] do not provide a numerically feasible procedure to compute the temporal stabilisability measure $\rho_{\min}(t_i,t_{i+1})$ except when $T(t)$, $t\in(t_i,t_{i+1})$ is time-invariant □

Remark 4

The computational procedure proposed in [1] does provide a general numerically feasible procedure to compute temporal stabilisability and detectability measures because it applies standard LQ control algorithms to the untransformed system □

6. NUMERICAL EXAMPLES

In this section numerical examples will be considered for which $T(t)$, $t\in(t_i,t_{i+1})$ is time-invariant. This enables numerical computation of the stabilisability and detectability measures according to Corollary 3 and Corollary 4. The results will be compared with the results obtained by the procedure proposed in [1], that is generally applicable. The examples are also used to illustrate the importance of selecting appropriately the numerical tolerances used to compute the d-Kalman decomposition and the temporal stabilisability and detectability measures.

The numerical examples considered here are examples 1-3 presented in [1] because these provide a constant $T(t)$, $t\in(t_i,t_{i+1})$, are easy to interpret and illustrate clearly the numerical issues. The examples concern the finite horizon optimal control and optimal LQG output feedback of a non-linear system. To check whether optimal LQG output feedback successfully stabilises the closed loop system temporal stabilisability and temporal detectability of the linearised system about the open loop optimal trajectory $u^*(t),\ x^*(t),\ y^*(t)$ are computed, as in [1]. The linearised system is described by,

$$\begin{aligned}\partial\dot{x}(t)&=A(t)\partial x(t)+B(t)\partial u(t)\\ \partial y(t)&=C(t)\partial x(t),\ t\in\left[0,t_f\right]\end{aligned}\tag{21}$$

where $\partial x(t)=x(t)-x^*(t)$ are state deviations, $\partial u(t)=u(t)-u^*(t)$ are control corrections and $\partial y(t)=y(t)-y^*(t)$ are output deviations from the optimal ones. For examples 1-3 in [1] we have,

$$A(t)=\begin{bmatrix}-1+0.05u^*(t) & -1\\ x_2^*(t) & x_1^*(t)\end{bmatrix},B(t)=\begin{bmatrix}1+0.05x_1^*(t)\\ 0\end{bmatrix},$$

$$C(t)=\left[x_2^*(t)\quad x_1^*(t)+12\right],\ t_f=4 \tag{22}$$

Note that $C(t)$ was inconsistently represented by a column vector in [1]. From equation (22) one can see that over intervals where $x_2^*(t)=0$, the linearised system is both temporal uncontrollable and temporal unreconstructable. This can be seen because the linearised system is already represented in the d-Kalman canonical form implying $T(t)=I$, $t\in\left(0,t_f\right)$. The examples in [1] where deliberately constructed this way. So over intervals where $x_2^*(t)=0$ it is necessary to check temporal stabilisability and temporal detectability.

According to equation (3), to detect the intervals (t_i,t_{i+1}) where the system is temporal uncontrollable or temporal unreconstructable, the numerical rank of W_t and M_t has to be determined. This requires selection of a small positive tolerance. For the linearised system (22) this reduces to selecting a small positive tolerance ε_{x_2}. If $\left\|x_2^*(t)\right\|<\varepsilon_{x_2}$, $x_2^*(t)$ is considered to be zero in equation (22) causing temporal uncontrollability as well as temporal unreconstructability. The values of $x_2^*(t)$ computed in [1] are represented by the curve in left Fig**.** 1. The height of each horizontal line in this figure represents a value of ε_{x_2} and the length the associated time-interval (t_i,t_{i+1}) over which $x_2^*(t)$ is considered to be zero. This shows how (t_i,t_{i+1}) depends on the choice of ε_{x_2}. Next for three values of ε_{x_2} in the range of left Fig**.** 1, Fig. 2 plots the temporal stabilisability measure $\rho_{\min}(t,t_{i+1})$, $t\in(t_i,t_{i+1})$ computed in three different ways. One calculates $\left\|S_\varepsilon(t)\right\|,\ \varepsilon=\sqrt{\varepsilon_{x_2}}$ which approximates $\left\|S^*(t)\right\|$ indicated by LQ in Fig. 2. The choice $\varepsilon=\sqrt{\varepsilon_{x_2}}$ is a numerical one that links the control penalty to the "uncontrollability margin" ε_{x_2}. The other two computations use equation (18) with t_i replaced by t. One computes $\Phi_r(t,t_{i+1})$ using equation (19) while the other uses equation (20), both with t_i replaced by t. These two computations comply with Corollary 3 and Corollary 4 respectively. They give almost identical results because $T(t)=T'(t)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $T_2(t)=\begin{bmatrix}0&1\end{bmatrix},T_2'(t)=\begin{bmatrix}0\\1\end{bmatrix},T_2'(t)T_2(t)=\begin{bmatrix}0&0\\0&1\end{bmatrix},t\in\left(0,t_f\right).$ since equation (22) is already represented in the d-Kalman canonical form. Then, according to Remark 2, identical results are expected.

As can be seen from Fig. 2 the temporal stabilisability measure $\rho_{\min}(t,t_{i+1})$ is highly sensitive to ε_{x_2}. This can be explained as follows. According to equation (22) and (11), when $x_2^*(t)=0$, the temporal uncontrollable autonomous system part $A'_{22}(t)$ equals $x_1^*(t)$. Now $x_1^*(t)$ is obtained from [1] and plotted in right Fig**.** 1. Observe that as the interval (t_i,t_{i+1}) gets larger with ε_{x_2} in left Fig**.** 1, ever larger positive values of $x_1^*(t)$ are incorporated at the end of (t_i,t_{i+1}) and ever larger negative values at the start. This explains the large sensitivity. Also note from Fig. 2 that the smaller $\varepsilon=\varepsilon_{x_2}$ the more accurate and equal the three different computations become while the interval (t_i,t_{i+1}) becomes smaller, as expected. The fact that the "LQ measure" comes out slightly smaller is most likely due to the fact that without performing transformations the system is only approximately temporal uncontrollable. Therefore it is slightly stabilised by the temporal controllable system part.

Similar dual results apply to the temporal detectability measure $\sigma_{\min}(t_i,t)$, $t\in(t_i,t_{i+1})$ depicted in Fig. 3. Observe that the linearised system is not temporal stabilisable over all subintervals (t,t_{i+1}) for which $\rho_{\min}(t,t_{i+1})>1$. The linearised system is temporal detectable over all sub-intervals (t_i,t), $t\in(t_i,t_{i+1})$ because $\sigma_{\min}(t_i,t)<1$. As in [1] we conclude that LQG output feedback suffers from temporal unstabilisability that will most likely cause temporal instability of the closed loop system.

7. CONCLUSIONS

Temporal properties of time-varying linear systems relate to *transient* state behaviour. As demonstrated in this paper, this leads to fundamental differences between ordinary and temporal stability and stabilisability. The latter can no longer be independent of the state representation as one might expect they should. The reason is that ordinary stability and stabilisability for time-varying linear systems only consider whether all states go to zero as time tends to infinity. This is obviously unaffected by similarity transformations. Transient behaviour on the other hand needs to be *measured* and the results and definitions will *necessarily depend on the selected measure*. The measure is a state vector norm. Similarity transformations and changes of the state vector norm, which were shown to be equivalent in this paper, will generally affect temporal stability and stabilisability measures and possibly the outcome of temporal stability and stabilisability. Dual results apply to temporal detectability.

As shown in this paper, selecting a state vector norm is equivalent to selecting the quadratic state penalty for LQ control. A simple, sensible choice is to select a constant diagonal matrix. Then each constant diagonal element weighs each corresponding state squared. The choice is greatly facilitated if all states have a clear meaning and interpretation. This provides another argument in favour of *first principles modelling*. Although in this paper we considered continuous time-varying linear systems only, similar arguments apply to time-varying linear discrete-time systems for which we developed temporal properties as well [4], [12].

Despite major differences between ordinary and temporal stabilisability, this paper showed how a well-known approach to determine ordinary stabilisability of time-invariant linear systems, may be used to determine temporal stabilisability of time-varying linear systems. However, the methodology relies on a similarity transformation that for time-varying linear continuous-time systems is difficult to compute numerically, because of continuity requirements that are not easily met in practice. Therefore we conclude that the numerical methods proposed in [1], [4] to compute measures of temporal stabilisability and detectability are most favourable for continuous-time systems, since they apply standard LQ control algorithms to the *untransformed* time-varying linear system. Since discrete-time is not dense, problems relating to continuity of similarity transformations do not occur for linear time-varying discrete-time systems.

Through simple, illustrative examples we showed that numerical tolerances to compute temporal uncontrollable and temporal unreconstructable time intervals as well as associated temporal stabilisability and detectability measures should be selected carefully because they can have a large influence on the outcome. As with ordinary numerical determination of controllability and reconstructability, the selection of tolerances should made carefully based on computer rounding errors and the ranges of state and control variables. Also temporal stabilisability and detectability measures may be sensitive to changes of the interval over which the (linearised) system is considered temporal uncontrollable or unreconstructable. Despite their possible sensitivity the measures give highly valuable information concerning the practical, temporal stabilisability and detectability of the linear or linearised time-varying continuous-time system.

From a mathematical perspective the dependence of temporal stabilisability and detectability on their associated measures may appear unsatisfactory. From an engineering and control system design point of view this dependence is natural and the properties are very practical and meaningful. The same applies to the possible sensitivity of temporal stabilisability and detectability measures to numerical tolerances.

REFERENCES

[1] L.G. van Willigenburg, W.L. De Koning, 2012, "Temporal and differential stabilisability and detectability of piecewise constant rank systems", *Optimal Control Applications & Methods*, 33, 302-317.

[2] M. Athans, 1971, "The role and use of the Linear-Quadratic-Gaussian problem in control system design", *IEEE Trans. Aut. Contr.*, 16, 6, 529-552.

[3] W.S. Oates, R.C. Smith, 2008, "Nonlinear Optimal Control Techniques for Vibration Attenuation Using Magnetostrictive Actuators", *Journal of Intelligent Material Systems and Structures*, 19, 193-209.

[4] L.G. van Willigenburg, W.L. de Koning, 2012, "Temporal and one-step stabilisability and detectability

of discrete-time linear systems", *IET Control Theory Appl.*, 2013, 7, 1, 151–159.
[5] L.G. Van Willigenburg, W.L. De Koning, 2007, “A Kalman decomposition to detect temporal linear system structure”, *Proceedings European Control Conference*, Kos, Greece, July 2-7, 1721-1726.
[6] L.G. van Willigenburg, W.L. De Koning, 2008, "Temporal linear system structure", *IEEE Transactions on Automatic Control*, 53, 5, 1318-1323.
[7] L.M. Silverman, H.E. Meadows, 1969, “Equivalent realizations of linear systems”, *SIAM J. Appl. Math*., 17, 2, 393-408.
[8] R.E. Kalman, 1960, “Contributions to the theory of optimal control”, *Boletin de la Sociedad Matematica Mexicana* 5, 102-119.
[9] G.H. Golub, C.F. Van Loan, 1996, Matrix computations, John Hopkins University Press.
[10] F. Amato, M. Ariola, M. Carbone, C Cosentiono, 2006, "Finite-Time Control of Linear Systems: A Survey", Current trends in nonlinear systems and control, Systems and Control: Foundations & Applications, 2006, Part II, 195-213.
[11] L.G. Van Willigenburg, 1992, "Computation of the digital LQG regulator and tracker for time-varying systems", *Opt. Contr. Appl. Meth.*, 13, 289-299.
[12] L.G. van Willigenburg, W.L. De Koning, "Temporal linear system structure: The discrete-time case", Proceedings of the ECC 2009, 23-26 August, 2009, Budapest, pp. 225-230.

Fig. 1: Temporal uncontrollability & unreconstructability interval

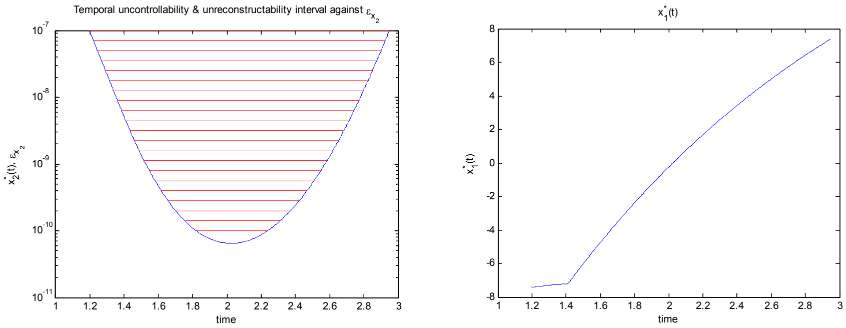

Fig. 2: Temporal stabilisability measures for different ε_{x_2}

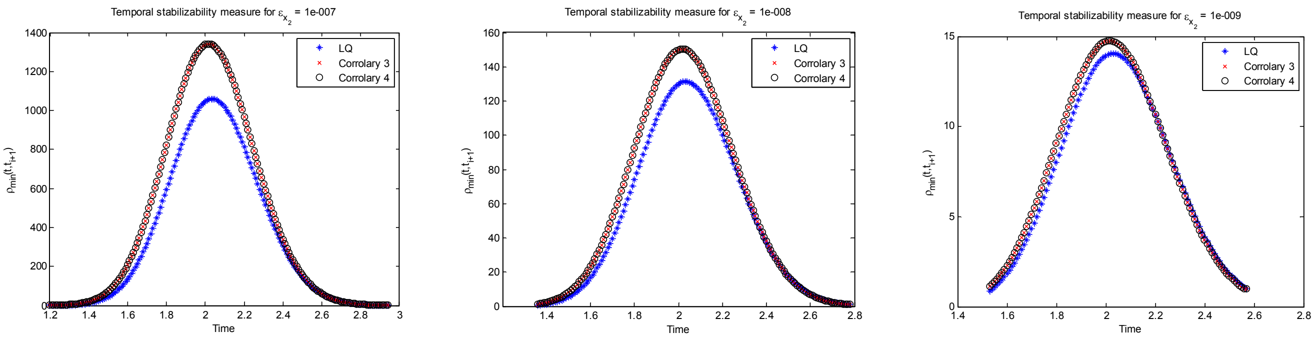

Fig. 3 : Temporal detectability measures for different ε_{x_2}