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Sensitivity matrices as keys to local structural system properties of large-scale nonlinear systems

L. G. Van Willigenburg · J. D. Stigter · J. Molenaar

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Abstract Sensitivities are shown to play a key role in a highly efficient algorithm, presented in this paper, to establish three fundamental structural system properties: local structural identifiability, local observability, and local strong accessibility. Sensitivities have the advantageous property to be governed by linear dynamics, also if the system itself is nonlinear. By integrating their linear dynamics over a short time period, and by sampling the result, a sensitivity matrix is obtained. If this sensitivity matrix satisfies a rank condition, then the local structural system property under investigation holds. This rank condition will be referred to in this paper as the sensitivity rank condition (SERC). Applying a singular value decomposition (SVD) to the sensitivity matrix not only determines its rank but also pinpoints exactly the system components causing a possible failure to satisfy the local structural system property. The algorithm is highly efficient because integration of linear systems over short time-periods and computation of an SVD are computationally cheap. Therefore, it allows for the handling of large-scale systems in the order of seconds, as opposed to conventional algorithms that mostly rely on Lie series expansions and a corresponding Lie algebraic rank condition (LARC). The SERC and LARC algorithms are mathematically and computationally compared through a series of examples.

Keywords Local controllability · Local strong accessibility · Local observability · Local structural identifiability · Large-scale nonlinear systems · Sensitivity matrix · Singular value decomposition · Lie algebraic rank condition (LARC) · Sensitivity rank condition (SERC)

1 Introduction

In establishing local structural identifiability of nonlinear dynamical systems [1–4], we found that parametric output sensitivity matrices turned out to be the key to a simple and efficient algorithm that determines local structural identifiability, even for large-scale nonlinear systems. Moreover, the signatures obtained from this algorithm pinpoint exactly to the parameters involved in total correlations that, in fact, cause the lack of identifiability. As compared to alternative algorithms, based on, e.g., Lie brackets [5-10] or directed graphs [11–13], this algorithm puts hardly any restrictions on system size or system structure. Moreover, in terms of efficiency, the algorithm is highly competitive. Together, this opens up a large field of applications, e.g., in mechanical engineering, systems biology, and chemical engineering where use

L. G. Van Willigenburg (⊠) · J. D. Stigter · J. Molenaar Mathematical and Statistical Methods Group, Wageningen University, P.O. Box 17, 6700 AA Wageningen, The Netherlands e-mail: gerard.vanwilligenburg@wur.nl

is made of nonlinear, possibly large-scale dynamic systems described by sets of ordinary differential equations.

The beneficial properties of the algorithm are largely due to the fact that parametric output sensitivities are governed by linear dynamics, irrespective of the possibly nonlinear dynamics of the underlying system. They also relate to the fact that the linear dynamics need only be integrated over a small timeinterval and the fact that, after sampling of the result, an SVD is the only additional computation required. Very recently, in an extensive overview and comparison of computational methods for identifiability [14], this algorithm was mentioned to be incredibly fast. In the earlier comparisons [15, 16], the computational complexity of establishing identifiability had already been noted. This motivates us to extend this algorithm, developed in [1-3] for local structural identifiability, to local observability and local strong accessibility, as presented in this paper. All three properties are highly fundamental structural properties of nonlinear systems.

Computational methods for local identifiability of parameters, often also including initial conditions, have received significant attention in the last decade, as can be seen from the comparison and large number of references in [14]. For nonlinear systems, a Lie algebraic rank condition (LARC) still plays a central role in almost all of them and is also used to check local observability [9]. LARC is also central to algorithms detecting local strong accessibility, which is closely related to local controllability [7, 17]. The computation of LARC requires repeated symbolic Lie differentiation or Lie Bracketing which are computationally expensive, especially with growing system dimensions. But even for small-scale systems, the computation may be very expensive [18]. In addition, no known stopping condition may exist as long as LARC is not satisfied [19]. Because local observability is almost dual to local strong accessibility [5, 7, 20], it is remarkable that a similar development of algorithms detecting local strong accessibility seems not to have taken place [17, 21, 22]. A possible reason for this has recently been discussed by us [18]. To extend the algorithm, we will further investigate and exploit duality in this paper.

The algorithm based on sensitivities culminates in computing what we refer to as the *sensitivity rank*

condition (SERC). The shorthand SERC will also be used to refer to the algorithm itself. We also compare SERC with a conventional algorithm establishing the same three properties by means of a Lie algebraic rank condition (LARC) [5–10]. This conventional algorithm, also denoted by the shorthand LARC, relies on Lie series expansion, differential geometry and symbolic computation. From a practical perspective, SERC is shown to heavily outperform LARC in terms of computational efficiency and feasibility for largescale systems. These results are illustrated with a series of examples, including both small- and largescale systems.

The outline of this paper is as follows. Starting from the local structural identifiability algorithm, Sect. 2 presents the extensions to assess local observability and local strong accessibility revealing that all three properties can be treated within one single framework. Section 3 presents a series of examples concerning both small- and large-scale systems, as well as details concerning implementation of the algorithm. One large-scale example demonstrates how the amount by which SERC outperforms LARC in computational efficiency, heavily grows with increasing system dimensions. Section 3 concerns an extensive comparison and discussion of SERC versus LARC. For local strong accessibility and local observability, generic equivalence of SERC and LARC is proved in appendix 2. A similar proof seems possible for local structural identifiability. But the additional technical issues involved are considered to be beyond the scope of this paper, that focusses on application of SERC. Section 5 presents conclusions, an important one stating that SERC reveals how linear dynamics are fundamental in determining local structural properties of dynamic systems, also if the systems themselves are nonlinear.

2 Algorithms establishing local structural identifiability, local observability and local strong accessibility

2.1 Systems and trajectories

In this paper, we consider dynamical systems represented by ordinary differential equations. These can be written in the state-space format

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u(t), \theta), \ f, x \in \mathbb{R}^n, \ u \in \mathbb{R}^r, \theta \in \mathbb{R}^p,$$
(2.1)

$$y(t) = h(x(t), \theta), \ y, h \in \mathbb{R}^m.$$
(2.2)

Here, variable t denotes continuous time, x(t) is a vector containing the state variables, u(t) is a vector containing the input variables of the system, θ represents the vector of parameters and f is an analytic vector function also referred to as the system dynamics. Furthermore, y(t) is a vector containing the output variables available for observation, identification and control and h is an analytic vector function that determines how the output variables depend on the state variables and parameters. Analytic vector functions f, h may be nonlinear and f need not be affine in uas is required in most conventional controllability, observability and structural identifiability analyses [5-10]. The algorithms to be presented will perform computations along an *a-priori* specified trajectory governed by Eqs. (2.1), (2.2) for $t \in [t_0, t_N]$, $t_N > t_0$. This trajectory is fixed if we fully specify the initial conditions, the parameter values and input:

$$x(t_0), \ \theta, \ u(t), \ t \in [t_0, t_N].$$
 (2.3)

All results obtained from our algorithms are therefore conditioned on *a choice of* (2.3). This choice will be further discussed in Sect. 3.

2.2 Local structural identifiability algorithm

Our starting point is the algorithm presented in [1] that addresses local structural identifiability. The so-called parametric state and output *sensitivity matrix functions* $\frac{\partial x}{\partial \theta}(t)$, $\frac{\partial y}{\partial \theta}(t)$ associated with system (2.1), (2.2) play a key role. For notational convenience, these matrix functions will be written as $x_{\theta}(t)$ and $y_{\theta}(t)$ having dimensions $n \times p$ and $m \times p$, respectively. At each time t, x_{θ} and y_{θ} indicate the sensitivity of state vector x(t) and output vector y(t) to the parameter vector θ . Straightforward differentiation of Eqs. (2.1), (2.2) with respect to parameter vector θ , and interchanging differentiation with respect to different, independent parameters, reveals that the dynamics of these sensitivity matrix functions is governed by

$$\frac{\mathrm{d}x_{\theta}}{\mathrm{d}t} = \frac{\partial f}{\partial x}x_{\theta} + \frac{\partial f}{\partial \theta},\tag{2.4}$$

$$y_{\theta} = \frac{\partial h}{\partial x} x_{\theta} + \frac{\partial h}{\partial \theta}.$$
 (2.5)

Simultaneous integration of differential Eqs. (2.1), (2.4) from the initial conditions $x(t_0)$ in (2.3) and,

$$x_{\theta}(t_0) = 0 \in \mathbb{R}^{n \times p}, \tag{2.6}$$

provides x(t), $x_{\theta}(t)$. Substitution of x(t), $x_{\theta}(t)$ in (2.5) then yields $y_{\theta}(t)$. Sensitivity matrix function $y_{\theta}(t)$ having p independent columns is a *necessary and sufficient* condition to uniquely determine θ from y(t)of system (2.1), (2.2), given a choice of (2.3) [23]. One way to check this rank condition is to evaluate $y_{\theta}(t)$, at discrete times $t_0 < t_1 < ... < t_N$, and to compute the rank of the resulting *parametric output sensitivity matrix:*

$$Y_{\theta} = \begin{bmatrix} y_{\theta}(t_0) \\ y_{\theta}(t_1) \\ \vdots \\ y_{\theta}(t_N) \end{bmatrix} \in \mathbb{R}^{(N+1)m \times p}.$$
(2.7)

To avoid obvious rank deficiency of Y_{θ} , the number of discrete time points N should be chosen such that,

$$(N+1)m \ge p. \tag{2.8}$$

In principle, the sampling instants $t_0 < t_1 < .. < t_N$ in (2.7) may be chosen arbitrarily close, thus requiring only a very small integration interval $[t_0, t_N]$. In practice, this interval should be selected large enough for all timescales of the system, especially the larger ones, to become manifest. To verify this, one may increase the length of the integration interval and check whether this increases the rank of sensitivity matrix Y_{θ} .

The rank condition, referred to as SERC, now reads as $rank(Y_{\theta}) = p$. It is a *necessary and sufficient* condition to uniquely determine θ from y(t) of system (2.1), (2.2), given a choice of (2.3). To determine SERC, it is convenient to compute a singular value decomposition (SVD) of Y_{θ} . If this matrix is rank deficient, its SVD yields one or more zero singular values. Corresponding to each zero singular value, a singular vector is obtained from the SVD. The nonzero elements of such a singular vector indicate the parameters that are totally correlated and therefore cannot be identified uniquely. A graphical representation of these singular vectors is called a *signature* [1].

2.3 Local observability algorithm

By considering the initial state $x(t_0)$ as the to be identified system parameters θ , local structural identifiability turns into local observability of state $x(t_0)$, i.e., local structural identifiability of the initial state $x(t_0)$ from measurements y(t), $t \ge t_0$. In this case, state vector $x(t_0)$ acts as parameter vector θ in Eqs. (2.1)– (2.5). For notational convenience, let x_0 denote $x(t_0)$. Because in Eqs. (2.4), (2.5), f and h do not depend on x_0 , these equations simplify to

$$\frac{\mathrm{d}x_{x_0}}{\mathrm{d}t} = \frac{\partial f}{\partial x} x_{x_0},\tag{2.9}$$

$$y_{x_0} = \frac{\partial h}{\partial x} x_{x_0}, \qquad (2.10)$$

where $x_{x_0}(t)$, $y_{x_0}(t)$ represent the *parametric sensitivity matrix functions* of x(t) and y(t) with respect to the initial state x_0 . Simultaneous integration of differential Eqs. (2.1), (2.9) from the initial condition (2.3) and initial condition

$$x_{x_0}(t_0) = I_n, (2.11)$$

yields x(t), $x_{x_0}(t)$. In Eq. (2.11), I_n denotes the $n \times n$ identity matrix, being the sensitivity of $x(t_0) = x_0$ with respect to itself. Substitution of x(t), $x_{x_0}(t)$ in (2.9), (2.10) then yields y(t), $y_{x_0}(t)$.

Sensitivity matrix function $y_{x_0}(t)$ having *n* independent columns is *necessary and sufficient* to uniquely obtain x_0 from y(t) of system (2.1), (2.2), given a choice of (2.3) [23]. As in (2.7), to check this rank condition, we concatenate evaluations of sensitivity matrix function $y_{x_0}(t)$ at times $t_0 < t_1 < ... < t_N$ to obtain the following sensitivity matrix:

$$Y_{x_0} = \begin{bmatrix} y_{x_0}(t_0) \\ y_{x_0}(t_1) \\ \vdots \\ y_{x_0}(t_N) \end{bmatrix} \in \mathbb{R}^{(N+1)m \times n}.$$
 (2.12)

To avoid obvious rank deficiency of Y_{x_0} , the number of discrete time points N should now be chosen such that

 $(N+1)m \ge n. \tag{2.13}$

Similar as for local structural identifiability, SERC for local observability now reads as $rank(Y_{x_0}) = n$. It is a *necessary and sufficient* condition to uniquely obtain x_0 from y(t) of system (2.1), (2.2), given a choice of (2.3). If Y_{x_0} does *not* have full rank *n*, the nonzero elements of singular vectors corresponding to zero singular values indicate state variables that are totally correlated and therefore cannot be identified uniquely [1].

2.4 Linking observability of linear and nonlinear systems through sensitivities

Here, using sensitivities, we will show how observability of linear systems can be used to asses local observability of system (2.1), (2.2), that may be nonlinear. Recall that sensitivity matrix functions $x_{x_0}(t)$, $y_{x_0}(t)$ have the advantageous property of being governed by the dynamics (2.9), (2.10) that are *linear*. This linearity is easily understood from the fact that the sensitivity of x(t) and y(t) with respect to $x(t_0) = x_0$ concerns the propagation of infinitesimal deviations $\delta x(t_0), \delta y(t_0)$ from $x(t_0), y(t_0)$ along a trajectory of the nonlinear system (2.1), (2.2) specified by (2.3). The corresponding dynamic equations of $\delta x(t), \delta y(t)$ are therefore linear dynamic equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x = \frac{\partial f}{\partial x}\delta x, \quad \delta y = \frac{\partial h}{\partial x}\delta x, \tag{2.14}$$

evaluated along the trajectory under consideration causing these linear dynamics to be *time-varying*. Associated with time-varying linear systems is the state transition matrix $\Phi(t, t_0)$, which for system (2.14) is defined in the following way [24]:

$$\delta x(t) = \Phi(t, t_0) \delta x(t_0). \tag{2.15}$$

By substituting (2.15) in (2.14), one immediately sees that the state transition matrix $\Phi(t, t_0)$ also satisfies matrix differential equation (2.9) if we identify $x_{x_0}(t)$ with $\Phi(t, t_0)$. Moreover, the initial condition (2.11) of sensitivity matrix function $x_{x_0}(t)$ is precisely the initial condition of state transition matrix $\Phi(t, t_0)$. Therefore,

$$x_{x_0}(t) \equiv \Phi(t, t_0), \qquad (2.16)$$

sensitivity matrix function $x_{x_0}(t)$ is nothing but the state transition matrix $\Phi(t, t_0)$ of the linear dynamics (2.14). The linear system (2.14) is observable if matrix

function $\frac{\partial h}{\partial x}(t)\Phi(t,t_0)$ has *n* independent columns [24]. Note that, in [24], this type of observability is called A-observability [25]. From Eqs. (2.10) and (2.16), we find that

$$y_{x_0}(t) = \frac{\partial h}{\partial x}(t)\Phi(t,t_0).$$
(2.17)

Therefore, the condition for observability of the linear dynamics (2.14) is identical to the condition that sensitivity matrix function $y_{x_0}(t)$, $t_0 \le t \le t_N$ has *n* independent columns, as required for local observability of x_0 from the trajectory. Thus, we arrive at the following important conclusion:

For nonlinear system (2.1), (2.2), local observability of x_0 from a trajectory specified by (2.3) is equivalent with observability of the linear dynamics (2.14) along that trajectory.

Observability of linear systems is very well developed as part of linear control system design [24–27]. Specifically, observability is an important property for the successful application of linear output perturbation feedback controller designs, to control nonlinear systems along optimal state and control trajectories [27]. In this context, however, the linear dynamics represent an *approximation* of the nonlinear dynamics (2.1), (2.2) *close* to the trajectory, whereas in our context, they are an *exact description* of the propagation of sensitivities *along* the trajectory.

2.5 Local strong accessibility algorithm

By considering sensitivities, in section 1.3, we found the important equivalence between observability of the linear dynamics along trajectories and observability of the initial state x_0 from such trajectories. Since for linear systems, observability and controllability are dual properties, this suggests a similar equivalence between local controllability and controllability of linear dynamics along trajectories. Based on this idea, in [20, 18], we already presented some preliminary results. In this section, the dual results are presented. We will use these in section 3 to argue that the equivalence holds if local controllability is replaced with local strong accessibility, which is a slightly weaker property [7]. Consider the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u. \tag{2.18}$$

describing the propagation of infinitesimal perturbations $\delta x, \delta u$ along trajectories of system (2.1). To pursue duality, the following substitutions must be made in section 1.2 [20, 18, 23–25]:

$$t \to t_N - t, \ t_0 \to t_N, \ y \to u, \ Y \to U, \ x(t_0)$$
$$= x_0 \to x(t_N) = x_N.$$
$$\frac{\partial f}{\partial x} \to \left(\frac{\partial f}{\partial x}\right)^T, \ \frac{\partial h}{\partial x} \to \left(\frac{\partial f}{\partial u}\right)^T, \ y_{x_0} \to u_{x_N}, \ \Phi \to \left(\Phi^T\right)^{-1}.$$
(2.19)

As dual equivalents of Eqs. (2.9)–(2.11), we now obtain

$$-\frac{\mathrm{d}x_{x_N}}{\mathrm{d}t} = \left(\frac{\partial f}{\partial x}\right)^T x_{x_N},\tag{2.20}$$

$$u_{x_N} = \left(\frac{\partial f}{\partial u}\right)^T x_{x_N},\tag{2.21}$$

$$x_{x_N}(t_N) = I_n.$$
 (2.22)

The dual equivalent of sensitivity matrix (2.12) is

$$U_{x_{N}} = \begin{bmatrix} u_{x_{N}}(t_{N}) \\ u_{x_{N}}(t_{N-1}) \\ \vdots \\ u_{x_{N}}(t_{0}) \end{bmatrix} \in \mathbb{R}^{(N+1)r \times n}, \qquad (2.23)$$

containing samples of $u_{x_N}(t)$. The dual equivalent of (2.17) becomes

$$u_{x_N}(t) = \left(\frac{\partial f}{\partial u}\right)^T (t) \left(\Phi^T(t_N - t, t_N)\right)^{-1}.$$
 (2.24)

Using $(\Phi^T(t_N - t, t_N))^{-1} = \Phi^T(t_N, t_N - t)$ [24], equation (2.24) becomes

$$u_{x_N}(t) = \left(\frac{\partial f}{\partial u}\right)^T (t) \Phi^T(t_N, t_N - t).$$
(2.25)

From (2.25), $u_{x_N}(t)$ having *n* independent columns is the dual equivalent of $\Phi(t_N, t_N - t) \frac{\partial f}{\partial u}$ having *n* independent rows which is precisely the condition for the linear dynamics (2.18) to be what is called Acontrollable in [24]. To avoid obvious rank deficiency in (2.23), *N* should now be chosen such that,

$$(N+1)r \ge n. \tag{2.26}$$

3 Algorithm implementation and examples

3.1 Implementation

The algorithms described in the previous sections rely on numerical integration of the dynamics (2.1), (2.2), (2.4), (2.5) for local structural identifiability, and (2.1), (2.2), (2.9), (2.10) for local observability along a trajectory determined by a choice of (2.3). In section 3, we will argue that, in general, selecting the input u(t), $t \in [t_0, t_N]$ in (2.3) arbitrarily, results in sufficient excitation of the linear sensitivity dynamics required for SERC to produce the same rank as LARC. The sampling instants at which the sensitivities are evaluated are taken to be equidistant here, i.e.,

$$t_i = i \frac{t_N}{N}, \ i = 0, 1, \dots, N.$$
 (3.1)

To satisfy (2.8), we select N = 2p, and to satisfy (2.13), (2.26) N = 2n.

The algorithm for local structural identifiability has been used earlier in [1, 3], on both small- and largescale systems to assess local structural identifiability of θ , sometimes with the initial condition of the state variables x_0 included as additional parameters. As explained in section 1.2, identifiability of the initial state variables x_0 alone is equivalent with local observability of x_0 . Since in [1–3], identifiability of parameters and initial states has already been covered extensively, most examples in this paper concern local strong accessibility. These examples require numerical integration of Eqs. (2.1), (2.20), (2.21) backwards in time. We will find that in all examples, numerical integration stays tractable and is efficient, even for large-scale systems, since we can restrict integration to relatively short time intervals. The only requirement for the length of this time interval is, of course, that all timescales of the system become manifest. The partial derivatives in the sensitivity equations are computed using automatic differentiation [28], since pure symbolic differentiation may become intractable for largescale systems [29]. Alternatively, one may use complex-valued algebra for a very accurate determination of the Jacobi matrices that appear in the sensitivity equations [30]. We used MATLAB R2019A for our computations on a PC having a 3.10 gigahertz Intel Core i5-8600 processor and running on Windows 10. For automatic differentiation, we used a method obtained from MATLAB Central, supplied by Martin de la Gorce, that supports sparse matrices. The latter improves the algorithm efficiency, especially for large systems. For numerical integration, we used MATLAB function ode45 with default settings. We deliberately do not select an integrator for stiff systems, since these may approximate fast transient behavior corresponding to small time scales, that are important for detecting structural system properties.

For local structural identifiability and local observability, a singular value decomposition (SVD) of sensitivity matrix Y_{x_0} in (2.7), (2.12) plays a key role in the analysis. Similarly, for local strong accessibility, an SVD of sensitivity matrix U_{x_N} in (2.23) plays this key role. The SVD of this matrix can be represented by

$$U_{x_N} = \sum_{i=1}^n u_i \sigma_i v_i^T, \ U_{x_N} \in \mathbb{R}^{(N+1)r \times n}, \ u_i \in \mathbb{R}^{(N+1)r}, \ \sigma_i \in \mathbb{R}^1, \ v_i \in \mathbb{R}^n.$$
(3.2)

This decomposition is a sum of equally sized matrices $u_i v_i^T$, weighted by their corresponding singular values $\sigma_i \ge 0$. If one or more σ_i are (numerically) zero, U_{x_N} is rank deficient. Column vectors v_i corresponding to zero singular values σ_i are basis vectors of the null space of U_{x_N} . Given that $(N + 1)r \ge n$, these vectors determine directions in state space along which state x_N cannot be controlled locally. In the examples presented in the next two sections, the column vectors v_i will be graphically presented as to clearly show these directions. As in [1], we call these graphs the *strong accessibility signature*. Of course, if the system satisfies SERC, then nonzero singular values are present and the strong accessibility signature is *empty*.

3.2 Small-scale system examples

Example 1 (Example 3 from [31]).

This artificial example is used in [31] to show how initial conditions $x(t_0)$ can influence strong accessibility of systems which in turn may influence identifiability. The system is represented by (2.1), (2.2) with

$$f(x, u, \theta) = \begin{bmatrix} -\theta_1 u_1 - \theta_2 x_1 - \theta_3 x_2 \\ (\theta_3 x_2 - \theta_4) x_1 \end{bmatrix},$$
(3.3)

$$h(x,\theta) = x_1. \tag{3.4}$$

The system (2.1), (3.3) is local strong accessible from all states except for states satisfying $\theta_3 x_2 - \theta_4 = 0.$ In that case from (3.3), $x_2(t) = x_2(t_0)$, irrespective of the input u(t), which causes the system to be not local strong accessible from these states. This in turn causes a lack of local identifiability of system (2.1), (2.2), (3.3), (3.4) that does not occur otherwise [31]. We will use our algorithms to confirm these results. Taking arbitrarily,

$$u(t) = 1 + 0.5 \sin(2\pi t), \ t \in [t_0, t_N], \ t_0 = 0, t_N = 1,$$

$$\theta = \begin{bmatrix} 5.1291e - 01 \\ 4.6048e - 01 \\ 3.5040e - 01 \\ 9.5046e - 02 \end{bmatrix}, \ x(t_N) = \begin{bmatrix} 7.0924e - 01 \\ 1.1597e - 01 \\ 1.1597e - 01 \end{bmatrix},$$

(3.5)

our algorithm for local strong accessibility produces the singular values $\sigma_1 = 9.3806e - 01$, $\sigma_2 = 2.1674e - 02$ of U_{x_N} in (2.23), confirming local strong accessibility. Next, we take (3.5) except for

$$x_2(t_N) = \frac{\theta_4}{\theta_3}.$$
(3.6)

From (3.3), the choice (3.6) implies $x_2(t) = \theta_4/\theta_3$, irrespective of the input u(t). Then, our algorithm produces the singular values $\sigma_1 = 9.3522e - 01$, $\sigma_2 = 0$. Clearly, the zero singular value indicates that local strong accessibility does *not* hold. The corresponding right singular vector in (3.2) is found to be $v_2 = [0, 1]^T$. This confirms that only the *second* state is not local strong accessible.

Suppose we want to estimate all p = 4 parameters of system (2.1), (2.2), (3.3), (3.4). Then, given (3.5), with $x(t_N)$ replaced by $x(t_0)$, our identifiability algorithm produces singular values 1.7372e + 00, 7.2723e - 02, 1.6213e - 02, 5.6044e - 04, indicating local identifiability. Next, take (3.5) except for,

$$x_2(t_0) = \frac{\theta_4}{\theta_3}.\tag{3.7}$$

Again, from (3.3), the choice (3.7) implies $x_2(t) = \theta_4/\theta_3$, irrespective of the input u(t). Furthermore, observe that (3.7) represents a very special case in which the initial state $x(t_0)$ depends on the

parameters θ . This requires a modification of the initial conditions (2.6) (initiating the sensitivity dynamics) of the identifiability algorithm

$$x_{\theta}(t_0) = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -\theta_4/\theta_3^2 & 1/\theta_3 \end{bmatrix}.$$
 (3.8)

Then, our algorithm produces the singular values 2.2712e + 00, 1.6340e - 01, 3.5852e - 02, 0 indicating that local identifiability is lost again. To the singular value $\sigma_4 = 0$ in (3.2), the right singular vector $v_4 = [0, 0, -1, 0]^T$ corresponds. It indicates that only the *third* parameter is unidentifiable. All these results completely match the ones found in [31]. It is important to note that, of these examples, the one with the largest computation time took *only* 0.04 seconds.

Example 2 (Example 6.4 from [8] where local strong accessibility is called local reachability).

This example concerns the classic car parking problem where the dynamics of the car are represented by

$$f(x,u) = \begin{bmatrix} \cos(x_3 + x_4) \\ \sin(x_3 + x_4) \\ \sin(x_4) \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2.$$
(3.9)

The dynamics show that in this case, there is no drift vector field, but instead two control vector fields (see section 3 for this terminology) can be manipulated. We arbitrarily selected $u(t) = \left[\sin\left(\frac{2\pi t}{l_N}\right), \cos\left(\frac{2\pi t}{l_N}\right)\right]^T$, $x(t_N) = [1.0129, 0.9605, 0.8504, 0.5950]^T$,

 $t \in [t_0, t_N]$, $t_0 = 0$, $t_N = 1$. The singular values of sensitivity matrix U_{x_N} in (2.23) are then computed to be 3.4230, 3.0000, 0.1001, 0.01782. Since no numerically zero singular values are found, local strong accessibility is established, as in [8]. The computation time required by the algorithm was 0.049 s. Selecting u(t) = 0, $t \in [t_0, t_N]$ clearly results in a steady-state trajectory. For u(t) = 0, $t \in [t_0, t_N]$ and $x(t_N) =$ [1.0129, 0.9605, 0.8504, 0.5950]^T the singular values are computed to be 3.4909, 3.0000, 2.6986e - 16, 4.8033e - 33. The two numerically zero singular values now indicate a rank deficiency. Trajectories like these are called *singular trajectories* in Sect. 4. Such trajectories do not excite the sensitivity dynamics sufficiently for SERC to obtain the same rank as LARC. For this particular example, this is further demonstrated in [18]. It will be argued in Sect. 4 that, in general, an arbitrary choice of u(t), $t \in [t_0, t_N]$ in (2.3) results in a nonsingular trajectory.

Example 3 (Example 6.1 from [8]).

$$f(x,u) = \begin{bmatrix} x_1 x_3 + x_2 e^{x_2} \\ x_3 \\ x_4 - x_2 x_3 \\ x_3^2 + x_2 x_4 - x_2^2 x_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3 \end{bmatrix} u.$$
(3.10)

From [8], we know that system (2.1), (3.10) has two modes that are not locally strong accessible. We arbitrarily selected $u(t) = \cos\left(\frac{2\pi t}{t_N}\right), t \in [t_0, t_N], t_0 = 0$, $t_N = 1$, $x(t_N) = [1.0129, 0.9605, 0.8504, 0.5950]^T$. Then, the singular values obtained from an SVD of the sensitivity matrix U_{x_N} in (2.23) are plotted in the left panel of Fig. 1 in descending order. Note the logarithmic scale and the large gap between the singular values clearly indicating that two singular values are numerically zero. They reveal the two modes that are not locally strong accessible as already found in [8]. The computation time required by the algorithm was 0.1 s.

The right panel of Fig. 1 shows what we call the *strong accessibility signature* showing all elements of the two column vectors v_i in (3.2) corresponding to the



Fig. 1 Singular values of sensitivity matrix U_{x_N} (left panel) and strong accessibility signature of Example 3 (right panel). Note the large gap between the second and third singular value, indicating two numerically zero singular values and a

two numerically zero singular values σ_3 , σ_4 . Element *j* of each singular vector corresponds to state variable *j*, j = 1, 2, ..., n. Elements of the one-but-last singular vector v_3 are denoted by 'o' and those of the last singular vector v_4 by 'x'. From this strong accessibility signature on the right, we conclude that state variable 1 is locally strong accessible while the two modes that are not locally strong accessible involve state variables 2, 3 and 4. The computation time required by the algorithm is 0.343 s.

3.3 Large-scale system examples

To demonstrate the high efficiency of our algorithm, and its applicability to large-scale systems, the first two examples presented in this section are those with the highest state dimension appearing in [32-34]. For one of these examples, it is very easy to extend the state dimension, as we will do. Furthermore, we will extend the analysis performed in [32-34] by checking local strong accessibility for different combinations of control inputs.

Example 4 A platoon of underwater robots (vehicles) [32].

In this example, we consider a platoon consisting of one robot that is the leader and several other robots following the leader. The leader is the only robot with full information of the terrain and mission. The robots



corresponding loss of local strong accessibility. In the right panel 'o' are elements of the singular vector corresponding to σ_3 and 'x' elements of the singular vector corresponding to σ_4

that follow maintain the same heading as the leading robot but have full control over their individual inputs. In the planar state-space model of the platoon, the leader and each follower are represented by five state variables. State variables 1 and 3 represent planar x, ycoordinates of the robot. State variables 2 and 4 and their derivatives model damping present in each robot. State variable 5 represents angular position with respect to a planar reference frame. The dynamics of the leader is given by

$$\begin{bmatrix} \dot{x}_{1} \\ \dot{x}_{2} \\ \dot{x}_{3} \\ \dot{x}_{4} \\ \dot{x}_{5} \end{bmatrix} = \begin{bmatrix} x_{2} \\ -2x_{2} \\ x_{4} \\ -2x_{4} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(x_{5}) \\ 0 \\ \sin(x_{5}) \\ 0 \end{bmatrix} u_{1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{2}.$$
(3.11)

The followers are numbered by j = 1, 2, ..., M. The dynamics of the j^{th} follower is governed by

$$\begin{bmatrix} \dot{x}_{5j+1} \\ \dot{x}_{5j+2} \\ \dot{x}_{5j+3} \\ \dot{x}_{5j+4} \\ \dot{x}_{5j+5} \end{bmatrix} = \begin{bmatrix} x_{5j+2} \\ -2x_{5j+2} \\ x_{5j+4} \\ -2x_{5j+4} \\ x_{5j-1} - x_{5j+4} \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(x_{5j+5}) \\ 0 \\ \sin(x_{5j+5}) \\ 0 \end{bmatrix} u_{2j+1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{2j+2}.$$
(3.12)

With *M* followers, the complete platoon has n =5(M+1) state variables and r = 2(M+1) inputs. By increasing M, arbitrary large state and control dimensions are obtained. Using a symmetry argument, it is proved in [32] that local strong accessibility for M = 1implies local strong accessibility for M > 1. For M =1, 4, 9, 19, 39, 49 Fig. 2 shows the singular values of U_{x_N} in (2.23) on the vertical axis. Since no singular value is numerically zero, this confirms local strong accessibility in each case. Numerical integration was performed starting at a terminal state x_N , having uniformly distributed components between -1 and 1. All controls were taken to be constant during backwards integration from terminal time $t_N = 1$ to time $t_0 = 0$. The components of the control vector were also uniformly distributed random numbers between -1and 1. The corresponding computation times required



Fig. 2 Singular values for M = 1, 4, 9, 19, 39, 49 confirming local strong accessibility in each case because no gaps and associated numerically zero singular values occur

by our algorithm are presented in Table 1. To show the high efficiency of the algorithm for large-scale systems, a comparison is made with the Mathematica package called *ProPac* [8]. Computation times to execute the function ControlDistribution within Mathematica 11.2 on the very same PC are also recorded in Table 1. This function uses Lie brackets to establish local strong accessibility (called local reachability in [8]). The factor by which our algorithm outperforms the one using Lie brackets is seen to dramatically increase with the state dimension.

The numerical efficiency of the algorithm allows extended analysis of large systems, e.g., we may easily remove some control inputs and check the effect on local strong accessibility. For the case M = 4, (n = 25) after removing control input u_4 , Fig. 3 shows the singular values of U_{x_N} and the corresponding strong accessibility signature. These reveal that in this case, one mode is not locally strong accessible. Its direction involves state variables 3, 8 and 10. Thus, the other state variables are still locally strong accessible. If we remove u_3 , instead of u_4 , four modes appear to be not locally strong accessible, as can be seen from Fig. 4. These modes involve state variables 6, 7, 8 and 9 that all belong to the first robot that follows. The other state variables and robots are therefore still locally strong accessible.

Finally, we consider the case M = 4 with u_6 removed. Figure 5 shows the resulting singular values and strong accessibility signature. As expected from symmetry arguments in [32], the effect of removing u_6

М	1	4	9	19	29	39	49
n	10	25	50	100	150	200	250
EXE SERC [s.]	0.183	0.282	0.53	0.981	1.507	1.988	2.525
EXE LARC [s.]	0.016	0.157	2.797	49.641	270.297	896.875	2216.41

Table 1 Algorithm execution times for different M and corresponding state dimension n

"EXE SERC" indicates the execution times required by our MATLAB algorithm implementation and "EXE LARC" those required by the ProPac function ControlDistribution that uses Lie brackets



Fig. 3 M = 4, u_4 removed; singular values (left) and strong accessibility signature (right)



Fig. 4 M = 4, u_3 removed; singular values (left) and strong accessibility signature (right). Corresponding to the 4 numerically zero singular values on the left, on the right the

is similar to removing u_4 . This can also be concluded from the structure of equation (3.12). By comparing Fig. 5 with Fig. 3, we observe that one mode is



components of the 4 singular vectors indicated by $o, x, *, \nabla$ respectively, make up the strong accessibility signature

obtained that is not locally strong accessible in both cases. Moreover, when u_6 is removed, the strong accessibility signature is shifted five indices to the

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Fig. 5 M = 4, u_6 removed; singular values (left) and strong accessibility signature (right)

right as compared to when u_4 is removed (apart from a possible sign reversal of the elements of each singular vector). Thus, when u_6 is removed, the modes that are not locally strong accessible involve the same combinations of state variables 8, 13 and 15 as of state variables 3, 8 and 10 when u_4 is removed. Applying the same symmetry argument, if instead of u_3 we remove u_5 , we expect four modes will pop up that are not locally strong accessible, and a shift of the strong accessibility signature of Figs. 4, 5 places to the right. This is confirmed by our algorithm. In computing the above cases, the algorithm CPU time varied between 0.252 and 0.255 s.

Example 5 A combined heat and power system of a hotel building (BCHP) [33, 34].

In this example, we consider the controllability of temperature in different rooms and parts of a Hotel building by adjusting flow rates in the system. The state-space representation of this system is given in appendix 1. The system has 14 state variables all representing different temperatures in the system. It has 12 control inputs that represent different flow rates in the system. In [33], it is reported that the system is locally strong accessible. Deactivating sets of control inputs and applying our algorithm revealed that local strong accessibility is still preserved when $C_1 = \{u_1, u_5, u_6, u_8, u_9\}$ are set to zero. The same holds for $C_2 = \{u_1, u_6, u_8, u_9, u_{12}\}$.

These results follow from Fig. 6 where C_0 indicates the case that no controls are excluded. Numerical



Fig. 6 Singular values obtained for 3 excluded sets of inputs. Since all of them are nonzero numerically this confirms local strong accessibility in each case

integration was performed starting from terminal state x_N having randomly generated components between 0 and 1. All controls were taken to be constant during integration from terminal time $t_N = 1$ to time $t_0 = 0$. The components of the control vector were generated randomly between -1 and 1. To the components of parameter vector θ , we assigned random values between 1 and 2. The algorithm execution time varied between 0.361 and 0.409 s.

Figures 7 and 8 show the singular values and strong accessibility signatures for four other sets of excluded control inputs, namely $C_3 = \{u_1, u_5, u_6, u_8, u_9, u_{12}\}, C_4 = \{u_1, u_5, u_6, u_7, u_9\}, C_5 = \{u_1, u_7, u_9, u_{11}, u_{12}\},$



Fig. 7 Singular values obtained for 4 excluded sets of inputs. In each case at least 1 numerically zero singular value occurs implying that local strong accessibility does not hold



and $C_6 = \{u_1, u_2, u_6, u_8, u_9, u_{12}\}$. Observe that for the excluded sets $C_3 - C_5$, one mode occurs that is not locally strong accessible. For C_3 , this mode coincides with state variable 11, for C_4 with state variable 6, and for C_5 with state variable 12. All other state variables remain locally strong accessible. For C_6 , two modes occur that are not locally strong accessible, coinciding with state variables 1 and 2.

The next and final example is a challenging one since it contains parameters that are ill-identifiable, i.e., practically very hard to identify. In addition, the system is also ill-conditioned in some cases. The example concerns large-scale mass-spring-damper systems that contain nonlinearities and in some cases flutter. Flutter implies stiffness of the system, which reduces the efficiency of numerical integration. Still, with some modification, our algorithm still comes up



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Fig. 8 Strong accessibility signatures for excluded sets C_3 , C_4 (top) and C_5 , C_6 (bottom)

0.8

0.6

0.4

0.2

with correct solutions in the order of seconds up to a minute.

Example 6: Identifiability of parameters of a mass-spring-damper system.

The system consists of n_M unit masses $m_i = 1, i = 1, 2, .., n_M$ connected in series with nonlinear springs and dampers. At each end of the series, a nonlinear spring and linear damper connect to a fixed wall, see Fig. 9. The state variables of this system are x_i , $i = 1, 2, .., n_M$ representing displacements of the masses from their equilibrium position and x_{n_M+i} , $i = 1, 2, .., n_M$ representing their velocities, with positive values pointing to the right. In Fig. 9, s_i , σ_i , d_i , $i = 1, 2, .., n_M + 1$ represent ordinary spring constants acting linearly, nonlinear spring constants, and damping coefficients acting linearly. The system dynamics are represented by (2.1) with

$$f_{i} = -s_{i}(x_{i} - x_{i-1}) - \sigma_{i}(x_{i} - x_{i-1})^{3} - d_{i}(x_{i+n_{M}} - x_{i+n_{M}-1}) - s_{i+1}(x_{i} - x_{i+1}) - \sigma_{i+1}(x_{i} - x_{i+1})^{3} - d_{i+1}(x_{i+n_{M}} - x_{i+n_{M}+1}).$$
(3.13)

for $i = 2, 3, ..., n_M - 1$. Because of the fixed walls, for i = 1, equation (3.13) applies with $x_0 = x_{n_M} = 0$, and for $i = n_M$, equation (3.13) applies with $x_{n_M+1} = x_{n_M+n_M+1} = 0$.

Assuming the displacement of only one mass is measured, we study the identifiability of parameters $s_i, \sigma_i, d_i, i = 1, 2, ..., n_M + 1$ for different values of n_M .

We selected n_M in between 1 and 15 and random parameter values s_i, d_i in between 0.2 and 1 and $\sigma_i = s_i/2, i = 1, 2, .., n_M + 1$. The initial conditions x_i , $i = 1, 2, .., 2n_M$ were also selected randomly between -1 and 1. Then, in each case, all $3(n_M + 1)$ parameters turned out to be identifiable regardless of which mass displacement is measured. On the other hand, parameters of springs and dampers close to the right wall turned out to be ill-identifiable from measurement of the position x_1 of mass m_1 on the left, especially for larger values of n_M .

Here, we only present results of some interesting, difficult cases. For the large-scale example with $n_M = 15$, while measuring the displacement x_1 of the first mass m_1 , we provide the singular values obtained from one trajectory in the left panel in Fig. 10 and of five trajectories in the right panel. In the latter case, the five sensitivity matrices (2.7) obtained from each trajectory, having identical parameters but different initial conditions, are stacked on top of one another resulting in a concatenated sensitivity matrix to which we apply the singular value decomposition (3.2). As explained in [3], [35], this concatenation can greatly improve the visibility of a possible gap in the singular values indicating a lack of identifiability. Since in this case all parameters are identifiable, the stacking only reduces the range of the singular values significantly. Note that, the ratio of singular values on the left is limited by the machine precision of 2.2×10^{-16} which may prevent locating a possible gap. This problem is thus resolved by concatenating the sensitivity matrices of five trajectories on the right. The execution time is approximately 3.3 s for each single trajectory.

As a final case, we present the results for $n_M = 5$ and $n_M = 15$ with flutter introduced by enlarging the spring constants s_1 , σ_1 , s_{16} , σ_{16} with a factor 1000 and at the same time reducing d_1 , d_{16} with a factor 1000 turning the system into a stiff one. Moreover, perfect symmetric motions are realized on each side of the middle mass m_3 for $n_M = 5$, and m_8 for $n_M = 15$. In the latter case, this is achieved by taking $s_{8+i} = s_{8-i}$, $d_{8+i} = d_{8-i}$, $x_{0,i+7} = -x_{0,i-7}$, i = 1, 2, ..., 7, $x_{0,i} = 0, i = 8, 16, 17, 18, ..., 30$, $\sigma_i = s_i/2$, i =



Fig. 9 Mass-spring-damper system with nonlinear springs and unit masses

100

10

Singular values 01. 10.

10-1

10-20

1

35 7

Fig. 10 Singular values for one trajectory (left panel) and five trajectories after concatenating their sensitivity matrices (right panel) obtained taking $t_N = 9$. Given the machine precision of

Index

Singular values sensitivity matrix

1, 2, ..., 16 and otherwise random. Due to this symmetry, masses m_3 and m_8 do not move. Measuring their position, as we do now, will cause unidentifiability of parameters.

In Fig. 11, the top panels present results of two individual symmetric trajectories, one for $n_M = 5$, with an execution time of 1.16 s, and one for $n_M = 15$. To obtain proper results for $n_M = 15$ that also includes flutter, sensitivity matrices of five trajectories are concatenated in the bottom left panel. The execution time for all five trajectories together was 37.8 s. A slight change of any of the parameters or initial conditions will destroy the symmetry of the motions on either side of the middle mass m_8 , resulting in identifiability of all parameters again. Like in Example 2, the symmetric trajectories in Fig. 11 are singular trajectories, as further explained and discussed in the next section.

4 Is SERC a free lunch?

The benefits of SERC as compared to LARC have been clearly demonstrated in the previous section. They mostly relate to an enormous gain in efficiency, enabling the handling of large-scale systems, for which LARC tends to be very inefficient or to break down. Furthermore, LARC is restricted to nonlinear systems (2.1), (2.2) that are affine in the input. These are represented by (2.1), (2.2) with



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 2.2×10^{-16} only from the right panel identifiability of all 48 parameters may be concluded

$$f(x, u, \theta) = f_0(x, \theta) + \sum_{i=1}^r f_i(x, \theta) u_i,$$
 (4.1)

where $f_i(x, \theta)$, i = 0, 1, ..., r are analytic vector functions. Note that, the property local strong accessibility, which is verified by both SERC and LARC, is a slightly weaker property than local controllability. If the so-called drift vector field $f_0(x, \theta) = 0$, local strong accessibility becomes equal to local controllability. The vector fields $f_i(x, \theta)$, i = 1, 2, ..., r that multiply u_i in (4.1), are called control vector fields [7].

SERC performs computations along the a-priori selected trajectory specified by parameters θ , terminal state $x(t_N)$ and input u(t), $t \in [t_0, t_N]$ in (2.3). Since system (2.1), (2.2) is assumed analytic, results concerning system structure obtained for terminal state $x(t_N)$ on the trajectory, hold for all states $x(t), t \in$ $[t_0, t_N]$ on the trajectory. Establishing the linear dynamics by which sensitivities propagate requires differentiation of the nonlinear system dynamics. This can be very accurately handled by automatic differentiation [28]. After equidistant sampling over time, a numerical sensitivity matrix is obtained of which the numerical rank must be determined. Apart from automatic differentiation, these numerical computations may be prone to numerical errors. As the examples in the previous section show, with default settings for numerical integration, and taking a still short, but long enough trajectory, leads generally to a clear gap in the singular values, if any, to properly



Fig. 11 Singular values for one symmetric trajectory obtained taking $t_N = 3$ and $n_M = 5$ (top left panel) and one in case $n_M = 15$ (top right panel). In the latter case concatenation of sensitivity matrices is required to find the correct gap halfway

determine the numerical rank. This gap can be increased, and the range of singular values decreased, by concatenating sensitivity matrices of different trajectories, as illustrated by Example 6. From this example, it is clear that when the range of singular values approaches the machine precision, our algorithm cannot locate a possible gap in the lower range. Since the singular values are actually *measures of identifiability*, when being close to the machine precision, they indicate *extreme ill-identifiability*. This is valuable information to engineers concerning



the total number of parameters (bottom left panel). The flutter and symmetry of motion in case $n_M = 15$ can be seen from the bottom right panel that shows all 15 mass displacements

1.5

2

-0.2

-0.4

-0.6 -0.8

0

0.5

practical identifiability but may prevent the algorithm from correctly establishing *structural identifiability* [36].

LARC on the other hand restricts analysis to a single state [7], denoted by x_L in Fig. 12, not requiring specification of a trajectory. It verifies local strong accessibility for state x_L , complying with local definitions of this property [7], [9]. LARC does not require specification of a trajectory since it *locally* considers *all possible trajectories* containing x_L . This is realized by considering a Lie series expansion of the nonlinear

x12

x13 x14

x15

3

2.5



Fig. 12 LARC evaluated at x_L , versus SERC evaluated along nonsingular state trajectories $x^{ns,i}(t)$, i = 1, 2, 3 and the singular trajectory $x^s(t)$, all containing x_L but having different inputs u(t), $t \in [t_0, t_N]$. LARC equals SERC along nonsingular trajectories. Along singular trajectories the rank obtained from SERC is less than the one from LARC. In practice, the number of nonsingular trajectories greatly outnumber the number of singular trajectories

system dynamics (2.1), (2.2) around state x_L , requiring Lie bracketing [5–10]. A disadvantage of LARC is that the number of Lie Brackets that are needed to reach the maximum rank is generally not known a-priori and may grow very large [19]. This causes the outcome to be *uncertain*, whenever the rank condition is not met and may cause the analysis to become infeasible [14–16], [18]. On the other hand, LARC determines structural identifiability and does not suffer from stiffness and other types of numerical ill-conditioning, such as ill-identifiability.

Only in exceptional cases, the difference between SERC and LARC may result in different outcomes. One example is the classic car parking problem (Example 2 in section 2.2, Example 1 in [18]) if $u(t) = 0, t \in [t_0, t_N]$. Then, the trajectories are steadystate trajectories with corresponding sensitivity matrices that are rank deficient, instead of full rank. Although SERC correctly indicates a failure of controllability of the linear dynamics (2.14), other trajectories containing x_L will not indicate this failure, see Fig. 12. This shows the importance of choosing the trajectory appropriately. In [37], *singular trajectories* are defined and recognized on which the equivalence between controllability of the linear dynamics (2.14) and local strong accessibility fails. This equivalence concerns precisely the one between SERC and LARC in this paper, when their rank is full. Moreover, in [37], trajectories are proved to be *generically nonsingular*. In Fig. 12, we have therefore adopted the same terminology. In appendix 2, using arguments from [37] and [5], the generic equivalence between SERC and LARC for local strong accessibility is also proved when LARC does not give full rank.

So far in this section, the discussion focused on local strong accessibility. Using similar arguments as for local strong accessibility, we also prove generic equivalence between LARC and SERC for local observability in appendix 2. Most likely, the proof for local structural identifiability can be obtained in a similar manner. But since this involves additional technicalities, it is considered out of the scope of this paper that focuses on application of SERC. Experimenting with SERC on different nonlinear systems and different trajectories, we found that if one selects the input u(t), $t \in [t_0, t_N]$ in (2.3) arbitrarily, SERC equaled LARC in all cases. This is in line with all examples in section 2 and [1-3]. In the case of local structural identifiability, the selection of a trajectory through an arbitrary choice of the input $u(t), t \in [t_0, t_N]$ in (2.3) may be identified with selecting an input that is "sufficiently exciting" for the output sensitivities, meaning that the input excites all relevant dynamics of the linear sensitivity system (2.4), (2.5) [38]. If one is in doubt about the nonsingularity of the trajectory containing x_L in Fig. 12, one may of course test with several trajectories containing x_L in Fig. 12 and select the one for which SERC gives the highest rank.

5 Conclusions

Sensitivity matrices and an associated rank condition (SERC) were shown to be keys to a unifying approach establishing three fundamental properties that characterize nonlinear system structure: local structural identifiability, local observability and local strong accessibility. This approach resulted in a highly efficient algorithm enabling the structural analysis of systems with a state dimension up to 250 within several seconds. Our algorithm opens the possibility to verify these properties for large-scale systems where other methods tend to fail or break down. The importance of sensitivity matrices has first been recognized in the context of system identification. In this paper, it was shown how it carries over to local observability and local strong accessibility.

The behavior of sensitivity matrix functions appears to be governed by time-varying linear dynamics. Integrating the linear sensitivity dynamics in parallel with the nonlinear system dynamics over short periods of time provides the sensitivity matrix functions. After appropriate sampling, a sensitivity matrix is obtained. A singular value decomposition applied to this matrix together with a sensitivity rank condition (SERC) yields enough information to decide whether the structural property holds. Moreover, the null space corresponding to zero singular values, also obtained from the singular value decomposition, reveals locally the cause of lack of identifiability, observability, or strong accessibility. It pinpoints the exact state variables and/or parameters that are involved in a total correlation. This null space can be graphically represented and is termed the identifiability/observability/strong accessibility signature.

For the examples presented in this paper and several references, the outcomes obtained along different, arbitrarily selected trajectories of the same system, are generally identical. This can be expected since local strong accessibility, local observability and local structural identifiability are properties of nonlinear systems that, in general, have a global character. Example 1 in this paper illustrated how, for a tiny set of initial conditions, structural properties may change. Last, but not least, we emphasize that the high efficiency of the algorithm based on sensitivities stems from their *linear dynamics* along trajectories. This holds even if the system itself is nonlinear.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose. Data will be made available on reasonable request.

Appendix 1: Dynamic model of the combined heat and power system of a hotel building [33, 34] used in Example 5

$$f(x, u, \theta) = \begin{cases} k_1(x_2 - x_1) + \left(\frac{b_1}{V_1} - \frac{x_1}{V_1}\right)u_1 \\ k_2(x_1 - x_2) + \left(\frac{x_4}{V_2} - \frac{x_2}{V_2}\right)u_2 \\ \left(\frac{x_2}{V_3} - \frac{x_3}{V_3}\right)u_2 + \left(\frac{x_4}{V_3} - \frac{x_3}{V_3}\right)u_3 \\ \left(\frac{x_6}{V_4} - \frac{x_4}{V_4}\right)u_4 \\ \left(\frac{x_3}{V_5} - \frac{x_5}{V_5}\right)u_4 \\ \left(\frac{x_7}{V_6} - \frac{x_6}{V_6}\right)u_7 + \left(\frac{x_9}{V_6} - \frac{x_6}{V_6}\right)u_9 + \left(\frac{x_{12}}{V_6} - \frac{x_6}{V_6}\right)u_5 \\ k_7(x_8 - x_7) + \left(\frac{x_5}{V_7} - \frac{x_7}{V_7}\right)u_7 \\ k_8(x_7 - x_8) + \left(\frac{b_2}{V_8} - \frac{x_8}{V_8}\right)u_8 \\ k_9(x_{10} - x_9) + \left(\frac{x_5}{V_9} - \frac{x_{10}}{V_9}\right)u_9 \\ k_{10}(x_9 - x_{10}) + \left(\frac{b_{11}}{V_{11}} - \frac{x_{11}}{V_{11}}\right)u_{12} \\ \left(\frac{x_{11}}{V_{11}} - \frac{x_{11}}{V_{12}}\right)u_{11} \\ k_{13}(x_{14} - x_{13}) + \left(\frac{x_{12}}{V_{13}} - \frac{x_{13}}{V_{13}}\right)u_{11} \\ k_{14}(x_{13} - x_{14}) + \left(\frac{b_4}{V_{14}} - \frac{x_{14}}{V_{14}}\right)u_6 \end{cases}$$
(A1)

$$\theta = [V_1, V_2, ..., V_{14}, k_1, k_2, k_7, k_8, k_9, k_{10}, k_{13}, k_{14}, b_1, b_2, b_3, b_4]^T$$
(A2)

Appendix 2: Proof of generic equivalence between SERC and LARC for local strong accessibility and local observability

Proof for local strong accessibility

Concerning local strong accessibility, the examples in this paper show that the rank given by SERC equals the one given by LARC, provided the selected trajectory (2.3) of system (2.1) is nonsingular, as shown in Fig. 12. For local strong accessible systems (2.1), it is proved in [37] that trajectories (2.3) of system (2.1) are *generically nonsingular*. Then, LARC gives full rank for x_L in Fig. 12 while SERC generically does, namely if the selected trajectory (2.3) of system (2.1) containing x_L is nonsingular. In this appendix, we will prove that if the rank obtained

from LARC is not full, SERC still generically produces the same rank for x_L in Fig. 12. To that end, we will first summarize the proof given in [37], when the rank obtained from LARC is full.

Theorem 2 in [37] states that for local strong accessible systems, trajectories are generically nonsingular. Its proof relies mostly on proposition 2.1 in [37]. This proposition concerns an observabilityrelated property developed in [39]. Recall that LARC for local strong accessibility is fully determined by the dynamics (2.1). The proof of theorem 2 in [37] requires extension of these dynamics with

$$\frac{dQ(t)}{dt} = A(t)Q(t) + Q(t)A(t) + B(t)B^{T}(t), \ t \ge t_{0}, \ Q(t_{0}) = 0$$
(A3)

$$A(t) = \frac{\mathrm{d}f}{\mathrm{d}x}(x(t), u(t)), \ B(t) = \frac{\mathrm{d}f}{\mathrm{d}u}(x(t), u(t)), \ Q(t)$$

$$\in R^{n \times n}$$
(A4)

In addition, the following output equation is required:

$$y_e(t) = h_e(x(t), Q(t)) = \det(Q(t)).$$
 (A5)

Note that, Q(t), $t > t_0$ in (A3) is the controllability matrix of the time-varying linear system (2.18) that describes the propagation of sensitivities. If Q(t), $t > t_0$ is not full rank, the linear dynamics (2.18) is not controllable and the trajectory is singular. In that case, the determinant of Q(t), $t > t_0$, being the output $y_e(t)$ in (A5), equals zero. System (2.1), extended with (A3), (A5), when written in the standard state-space format becomes

$$\frac{\mathrm{d}x_e(t)}{\mathrm{d}t} = f_e(x_e(t), u(t), \theta) = \begin{bmatrix} f(x(t), u(t), \theta) \\ st(A(t)Q(t) + Q(t)A(t) + B(t)B^T(t)) \end{bmatrix}$$
$$x_e(t) = \begin{bmatrix} x(t) \\ st(Q(t)) \end{bmatrix},$$

$$y_e(t) = h_e(x_e(t)) = \det(Q(t)), \tag{A7}$$

where *st* denotes the stack operator that stacks all columns of the matrix argument, starting with the first one. The proof of theorem 2 in [37] comes down to show, by means of proposition 2.1, that trajectories having initial state $x_e = \begin{bmatrix} x \\ 0 \end{bmatrix}$, generically, do not

produce a zero output $y_e(t)$ in (A5), and therefore are nonsingular. To obtain this result, local strong accessibility of system (2.1) is required to guarantee that every state $x_e = \begin{bmatrix} x \\ 0 \end{bmatrix}$ of the extended system belongs to the set *G*, mentioned in proposition 2.1 of [37].

Here, we would like to establish generic equivalence for any rank given by LARC, for any x_L in Fig. 12. This can be achieved since f in (2.1) is assumed analytic. Then, the Hermann Nagano theorem [5] holds, according to which the state space separates into one or more manifolds having equal or different dimensions. Then by Chow's theorem [5], a realization of each manifold exists, *that is local strong accessible*, having a state dimension equal to the rank given by LARC. Given their local strong accessibility, we are allowed again to apply the proof given in [37], to these realizations. This proves the generic equivalence between SERC and LARC for any rank, and for any state x_L in Fig. 12.

Proof for local observability

For local observability, to show that the rank from LARC generically equals the one from SERC, we can again apply proposition 2.1 from [37]. Now, LARC is obtained from Lie differentiation instead of Lie bracketing [5], [7], [8] and nonsingular trajectories are those leading to an observable linear system (2.14). Therefore, we must extend the dynamics (2.1) with dynamics describing the observability matrix propagation of the time-varying linear system (2.14),

$$\frac{dQ(t)}{dt} = A^{T}(t)Q(t) + Q(t)A^{T}(t) + C^{T}(t)C(t), \ t \ge t_{0}, \ Q(t_{0}) = 0,$$
(A8)

$$A(t) = \frac{\mathrm{d}f}{\mathrm{d}x}(x(t), u(t)), \ C(t) = \frac{\mathrm{d}h}{\mathrm{d}x}(x(t), u(t)), \ Q(t)$$

$$\in R^{n \times n},$$
(A9)

while taking as the output,

$$y_e(t) = h(x(t), Q(t)) = \det(Q(t)).$$
 (A10)

If Q(t), $t > t_0$ is not full rank, the linear dynamics (2.14) are not observable and the trajectory is singular. In that case, the determinant of Q(t), $t > t_0$, being the output $y_e(t)$ in (A10), equals zero. The state-space representation of the extended system now becomes

$$\begin{aligned} \frac{\mathrm{d}x_e(t)}{\mathrm{d}t} &= f_e(x_e(t), u(t), \theta) = \begin{bmatrix} f(x(t), u(t), \theta) \\ st(A^T(t)Q(t) + Q(t)A^T(t) + C^T(t)C(t)) \end{bmatrix}, \\ x_e(t) &= \begin{bmatrix} x(t) \\ st(Q(t)) \end{bmatrix}, \end{aligned}$$
(A11)

$$y_e(t) = h_e(x_e(t)) = \det(Q(t)).$$
(A12)

Again the proof follows from proposition 2.1 of [37] if all states $x_e = \begin{bmatrix} x \\ 0 \end{bmatrix}$ of the extended system

belong to the set *G*, mentioned in proposition 2.1. Now, this requires the original system (2.1), (2.2) to be local observable. That this is so follows from application of the same proposition 2.1, now taking (2.1) as the system and (2.2) as the output. So if LARC gives full rank for any x_L in Fig. 12, SERC generically does, namely if the selected trajectory (2.3) of system (2.1) is nonsingular.

To establish generic equivalence between LARC and SERC for any rank, we rely on our assumption that f in (2.1) and h in (2.2) are analytic. Then, the Hermann Nagano and Chow theorems [5] provide us with local accessible realizations of each manifold in which the state space separates. Moreover, these realizations each preserve the observation space, the dimension of which equals the rank obtained from LARC. Next, for each of those realizations, we may obtain a minimal realization that also preserves the observation space [7]. This minimal realization is locally observable and has a state dimension equal to the one of the observation space given by the rank obtained from LARC. Because these minimal realizations are locally observable, we may apply proposition 2.1 of [37] again, which proves the generic equivalence between LARC and SERC for any rank, and any state x_L in Fig. 12.

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