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An Efficient Method to Assess Local Controllability and Observability for Non-Linear Systems

J.D. Stigter * **L.G. van Willigenburg** * **J. Molenaar** *

* *Biometris, Department of Mathematical and Statistical Methods, Wageningen University and Research, The Netherlands.*

Abstract: Controllability and observability are two fundamental system properties that yield insight in the structural relationships between state-, input-, and output variables. It is well-known that for non-linear type of system models these structural properties are difficult to analyse, especially if the size of the corresponding model is large. A sensitivity based method to assess local controllability and observability for non-linear systems in a simple way is discussed here and its validity is demonstrated with some examples taken from various applications in the control literature. A major advantage of our sensitivity based approach is that no Lie-derivatives or Lie-brackets have to be calculated and this results in a fast algorithm for structural system analysis. An optional second step in the analyses is the verification of a *lack of observability/controllability*, once obtained in the first step, with a simplified symbolic computation. Finally, the so-called *observability- and controllability signatures* of a given system are presented that characterize local observability/controllability and yield a succinct visual summary of these fundamental system properties.

Keywords: Local Controllability, Local System Structure, Observability, Sensitivity Analysis

1. INTRODUCTION

Controllability and observability are fundamental properties that characterize the structure of a given system, (Hermann and Krener, 1977; Isidori, 1989; Nijmeijer and Van der Schaft, 1990; Kwatny and Blankenship, 2000). In this paper we aim for an easy-to-evaluate criterion to assess the local controllability/observability/identifiability properties of a general, possibly non-linear, system model in state-space format, i.e.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), u(t), \theta) \tag{1}$$

$$x(0) = x_0 \tag{2}$$

$$y(t) = h(x(t), u(t), \theta) \tag{3}$$

with $x(t)$ the state vector ($\dim(x) = n$), $u(t)$ the input vector ($\dim(u) = r$), $y(t)$ the output vector ($\dim(y) = m$), and θ a vector of parameters ($\dim(\theta) = p$). The vector functions f and h characterize the system dynamics and output or observation function, respectively. In Stigter and Molenaar (2015); Stigter et al. (2017a,b) a fast algorithm is introduced to assess local structural identifiability for model (1)-(3), meaning that the question of *locally unique* parameter estimates on the basis of a pre-specified set of observation signals (without noise) is solved. Since identifiability is just a special case of observability and, moreover, controllability is the *dual* problem of observability, the same ideas may be applied to assess the controllability property for the non-linear system model (1)-(3). This is the idea underlying the current paper.

1.1 Sensitivity dynamics and structural system properties

The main idea we pursue here is that we analyse *all* structural properties of the general non-linear state space model (1)-(3) on the basis of its sensitivity dynamics. For the local structural identifiability property this is well known, e.g. (Miao et al., 2011; Stigter and Molenaar, 2015), and the forward sensitivity dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x(t)}{\partial \theta}\right) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial \theta} \tag{4}$$

$$\frac{\partial y(t)}{\partial \theta} = \frac{\partial h}{\partial x}\frac{\partial x(t)}{\partial \theta} + \frac{\partial h}{\partial \theta} \tag{5}$$

can be straightforwardly related to structural identifiability properties of the system model through a rank test on the parametric output sensitivity matrix $S(t_0, \ldots, t_N, \theta)$. This matrix can simply be constructed by vertical concatenation of parametric output sensitivities at different time instances $t_0, t_1, \ldots, t_N$, as follows:

$$S(t_0, \ldots, t_N, \theta) = \begin{pmatrix} \frac{\partial y_1(t_0)}{\partial \theta_1} & \cdots & \frac{\partial y_1(t_0)}{\partial \theta_n} \\ \vdots & & \vdots \\ \frac{\partial y_m(t_0)}{\partial \theta_1} & \cdots & \frac{\partial y_m(t_0)}{\partial \theta_n} \\ \vdots & & \vdots \\ \frac{\partial y_1(t_N)}{\partial \theta_1} & \cdots & \frac{\partial y_1(t_N)}{\partial \theta_n} \\ \vdots & & \vdots \\ \frac{\partial y_m(t_N)}{\partial \theta_1} & \cdots & \frac{\partial y_m(t_N)}{\partial \theta_n} \end{pmatrix} \tag{6}$$

Alternatively, one may study relative sensitivities that yield a so-called relative sensitivity matrix with the same rank. At this point it must be mentioned that the non-linearities in the model structure are *not* approximated in some way whilst deriving the sensitivity dynamics (4)–(5). Instead, the sensitivity dynamics are *exact* and testing these sensitivities for linear dependencies is the essential key that allows an *exact* test for the structural properties observability, identifiability, and controllability.

The sensitivity matrix (6) can be written in its dyadic form as

$$S(t_0, \ldots, t_N, \theta) = u_1\, \sigma_1\, v_1^T + \ldots + u_p\, \sigma_p\, v_p^T \tag{7}$$

using a singular value decomposition (SVD) with $\sigma_1, \ldots, \sigma_p$ the p singular values, u_i and v_i, $i = 1, \ldots, p$, the columns of the unitary matrices U and V that follow from the SVD (Golub and van Loan, 1996; Kalman, 2002). Rank deficiency of the sensitivity matrix can be checked by inspection of the *last* singular value that equals zero for a rank deficient matrix. Detection of one or several zero singular(s) values is clearly an important task in the analysis of the structural properties of a control system. More specifically, it will become clear in the examples section that a *gap* (or sudden jump of several decades on a logarithmic scale) in the spectrum of singular values is a clear indicator of a lack of local controllability or local observability.

We further remark that structural controllability is a classical topic that has been investigated from many different angles. One of these is to study the properties of a *directed graph* associated with a *linear* control system and to apply graph-theory for the analysis of its structural properties (Murota, 1987; Lin, 1974). Recent work by Liu and co-workers also points in this direction, but now for *non-linear* systems as well (Liu et al., 2012; Liu and Barabasi, 2016). The present results are suggested to hold for non-linear systems as well, but our approach takes into account the algebraic relations in the model structure from the very beginning, rather than using an adjacency matrix only. It is, of course, interesting to study in more detail how these different approaches relate to one-another.

2. FROM IDENTIFIABILITY TO STATE-OBSERVABILITY/CONTROLLABILITY

2.1 Introduction

Extension of the identifiability question to local controllability and observability is straightforward once it is realised that initial conditions can be treated as simple time-invariant parameters. Hereto, first consider the case where we only analyse the *parametrised initial conditions* (2) so that $\theta = x_0$, and where we do *not* consider any system parameters θ that appear on the right hand side of (1) and (3). Consequently the term $\frac{\partial f}{\partial \theta}$ in (4) vanishes. The same holds for the term $\frac{\partial h}{\partial \theta}$ in (5). Viewing the initial conditions as the only parameters in the system therefore yields the following sensitivity system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x(t)}{\partial \theta}\right) = \frac{\partial f}{\partial x}\,\frac{\partial x(t)}{\partial \theta} \tag{8}$$

$$\frac{\partial y(t)}{\partial \theta} = \frac{\partial h}{\partial x}\,\frac{\partial x(t)}{\partial \theta} \tag{9}$$

Integrating the sensitivity system *forwards* in time, together with the original system model that can be viewed as its seed of dynamics, allows analysis of state observability for the non-linear system model in a simple way by means of (6). Initialization of the forward integration is carried out with $\frac{\partial x(t_0)}{\partial \theta} = I_n$ with I_n the $n \times n$ identity matrix. This follows easily from the fact that $\frac{\partial x_i(0)}{\partial \theta_j}$ equals zero for $i \neq j$, and one for $i = j$. Interesting to mention is that the above sensitivity dynamics are exactly equivalent to the solution of the fundamental matrix of the local, linearised system dynamics as given in, for example (Krener and Ide, 2009), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \frac{\partial f}{\partial x}\,\Phi(t) \tag{10}$$

$$\Phi(0) = I_n \tag{11}$$

Associated with these linearised system dynamics is the *observability Gramian*

$$P(x_0) = \int_0^T \Phi^T(t)\,\frac{\partial h^T}{\partial x}\,\frac{\partial h}{\partial x}\,\Phi(t)\,dt \tag{12}$$

Since local observability and controllability are *dual* concepts, consider now the controllability Gramian $Q(x_0)$ for the local linearised system dynamics:

$$Q(x_0) = \int_0^T \Phi(t)\,\frac{\partial f}{\partial u}\,\frac{\partial f}{\partial u}^T\,\Phi^T(t)\,dt \tag{13}$$

We now state the following conjecture:

Conjecture. *Local controllability of the system* (1)–(3) *can be analysed based on the dual sensitivity system:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x(t)}{\partial \theta}\right) = -\left(\frac{\partial f}{\partial x}\right)^T\,\frac{\partial x(t)}{\partial \theta} \tag{14}$$

$$y_\theta(t) = \left(\frac{\partial f}{\partial u}\right)^T\,\frac{\partial x(t)}{\partial \theta} \tag{15}$$

with $\theta = x(T)$.

More specifically, if we integrate the dual sensitivity system (14)–(15) backwards in time, then we are in a position to analyse local controllability properties of the dynamic system (1)-(3) in a very simple way. To see this intuitively, we only have to look at the controllability Gramian (13). The term $\frac{\partial h^T}{\partial x}$ in the observability Gramian has been replaced with $\frac{\partial f}{\partial u}$ and so the latter naturally appears in the output equation of the dual sensitivity system. In addition, a full rank of the controllability Gramian (13) can be interpreted as the existence of a unique mapping between the terminal state variables in $x(T)$ and the 'output' variables that is checked via the sensitivities (15). If this mapping from terminal condition parameters and observed 'output variables' is unique, then clearly independent control directions exist that allow the state vector $x(T)$ to be steered into n independent directions in the forward problem.

The state sensitivities $\frac{\partial x}{\partial \theta}$ in (14) are initialized as $\frac{\partial x(T)}{\partial \theta} = I_n$ with I_n the $n \times n$ identity matrix. Further note that the minus sign in (14) indicates a *backwards* integration in time. As in the forward case, initialisation of the sensitivities with the identity matrix follows easily from the fact that the terminal conditions $x(T)$ are viewed as *parameters*.

In summary, state controllability of (1) can be analysed using a combined backward integration of the non-linear system, together with the dual system of parametric state sensitivities (14) where the only parameters are the terminal conditions $x(T)$. Essentially, a rank test on the sensitivity matrix that is constructed via a sampled version of the dual output equation (15) is all that is needed.

2.2 Linear system case

For a linear-time-invariant (LTI) system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A\,x(t) + B\,u(t) \tag{16}$$

$$y(t) = C\,x(t) \tag{17}$$

the dual sensitivity system (14)-(15) reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x(t)}{\partial\theta}\right) = -A^T\,\frac{\partial x(t)}{\partial\theta} \tag{18}$$

$$y_\theta = B^T\,\frac{\partial x(t)}{\partial\theta} \tag{19}$$

and this dual sensitivity matrix is initialized at the terminal time T with the identity matrix. The corresponding Gramian (or 'Fisher Information matrix' for the terminal condition parameters $x(T)$, see Stigter et al. (2017a)) for this case is easily derived as

$$\int_0^T \Phi(t)\,B\,B^T\,\Phi^T(t)\,dt \tag{20}$$

and this equals the controllability Gramian of a linear time-invariant system (Kailath, 1980). An interesting remark to make for this case is that from the structure of the dual sensitivity equations we can clearly see that for LTI-systems the state vector $x(t)$ does *not* enter the sensitivity dynamics and so the state sensitivities are completely autonomous, once initialized with the identity matrix.

3. EXAMPLES

3.1 Example – confined movement on a sphere

Consider the following smooth affine control system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix} = \begin{pmatrix}x_2(t)\\-x_1(t)\\0\end{pmatrix} + \begin{pmatrix}0\\x_3(t)\\-x_2(t)\end{pmatrix}u(t) \tag{21}$$

If we calculate the product $x^T(t)\,\frac{\mathrm{d}}{\mathrm{d}t}x(t)$, we quickly see that it equals zero for any input $u(t)$ and, therefore, the integral $\frac{1}{2}\,x^T\,x$ is a constant of motion. In other words, once started at a fixed point x_0 in state space, the dynamics of this system continue to evolve on the sphere $x_0^T\,x_0 = R^2$, with R the radius of the sphere. Since the motion is constrained, we expect a non-controllable system for the simple reason that, once started at x_0, the system cannot reach *all* points in three dimensional space ($\mathbb{R}^3$).

The dual sensitivity system (14)-(15) for this example is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x(t)}{\partial\theta}\right) = -\begin{pmatrix}0 & -1 & 0\\1 & 0 & -u(t)\\0 & u(t) & 0\end{pmatrix}\frac{\partial x(t)}{\partial\theta} \tag{22}$$

$$y_\theta(t) = (0\ \ x_3(t)\ \ -x_2(t))\,\frac{\partial x(t)}{\partial\theta} \tag{23}$$

with $\frac{\partial x(t)}{\partial\theta}$ a (3×3) matrix that contains the sensitivities of the state variables with respect to the parametrised final conditions. Note that for controllability the original system, together with its dual sensitivity system, is integrated *backwards in time*. Since there are three state variables in this example, we have three terminal conditions for the parametric sensitivities that correspond to these terminal conditions, so that

$$\frac{\partial x(T)}{\partial\theta} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \tag{24}$$

In figure 1 the three singular values, corresponding to the three terminal condition parameters are summarized. As expected, a clear *gap* is observable indicating a lack of controllability. In addition, the bottom graph shows that all three state variables are involved in the nullspace of the sensitivity matrix, indicating that there is a structural relationship between the three state variables that causes the system to be non-controllable.

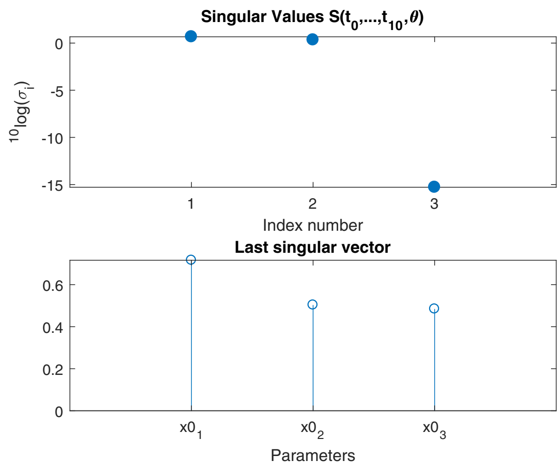

Fig. 1. Singular values and corresponding nullspace for the sensitivity matrix in example 1.

3.2 Example – cyclopentenol synthesis

In (Doyle III, 1997) the following reaction scheme, that represents the synthesis of cyclopentenol, is analysed for controllability:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C \tag{25}$$

$$2\,A \xrightarrow{k_3} D \tag{26}$$

with A cyclopentadien, B cyclopentenol, C cyclopentandiol, and D dicyclopentadien, and $k_i, i = 1,2,3$, the associated reaction constants. Applying a mass-balance over the reactor for species A and B and denoting their concentrations as $x_1(t)$ and $x_2(t)$, respectively, we can derive the following non-linear state space model:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -k_1\, x_1(t) - k_3\, x_1^2(t) + u(t)\,(x_{10} - x_1(t)) \tag{27}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = k_1\, x_1(t) - k_2\, x_2(t) - u(t)\, x_2(t) \tag{28}$$

with $u(t)$ the controllable dilution rate and x_{10} the inlet concentration of cyclopentadien. Figure 2 shows the singular values and last singular vector in one graph. Clearly, there is no detection of a zero in the singular values, indicating that the system is controllable. Symbolic computations of the Lie algebra confirm that, indeed, the controllability matrix is full rank.

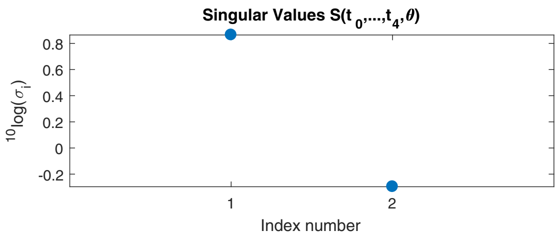

Fig. 2. Singular values for the sensitivity matrix in example 2. No zeros are detected, implying that a sufficient condition for controllability is satisfied.

3.3 Example – the parking problem

A by now classical example in non-linear control theory is the parking problem of a vehicle, see e.g. (Kwatny and Blankenship, 2000), whose simplified state dynamics are given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & 0 \\ \sin(\theta+\phi) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \tag{29}$$

with (x, y) the location of the centre of mass in a two-dimensional frame of reference, ϕ the vehicle's orientation, and θ the steering angle. The inputs v and w are the 'drive' and 'steer' controls that allow, as it turns out, complete manoeuvrability of the vehicle. Linearisation of this model, however, yields a non-controllable system that cannot be steered to any point in state space. Yet, the vehicle *is* controllable and this is confirmed by the non-linear controllability matrix $\mathcal{C}(x_0)$, which is full rank. This is because the Lie-bracket between the control vector fields 'steer' and 'drive' yield a so-called 'wriggle' vector field and, in addition, the bracket between 'wriggle' and 'drive' yields the control direction 'slide' that completes the basis of vector fields that allow complete control of the vehicle (Kwatny and Blankenship, 2000).

To see whether the sensitivity based approach yields the same conclusion, the dual sensitivity system (14)-(15) was integrated and the controllability signature was calculated. Clearly, the dual sensitivity matrix is full rank for arbitrary values of the terminal condition parameters, meaning that the system is locally controllable.

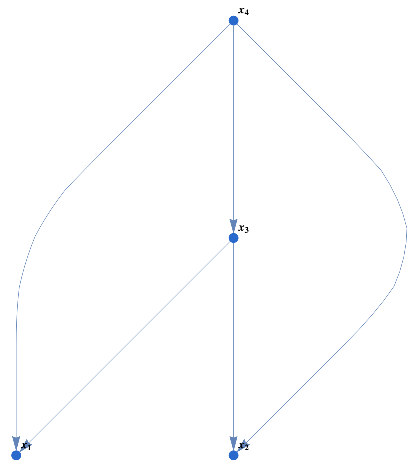

Fig. 3. Directed graph for the parking problem example that shows the information flow between the four state variables.

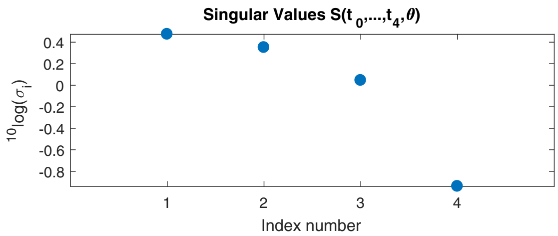

Fig. 4. Singular values for example 3, the parking problem. Since no zeros are detected, there is no non-trivial nullspace in this example. The system is controllable and no un-controllable modes are detected with the sensitivity based analysis.

3.4 Observability of the Chinese Hamster Ovary model

We now turn to an interesting case of an *observability* question for both state variables *and* time-independent parameters in a large model that was published as a benchmark case study for (symbolic) identifiability analysis (Villaverde et al., 2016). The model stems from a systems biology application and describes the dynamic behaviour of 34 metabolites in three compartments (fermenter, cytosol, and mitochondria) of a Chinese Hamster Ovary cell. Thirteen of these metabolites can be measured directly, corresponding to 13 measurable state variables, namely $(x_1, x_2, x_3, x_4, x_5, x_{11}, x_{13}, x_{15}, x_{21}, x_{27}, x_{29}, x_{30}, x_{32})^T$. Since there are 34 state variables, whose initial conditions are parametrised, we have 34 additional parameters on top of the 117 system parameters, totalling to a total of 151 parameters (!) whose values need to be determined from the sensor readings.

To appreciate the complexity of this question, a directed graph is presented in figure 5 that shows the connections between the 34 nodes (state variables) that interact with one another. Verifying identifiability with a symbolic algebra package (using, e.g., Sedoglavic' algorithm (Sedoglavic, 2002)) takes hours, if not days. Using our sensitivity based algorithm, we found that the speed of our calculations is orders of magnitudes faster in comparison with symbolic algorithms – within approximately 30 seconds the observability analysis is completed on basis of a compiled

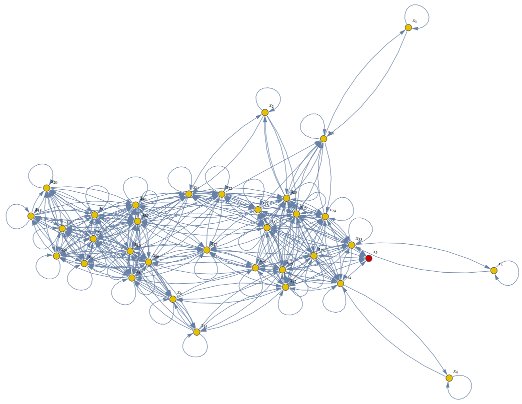

Fig. 5. Directed graph Chinese Hamster Ovary model with 34 state variables and 151 parameters

version of the CHO model that was run from the Systems Biology toolbox in Matlab. In addition, the accuracy on the calculated sensitivities was set to 10^{-20} which could easily be achieved on the basis of complex valued algebra for an accurate determination of the Jacobi matrices in the dynamic sensitivity equations. Furthermore, it was found that no matter what sensor set is used for identification of the 151 parameters in the model, there is *always* a lack of identifiability in this particular model that concerns two groups of two parameters each. Moreover, even if *all state variables* are directly measured, there still is a lack of identifiability for these four parameters.

To further visualize our results, we calculated the *observability signature* in figure 6 – a graph of the singular values of the sensitivity matrix, together with a graph of the entries of the last two singular vectors v_{150} and v_{151} in (7) that correspond to two zero singular values. From this graph it is immediately obvious that there is a lack of identifiability in the model. The bottom graph that visualizes all entries of the last two singular vectors, clearly shows that *four* parameters are involved in the two correlations with two corresponding singular values. These are the parameters p_{47}, p_{48} in one correlation (the red spikes in figure 6, bottom graph), and the parameters p_{55}, and p_{57} in the other correlation (the blue spikes in the same figure).

Interestingly enough, the above SVD-results can immediately be utilized in a complementary symbolic computation that validates our preliminary SVD findings. Hereto, we only need to realize that the SVD analysis clearly shows *which parameters* are involved in a correlation and so there is no need for a symbolic test for correlations between all 151 parameters in the model. This yields a substantial saving in computation time.

Since the four correlated parameters in the CHO model were already clearly shown in the above observability signature graph, we now continue with the calculation of a few Lie-derivatives for the output equation that includes all the previously mentioned measured state variables. We found the following Jacobi matrix, where columns 1–4 correspond to parameters p_{47}, p_{48}, p_{55}, and p_{57}, respectively:

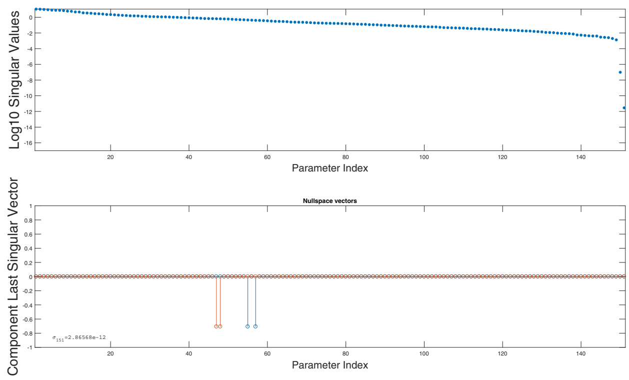

Fig. 6. Observability signature Chinese Hamster Ovary model. Two zero singular values are detected in the top graph corresponding to a gap of 4 decades on the logarithmic scale. These two zero singular values correspond to two unidentifiable combinations of two parameters each.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\log(x_{27}(0)) & \log(x_{27}(0)) \\ \log(x_{11}(0)) & -\log(x_{11}(0)) & \log(x_{27}(0)) & -\log(x_{27}(0)) \\ \log(x_{11}(0)) & -\log(x_{11}(0)) & 0 & 0 \\ -\log(x_{11}(0)) & \log(x_{11}(0)) & -\log(x_{27}(0)) & \log(x_{27}(0)) \\ -\frac{\log(x_{11}(0))}{2} & \frac{\log(x_{11}(0))}{2} & 0 & 0 \\ 0 & 0 & -\log(x_{27}(0)) & \log(x_{27}(0)) \\ 0 & 0 & \log(x_{27}(0)) & -\log(x_{27}(0)) \\ 0 & 0 & \log(x_{27}(0)) & -\log(x_{27}(0)) \\ \vdots & & & \vdots \end{pmatrix}$$

The null-space for the above matrix can easily be calculated as

$$\mathcal{N}(G(\theta)) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \tag{30}$$

This result clearly shows that there are indeed two separate groups of parameters in the observed correlations, namely p_{47} and p_{48} as the first group, and p_{55} and p_{57} as the second group. Put differently, we have validated the numerical results *symbolically* and have demonstrated a lack of identifiability for a model with 151 parameters in its structural definition.

4. CONCLUDING REMARKS

We have demonstrated that (dual) sensitivity equations can be used to reveal system structure in a simple way. Both controllability and observability properties can be calculated rapidly without the use of Lie-brackets or Lie-derivatives. The approach taken here provides an interesting perspective on local controllability and observability for non-linear systems that works for case studies that previously have been analysed in the literature on the basis of symbolic methods only. Of course, the results we present are work-in-progress, and a more solid footing of the controllability conjecture presented here must be given. We further mention that our method has been tested on numerous other case studies, not reported in this paper, as well, and in all cases the sensitivity based results confirmed the outcome of the symbolic calculations based on Lie-derivatives.

Further comparison and benchmarking of the methods presented here is, of course, an interesting research topic, especially so for the case of *non-regular or singular points* that can change observability and/or controllability properties of a given control system. This question relates to the work of Saccomani *et al.* on identifiability of parameters. In her work it has been clearly demonstrated that initial conditions in an identifiability analysis can change the identifiability outcome completely (Saccomani et al., 2003). Translated to the sensitivity based method we pursue here this means that the trajectory of our choice, and more specifically the *initial condition*, can make a crucial difference in a structural analysis of a non-linear control system.

Finally, of course, there are numerous examples that demonstrate the need for a rapid evaluation of observability and controllability properties for a given model. E.g. Krener and Ide (2009), demonstrate a measure of observability for a dynamic flow example that facilitates the choice for an optimal sensor location. Another example is in the field of aerospace engineering, where observability/controllability are, obviously, very important system properties that are in need for a rapid evaluation – see e.g. (Kaufman et al., 2016). In addition, the recent developments in Systems Biology show that there is a growing need for a rapid evaluation of structural properties for *large system models*. Our claim is that our sensitivity based algorithm allows us to analyse these properties efficiently and provides an interesting and alternative perspective on these fundamental system properties.

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