A UNIFIED APPROACH TO THE THEORY OF SAMPLING SYSTEMS*

BY

R. E. KALMAN[1] AND J. E. BERTRAM[2]

ABSTRACT

This paper presents a new method for the analysis of sampling (or sampled-data) systems. Unlike methods currently in use, it is not necessary to assume that the sampling operations are synchronous, performed at constant rate, and representable by means of an impulse modulator. Several different types of sampling operations are considered in detail, with the analysis proceeding similarly in each case. The paper concludes with a brief study of the stability of sampling systems and a generalization of Floquet's theorem.

INTRODUCTION

A dynamic system in which one or more variables are allowed to to change only at discrete instants of time (called *sampling instants*) is known as a *sampled-data* or more simply as a *sampling* system (**1**).[3] Sampling systems may be regarded as a subclass of nonstationary (time-varying) dynamic systems. Such systems arise naturally when a digital computer, a pulse or counting circuit, or any other discontinuously acting element is incorporated in a system.

Conventional Sampling

At the present time, most of the literature (**1–5**) dealing with the analysis and design of sampling systems is based on the following simplifying assumptions:

(*a*) All sampling operations can be adequately described by introducing a fictitious idealized component, called an *impulse modulator* (**2**) into the system.
(*b*) All sampling operations are performed synchronously.
(*c*) The time interval between successive samples is fixed.

In addition, it is almost always assumed that the system is linear.

A sampling system whose operation (within the accuracies demanded by engineering considerations) satisfies these simplifying assumptions will be referred to in this paper as a *conventional sampling system.*

The analysis and synthesis of conventional sampling systems have

* This work was done while both authors were in the Department of Electrical Engineering at Columbia University. The work was supported in part by the IBM Research Laboratories.

[1] RIAS, Baltimore, Md.

[2] IBM Research Laboratory, Ossining, N. Y.

[3] The boldface numbers in parentheses refer to the references appended to this paper.

been well developed and standardized in recent years with the aid of the z-transform method (1, 3–5).

Recently there has been increased interest in systems in which the sampling operation does not satisfy one or more of Assumptions (a), (b), (c). Examples of such systems are:

Nonsynchronous Sampling

All sampling operations are repeated at the same rate but occur at different instants of time. Such problems can be analyzed with the aid of the "modified" z-transform method (1, 4, 5) due to Barker. In all except the simplest cases, however, the algebraic details of the analysis become exceedingly laborious.

Multiple-Order Sampling

All sampling operations in the system are performed synchronously and a group of sampling operations is repeated every T seconds, the intervals between successive samples in the group being unequal. For example, *m^{th}-order sampling* means that once very T seconds samples are taken at times

$$t = kT, \quad kT + \tau_1, \cdots, \quad kT + \tau_{m-1}, \quad (k+1)T, \cdots \quad (k = 0, 1, \cdots)$$

where the τ_i are arbitrary constants subject to the condition $0 < \tau_1 < \tau_2 < \cdots < \tau_{m-1} < T$. Such sampling operations may arise when a digital computer is time-shared by several feedback control systems. Systems with multiple-order sampling have not been studied previously as far as the authors are aware. (Added in proof: See 20, 21.)

Multi-Rate Sampling

There may be several sampling operations in the system with fixed but unequal sampling intervals. Kranc (6) has made an extensive study of multi-rate systems in which the ratios of the various sampling periods are rational numbers. His analysis is based on a special transform method.

Noninstantaneous Sampling

Here the sampling interval is too long with respect to the time-scale of the dynamic components of the system to be adequately described by impulse modulation. An approximate solution to this problem has been obtained by Kranc (7), using transform methods. An exact solution has been published by Farmanfarma (8). See also the writers' discussion of Farmanfarma's paper.

Random Sampling

In some cases, the intervals between successive samples may be thought of as being selected according to some random scheme. Approximately random sampling may be caused by the desire to reduce susceptibility of the system to jamming or similar interference; by random time delays in the sampling operation depending on a physical measurement; by a random time delay which must elapse before a digital computer again becomes available to carry out control computations required by a particular feedback loop of a multi-loop system; and finally by inaccuracies in equipment which is designed to sample at constant intervals.

A comprehensive study of systems with random sampling has been carried out by one of the writers using mainly the methods of this paper; these results are discussed elsewhere (**18**). (Added in proof: See **22**.)

Evidently, all five types of sampling mentioned above may be encountered in a single system.

The New Method

The objective of this paper is to present a uniform method of analysis which can be applied in a straightforward fashion to sampling systems of the types mentioned. The same method has been used recently by the writers in connection with:

(i) design of conventional sampling systems for "deadbeat" response (**9**);
(ii) design of an optimal relay servo operating with conventional sampling (**10**);
(iii) design of conventional sampling systems to be optimal in the Wiener sense (**11**);
(iv) systems with random sampling (**18**);
(v) analysis of effect of round-off errors on the dynamic performance of sampling systems incorporating a digital element (**12**);
(vi) analysis of conventional sampling systems; and
(vii) design of Wiener filters operating on sampled data (**19**).

Transform theory as used in the past to study conventional sampling systems has proved to be either unwieldy or has required elaborate modifications before it could be applied to systems of the type mentioned. By using the concepts of *state* and *transition matrix* (see the next two sections) it is possible to handle systems of the general type in a clear and uniform way. The new method yields simplifications even in the analysis and synthesis of conventional sampling systems. The method automatically eliminates one of the chief difficulties of the transform method, namely that with the latter it is difficult or cumbersome to obtain information about the behavior of the system at any time other than the sampling instants. The method is in accordance with modern trends in engineering science; it devotes paper-and-pencil studies only to the analytic aspects of system problems, leaving the drudgery of numerical computation to be performed by a digital computer.

Finally, the formulation of system equations according to the point of view of this paper is a natural and necessary step preliminary to the study of nonlinear sampling systems.

ELEMENTS OF LINEAR SAMPLING SYSTEMS

Because of the generality of the method of analysis to be presented, careful attention must be paid to the definition of the individual elements of the system. The elements of a sampling system to which the method applies are of the following type:

Dynamic Elements

These are elements whose present output depends not only on the present value of the input but also on past values of the input. We will be concerned with two particular classes of dynamic elements:

Linear Continuous Dynamic Elements. These are assumed to be described by ordinary linear differential equations with constant coefficients. They represent, in an idealized fashion, such diverse physical entities as motors, ships, airplanes, missiles, chemical reactors, electrical networks, etc.

Linear Discrete Dynamic Elements. These are governed by linear difference equations with constant coefficients. Such elements are peculiar to sampling systems. They are an idealization of linear numerical computations performed by digital computers incorporated in the system. (Networks of delay lines used, for instance, to process radar data can be interpreted similarly.) A digital computer cannot operate directly on a continuous signal, but only on sequences of numbers, that is, on a *discrete signal.* Thus discrete dynamic elements inherently involve the operation of *sampling*, namely the conversion of continuous signals into discrete signals.

Sample-and-Hold Elements

These represent, again in an idealized fashion, the operation of "sampling" various types of information (measuring a radar return pulse, sampling the values of any continuous signal, etc.) and then applying the sampled information through a "smoothing" filter to the input of some continuous dynamic element. A sample-and-hold element is necessary whenever the output of a discrete dynamic element is to be applied to the input of a continuous dynamic element, since the output of a discrete dynamic element can only be observed through the process of sampling it (that is, instructing the digital computer to "write" out numbers at specific instants of time) and the input of a continuous dynamic element cannot be a discrete signal. Sometimes a sample-and-hold element is inserted in the system because, paradoxically, sampling may improve the dynamic performance of the system (1, 9).

The simplest "smoothing" operation consists simply of holding the last sampled value until the next sample is available. If the input to the sample-and-hold element is $u(t)$, its output $v(t)$ is given by

$$v(t_k + \tau) = u(t_k), \quad 0 < \tau \leqq t_{k+1} - t_k \tag{2}$$

where t_k ($k = 0, 1, 2, \cdots$) is the k^{th} *sampling instant* and τ is the time elapsed since the last sample was taken.

The manner in which the sampling instants are to be selected will be discussed in detail in the next section.

Imperfect Hold. In some cases, it is difficult to maintain the output of the sample-and-hold element exactly at the sampled value as required by Eq. 2. Instead of (2), the input-output relations of the sample-and-hold element may take the form

$$v(t_k + \tau) = (\exp \tau/T_s)u(t_k), \quad 0 < \tau \leqq t_{k+1} - t_k \tag{2-A}$$

where T_s is the time-constant of the sample-and-hold element. Of course T_s should be as large as possible compared to the average time interval between successive samples.

The modification represented by (2-A) can be easily incorporated in the general analysis which follows and therefore need not be discussed further.

Noninstantaneous Sampling. In this case, the sample-and-hold element transmits the input signal without change for U seconds following t_k; thereafter, the output is held constant at the last sampled value of the input. Therefore in this case we have

$$\begin{aligned} v(t_k + \tau) &= u(t), & 0 < \tau \leqq U \\ &= u(t_k + U), & U \leqq \tau \leqq t_{k+1} - t_k. \end{aligned} \tag{2-B}$$

Better Smoothing. Sometimes the smoothing provided by the sample-and-hold element is inadequate. In such cases, additional smoothing may be provided by cascading a linear dynamic element (say, n cascaded integrators) after the sample-and-hold element. This modification obviously requires no special consideration.

Interconnecting Elements

These represent idealized operations on the outputs of the dynamic elements, such as multiplication by a constant, differentiation, as well as linear combinations of these operations. Differentiation can occur only in connection with the output of continuous dynamic elements. See in this connection, *Remark* (a, ii), p. 413.

The preceding classification of elements is valid also in the case of nonlinear sampling systems. The latter are characterized by the fact that the interconnecting elements have nonlinear input-output relations; in other words, the principle of superposition is not valid.

To illustrate the manner in which the above elements are interconnected, consider the hypothetical feedback system shown in Fig. 1. The system consists of two continuous dynamic elements (*CDE*), one discrete dynamic element (*DDE*) and two sample-and-hold elements (*SHE*). The boxes with the symbols K, d/dt, and Σ denote interconnecting elements corresponding to multiplication by a constant, differentiation, and summation. The rest of the interconnections are indicated simply by the lines connecting the various boxes. The quantities x_7 (the output of the discrete dynamic element) and x_8, x_9 are discrete signals. All other signals are continuous. The *system input* is denoted by $r(t)$ and the *system output* by $x_1(t)$.

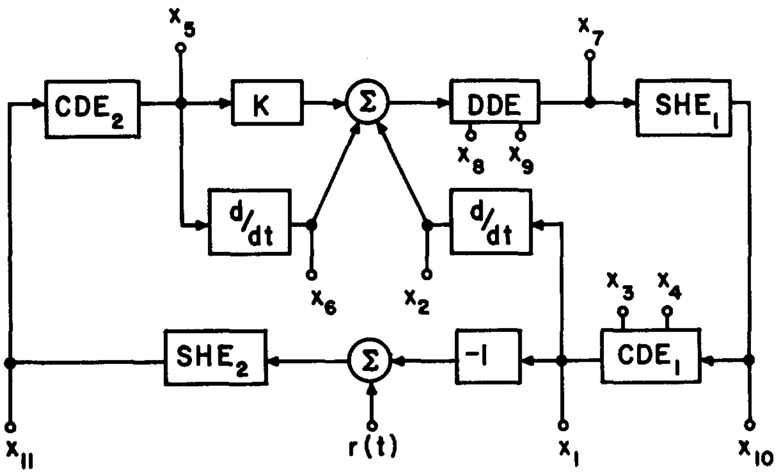

FIG. 1. A typical sampling system.

DESCRIPTION OF THE SAMPLING OPERATION

As mentioned in the preceding section, sampling operations occur in discrete dynamic elements as well as in sample-and-hold elements. Let us consider a single sampling operation. The k^{th} *sampling instant* is denoted by t_k $(k = 0, 1, \cdots)$. The interval between successive samples

$$T_k = t_{k+1} - t_k \tag{3}$$

is called the k^{th} *sampling period*. The various sampling operations may be characterized as follows:

Conventional Sampling. $T_k =$ const. for all k.

Nonsynchronous Sampling. If t_k, t_k' denote the instants at which two different sampling operations occur, then $t_k' = t_k + U$ for all k, where U is some positive constant.

Multiple-Order Sampling. The sampling period is a periodic function of k; in other words, $T_k = T_{k+q}$ where q is a positive integer.

Multi-Rate Sampling. If T_k, T_k' denote the sampling periods of two different sampling operations, T_k = const. and T_k' = const. for all k, but $T_k \neq T_k'$.

Noninstantaneous Sampling. This can occur only in connection with the sampling operations of the sample-and-hold elements, whose input-output relations in this case are given by (2-*B*).

Random Sampling. Both t_k and T_k are random variables.

The various types of sampling operations are depicted schematically in Fig. 2.

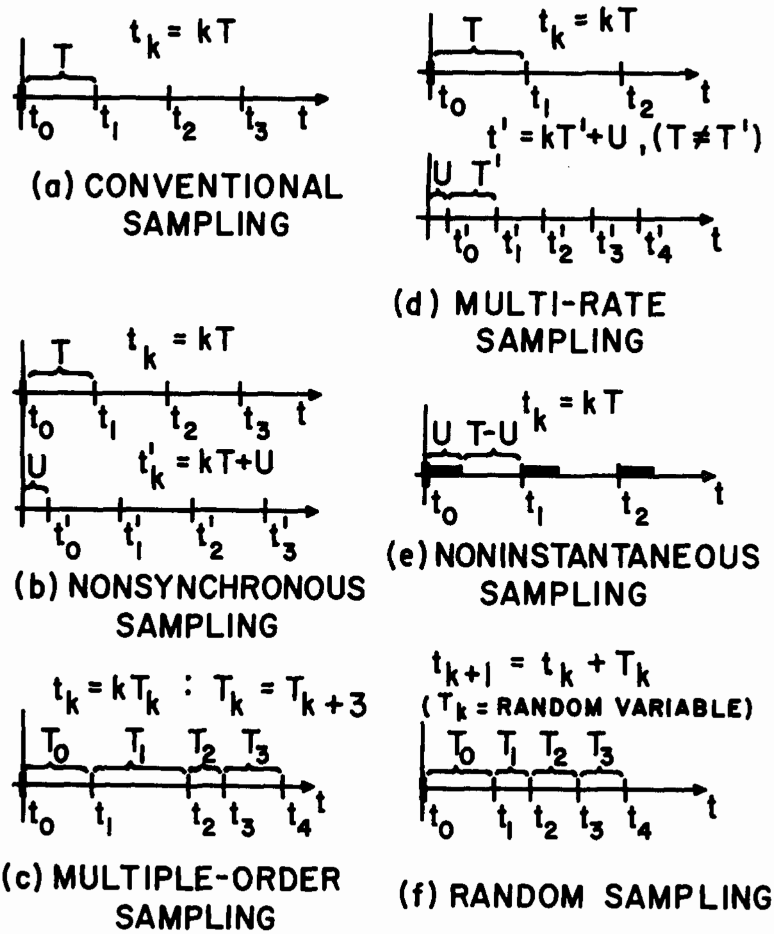

FIG. 2. The sampling operation.

Notation Convention. The following convention is used to denote the change in various quantities as a result of the sampling operations:

If $x(t)$ is some quantity whose value changes due to a sampling operation, and if t_k is the sampling instant, then we denote by $x(t_k)$ the value of $x(t)$ just before and by $x(t_k^+)$ the value of $x(t)$ just after the sampling operation. In other words, the sampling operation is thought of as taking place in an interval $t_k < t < t_k + \epsilon$, where ϵ is an arbitrarily small positive number. Note that this convention is in strict accordance with the inequalities used in (2), (2-A), and (2-B).

MATHEMATICAL DESCRIPTION OF DYNAMIC ELEMENTS

The mathematical description and analytic study of a dynamic system is greatly facilitated by focusing attention on the concept of *state*.

> *Intuitive Definition*: The *state* of a dynamic element is a set of numbers (called *state variables*) which contain as much information regarding the past history of the element as is required for the calculation of the entire future behavior of the element.

The evolution of a dynamic system through time may be visualized as a succession of *state transitions*. When the system is linear, these transitions are linear transformations of the state. The restriction to linearity greatly simplifies the analysis.

The state variables of a sampling system fall into three categories:

(i) The state variables of the continuous dynamic elements.
(ii) The state variables of the discrete dynamic elements.
(iii) The state variables of the sample-and-hold elements.

In the last two cases, the state transitions may be thought of as taking place at the sampling instants only; in the first case, the state transitions occur continuously.

We now derive the equations governing state transitions. These derivations are quite elementary; however, they require a certain amount of careful "bookkeeping." In the next section, we show how the transition equations of the entire system are obtained by combining the transition equations of the various dynamic elements. The manner in which the equations are combined is determined by (a) the interconnections of the various elements in the system, (b) the various sampling operations in the system. This completes the analysis; any information regarding the behavior of the system can be obtained from the transition equations.

State of Continuous Dynamic Elements

Since these elements are assumed to be described by ordinary differential equations (linearity is immaterial for the moment), they may be simulated by means of an analog computer. This simulation, on paper or in reality, may be done in a variety of ways; in each case, exactly γ integrators will be required, γ being the sum of the orders of the differential equations governing the various continuous dynamic elements. The outputs of the γ integrators at time t are denoted by the γ-vector ($\gamma \times 1$ matrix),

$$\mathbf{x}^c(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_\gamma(t) \end{bmatrix}. \tag{4}$$

In the absence of inputs, the signals generated in the analog computer after $t = t_0$ depend only on the voltages stored on the integrators at $t = t_0$ (that is, on $\mathbf{x}^c(t_0)$); thus $\mathbf{x}^c(t)$ is clearly a correct definition of the state of the continuous dynamic elements (13).

In Fig. 1, the state variables of the first continuous dynamic element are denoted by $x_1, \cdots, x_4$; those of the second by x_5, x_6. Thus $\gamma = 6$.

Remark a. It can be easily shown from the elementary theory of linear differential equations that

(i) Any output of a continuous dynamic element is a linear function of the state variables. In particular, the output may always be chosen as one of the state variables, which is what was done in Fig. 1.

(ii) Only those derivatives of outputs of continuous dynamic elements are "admissible" which can be expressed as a linear combination of the state variables of and the inputs to the dynamic element. Thus if v (output) and u (input) are related by

$$\frac{d^p v}{dtp} + b_{p-1}\frac{d^{p-1}v}{dt^{p-1}}, + \cdots + b_0 v = a_q \frac{d^q u}{dt^q} + \cdots + a_0 u$$

then it follows easily that $dv/dt, \cdots, d^{p-q}v/dt^{p-q}$ are admissible derivatives of v.

State of Discrete Dynamic Elements

As mentioned, these elements can be thought of as idealizations of numerical computations performed by digital computers incorporated in the system. Excluding programming instructions, fixed parameters, etc., suppose that δ numbers must be stored in the computer at time t in order to be able to perform a complete cycle of computations. These δ numbers are evidently the state variables of the discrete dynamic elements. They are denoted by the δ-vector ($\delta \times 1$ matrix)

$$\mathbf{x}^d(t) = \begin{bmatrix} x_{\gamma+1}(t) \\ x_{\gamma+2}(t) \\ \vdots \\ x_{\gamma+\delta}(t) \end{bmatrix}. \tag{5}$$

In Fig. 1, the state variables of the discrete dynamic element are denoted by x_7, x_8, x_9. Thus $\delta = 3$.

State of Sample-and-Hold Elements

From (2) we see that the state of a sample-and-hold element is identical with its output. Assuming there are σ sample-and-hold elements in a given system, we denote these state variables by the σ-vector

($\sigma \times 1$ matrix)

$$\mathbf{x}^s(t) = \begin{bmatrix} x_{\gamma+\delta+1}(t) \\ x_{\gamma+\delta+2}(t) \\ \vdots \\ x_{\gamma+\delta+\sigma}(t) \end{bmatrix} \tag{6}$$

which represents the state of the entire sampling system.

In Fig. 1, the state variables of the sample-and-hold elements are x_{10}, x_{11}. Thus $\sigma = 2$.

State of Entire System

By combining the state variables defined by (4), (5), and (6), we obtain the n-vector ($n \times 1$ matrix)

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}^c(t) \\ \mathbf{x}^d(t) \\ \mathbf{x}^s(t) \end{bmatrix}$$

which represents the state of the entire sampling system. The integer $n = \gamma + \delta + \sigma$ is called the *order* of the sampling system.

Transition Equations of Sample-and-Hold Elements

The state transitions take place (discontinuously) only at sampling points (see (2)):

$$x_i(t_k^+) = \sum_{j=1}^{\gamma+\delta} s_{ij}x_j(t_k) + s_{i,n+1}r(t_k), \quad (i = \gamma + \delta + 1, \cdots, n) \tag{7-i}$$

where the s_{ij} are constants; t_k refers to the sampling instant of the i^{th} sample-and-hold element (which may or may not coincide with the sampling instants of the other elements). Equations 7 express the fact that, in general, the input to a sample-and-hold element is a linear combination of the state variables of the continuous and discrete dynamic elements as well as the system input $r(t)$.

Transition Equations of Discrete Dynamic Elements

Here

$$x_i(t_k + \tau) = \sum_{j=1}^{\gamma+\delta} d_{ij}x_j(t_k) + d_{i,n+1}r(t_k),$$

$$\tau \geqq \tau_i \; (i = \gamma + 1, \cdots, \gamma + \delta) \tag{8-i}$$

where the d_{ij} are constants and τ_i is the time required for the computations indicated by (8) to be completed by a digital computer; t_k is the instant of time at which the variable quantities entering on the right-hand side of (8) have been sampled (t_k may be the same for several of the Eqs. 8-i).

If the results of the computations indicated by (8-i) are not to be called for (as must be the case in a well-designed system) until more than τ_i seconds after the sampling instant, t_k, then we may just as well assume that the state transitions take place instantaneously at t_k, in other words, the left-hand side of (8-i) may be denoted by $x_i(t_k^+)$ (see "Notation Convention," p. 411). Often the computing time τ_i is very short compared to the system dynamics and may be disregarded altogether.

Transition Equations of Continuous Dynamic Elements

Let us define, in conjunction with the analog computer setup introduced in a previous section (see p. 412), the constants

a_{ij} = coefficients with which the output of the j^{th} integrator is fed back to the input of the i^{th} integrator. (See Fig. 3.)

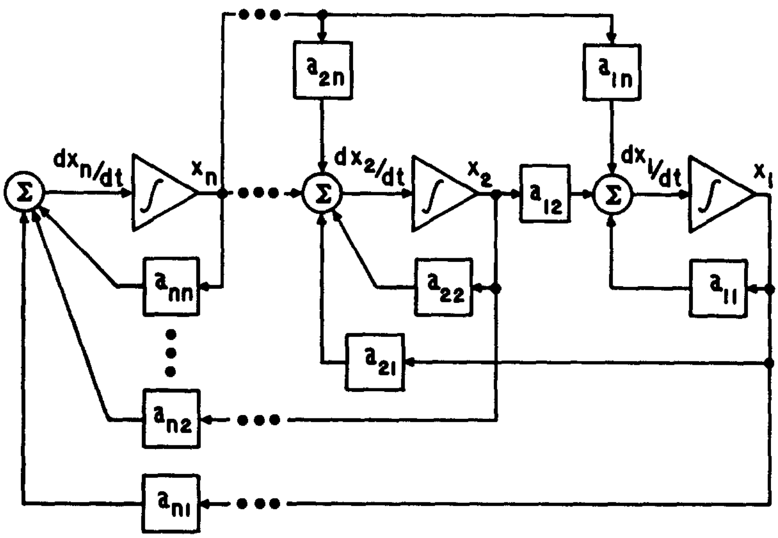

FIG. 3. An analog computer simulation.

In the absence of any external inputs to the continuous dynamic elements (13) we have by inspection of Fig. 3,

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{\gamma} a_{ij}x_j(t), \quad (i = 1, \cdots, \gamma). \tag{9-i}$$

This equation is a quantitative description of the state transition $x_i(t) \rightarrow x_i(t) + dx_i(t)$ due to the infinitesimal time change $t \rightarrow t + dt$. To calculate the state transition for a finite change in time, it is necessary to solve the system of differential equations (9). We first cast (9) into the standard vector-matrix notation:

$$\frac{d\mathbf{x}^c(t)}{dt} = A\mathbf{x}^c(t) \tag{10}$$

where $\mathbf{x}^c(t)$ is a vector defined by (4) and A is a $(\gamma \times \gamma)$ constant matrix whose elements are defined above.

It is well known (14) that solutions of (10) starting at any time t_0 and observed at any time t are given by

$$\mathbf{x}^c(t) = (\exp\,(t - t_0)A)\mathbf{x}^c(t_0) = \Phi^{cc}(t - t_0)\mathbf{x}^c(t_0) \tag{11}$$

where

$$\exp tA = \sum_{k=0}^{\infty} t^k A^k/k! \quad (A^0 = \text{unit matrix}). \tag{12}$$

The infinite series (12) converges for all finite values of t. $\Phi^{cc}(t)$ is called the transition matrix (11) of the continuous dynamic element.

The form of solution given by (12) is quite convenient if t is some fixed constant (as will frequently be the case in the examples) because it does not require finding the roots of the characteristic equation of the matrix A. On the other hand, it may be inefficient to use (12) for finding $\Phi^{cc}(t)$ for a large number of values of t.

A convenient way of obtaining a closed expression for $\Phi^{cc}(t)$ is the following. Let

$g_{ij}(s)$ = transfer function from input of j^{th} integrator to output of i^{th} integrator in the analog-computer setup of Fig. 3.

Then

$$\Phi^{cc}(t) = \mathfrak{L}^{-1}\begin{bmatrix} g_{11}(s) & \cdots & g_{1\gamma}(s) \\ \vdots & & \vdots \\ g_{\gamma 1}(s) & \cdots & g_{\gamma\gamma}(s) \end{bmatrix} = \mathfrak{L}^{-1}[G(s)] \tag{13}$$

where the inverse Laplace transformation $\mathfrak{L}^{-1}$ is to be applied separately to each element of the matrix. The calculation of transfer functions and their inverse Laplace transforms can be carried out by well known methods of linear system engineering. In the examples which follow, calculation of $\Phi^{cc}(t)$ by means of (13) will therefore not require additional explanation.

The following important properties of $\Phi^{cc}(t)$ should be noted:

$$\Phi^{cc}(t_1 + t_2) = \Phi^{cc}(t_1)\Phi^{cc}(t_2) \quad \text{for all } t_1, t_2. \tag{14}$$

This follows directly from (12). Therefore, in particular, we have

$$\Phi^{cc}(0) = \text{unit matrix}; \quad (\Phi^{cc}(t))^{-1} = \Phi^{cc}(-t). \tag{15}$$

To complete the derivation of the transition equations for the continuous dynamic elements, consider now the inputs to these elements. As mentioned previously, the external inputs to these elements are applied through sample-and-hold elements. (There may be inputs from one continuous dynamic element to another but these are implicitly taken care of in the definition of the A matrix in (9).)

When the sampling operations are nonsynchronous, multiple-order, multi-rate or random, the output of the sample-and-hold element is a constant between successive sampling instants. Therefore the inputs to the continuous dynamic elements will be a series of steps.

Let us replace, for a moment, the j^{th} sample-and-hold element by means of a generator whose output at time $t = t_k$ is a unit impulse. The effect of the impulse is to change instantaneously the state of the continuous dynamic elements from $\mathbf{x}^c(t_k)$ to $\mathbf{x}^c(t_k{}^+) = \mathbf{x}^c(t_k) + \Delta\mathbf{x}^c(t_k)$. Let us denote the change by the γ-vector ($\gamma \times 1$ matrix)

$$\mathbf{f}^j = \Delta\mathbf{x}^c(t_\kappa).$$

In view of the linearity, the effect of the unit impulse on the succeeding time-variation of the state variables is expressed by

$$\mathbf{x}^c(t) = (\exp\,(t - t_k)A)\mathbf{f}^j, \quad t > t_k.$$

If instead of applying a unit impulse, the generator in the place of the j^{th} sample-and-hold element applies a step of magnitude $x_j(t_k{}^+)$, we get, again using linearity,

$$\begin{aligned} \Delta\mathbf{x}^c(t) &= \int_{t_k}^{t} x_j(t_k{}^+)(\exp\,(t - u)A)\mathbf{f}^j du \quad t > t_k \\ &= \int_{0}^{t-t_k} x_j(t_k{}^+)(\exp uA)\mathbf{f}^j du = x_j(t_k{}^+)\mathbf{h}^j(t), \quad t > t_k. \end{aligned} \tag{16}$$

Using (12) and (13), the following explicit formulas are obtained for the vector $\mathbf{h}^j(t)$ defined by (16):

$$\mathbf{h}^j(t) = \sum_{k=0}^{\infty} (A^k t^{k+1}/(k+1)!)\mathbf{f}^j \tag{17}$$

$$= \mathfrak{L}^{-1}\left[\frac{G(s)\mathbf{f}^j}{s}\right]. \tag{18}$$

We can now write down the complete transition equations for continuous dynamic elements. Let t_k be the time at which any one of the sample-and-hold elements changes it state. Let t_{k+1} be the instant at which the next change of state of any one of the sample-and-hold element occurs. Then

$$x_i(t_k + \tau) = \sum_{j=1}^{\gamma} \phi_{ij}^{cc}(\tau)x_j(t_k) + \sum_{j=\gamma+\delta+1}^{n} h_i{}^j(\tau)x_j(t_k{}^+),$$

$$0 < \tau \leqq t_{k+1} - t_k \quad (i = 1, 2, \cdots, \gamma) \tag{19-i}$$

It remains to consider sample-and-hold elements in which the sampling operation is of the noninstantaneous type described on p. 406.

The output of such elements after $t_k + U$ (see (2-B)) is constant, so that the previous analysis applies. During the interval $t_k < t \leqq t_k + U$, the output of the sample-and-hold element is equal to the input. The input to the sample-and-hold element must consist of combinations of state variables of continuous dynamic elements and the system input $r(t)$; no state variables of discrete dynamic elements can appear.

Let the $x_n(t)$ be the state variable of the *only* sample-and-hold element of the noninstantaneous type described on p. 409. By (7), and bearing in mind the above remarks, we have

$$x_n(t) = s_{n1}x_1(t) + s_{n2}x_2(t) + \cdots + s_{n\gamma}x_\gamma(t) + s_{n,\,n+1}r(t) \qquad (7\text{-}n)$$

during the time interval $t_k < t \leqq t_k + U$. We now form a matrix $\bar{A}$ by defining its elements as

$$\bar{a}_{ij} = a_{ij} + f_i{}^n s_{nj}.$$

Similarly, we define a vector $\mathbf{f} = s_{n,\,n+1}\mathbf{f}^n$. Then the transition equations in the interval $t_k < t \leqq t_{k+1} + U$ are obtained by solving the differential equation

$$d\mathbf{x}^c/dt = \bar{A}\mathbf{x}^c + \mathbf{f}r(t) \qquad (20)$$

which yields

$$\mathbf{x}^c(t_k + \tau) = (\exp t\bar{A})\mathbf{x}^c(t_k) + \int_0^\tau r(u)(\exp(\tau - u)\bar{A})\mathbf{f}du, \qquad 0 < \tau \leqq U. \qquad (21)$$

An analogous procedure may be employed when there is more than one noninstantaneous sampling element. The tedious, but elementary, details of setting up equations in those cases are left to the interested reader.

SYSTEM TRANSITION EQUATIONS

The system transition equations can now be written down very simply. The state transitions of each one of the three types of elements are represented by a matrix acting on the combined state variables. By multiplying together the various transition matrices, we obtain the transition matrix for the system for any two *successive* values of time.

We assume first that $r(t) = 0$.

Sample-and-Hold Elements

Let S_i ($i = \gamma + \delta + 1, \cdots, n$) be the matrix obtained by replacing the i^{th} row of the $n \times n$ unit matrix by $s_{i1}, s_{i2}, \cdots, s_{i,\,\gamma+\delta}, 0, \cdots, 0$. (See (7-$i$).)

$$
S_i = \left[\begin{array}{c:c:c}
1 & 0 & 0 \\ \hdashline
0 & 1 & 0 \\ \hdashline
0 \cdots 0 & 0 \quad \cdot \quad \cdot \quad \cdot \quad 0 & 1 \\
 & & \ddots \\
 & & \quad 1 \\
s_{i1} \cdots s_{i\gamma} & s_{i,\gamma+1} \cdots s_{i,\gamma+\delta} & 0 \cdots 0 \cdots 0 \\
 & & \quad 1 \\
 & & \qquad \ddots \\
0 \cdots 0 & 0 \quad \cdot \quad \cdot \quad \cdot \quad 0 & \qquad 1
\end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow i^{\text{th}} \text{ row} \\ \\ \\ \\ \end{array} \tag{22-i}
$$

If the state $\mathbf{x}(t_k)$ of the over-all system is multiplied by the $n \times n$ matrix S_i, then all states remain the same except the i^{th} which changes in accordance with (7-i). Note also the relations

$$S_iS_j = S_jS_i \qquad \text{and} \qquad S_iS_i = S_i.$$

Discrete Dynamic Elements

Let $D_{ij\cdots}(i, j, \cdots = \gamma + 1, \cdots, \gamma + \delta)$ be the matrix obtained by replacing the i^{th} row of the $n \times n$ unit matrix by $d_{i1}, \cdots, d_{i,\gamma+\delta}, 0, \cdots, 0$; replacing the j^{th} row by $d_j, \cdots, d_{j,\gamma+\delta}, 0, \cdots, 0$; etc. (See (8-$i$), 8-$j$)):

$$
D_{ij\cdots} = \left[\begin{array}{c:c:c}
1 & 0 & 0 \\ \hdashline
0 \cdots 0 & 1 \quad \cdot \quad \cdot \quad \cdot \quad 0 & \\
 & & \\
d_{i1} \cdots d_{i\gamma} & d_{i,\gamma+1} \cdots d_{i,\gamma+\delta} & 0 \\
 & & \\
d_{ji} \cdots d_{j\gamma} & d_{j,\gamma+1} \cdots d_{j,\gamma+\delta} & \\
 & & \\
0 \cdots 0 & 0 \quad \cdot \quad \cdot \quad \cdot \quad 1 & \\ \hdashline
0 & 0 & 1
\end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow i^{\text{th}} \text{ row} \\ \\ \leftarrow j^{\text{th}} \text{ row} \\ \\ \\ \\ \end{array} \tag{23-$ij\cdots$}
$$

If the state $\mathbf{x}(t_k)$ of the over-all system is multiplied by the $n \times n$ matrix $D_{ij\cdots}$ then all the states remain the same, except $x_i(t)$, $x_j(t)$, $\cdots$ which undergo a transition in accordance with (8-i), (8-j), $\cdots$, all at the *same* time $t_i{}^+ = t_j{}^+ = \cdots$.

Continuous Dynamic Elements

Let $\Phi(\tau)$ be the matrix defined by:

$$
\Phi(\tau) = \left[\begin{array}{c:c:c}
\Phi^{cc}(\tau) & 0 & \begin{matrix} h_1{}^{\gamma+\delta+1}(\tau) & \cdots & h_1{}^n(\tau) \\ \vdots & & \vdots \\ h_\gamma{}^{\gamma+\delta+1}(\tau) & \cdots & h_\gamma{}^n(\tau) \end{matrix} \\ \hdashline
0 & 1 & 0 \\ \hdashline
0 & 0 & 1
\end{array}\right] \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{first } \gamma \text{ rows} \\ \\ \\ \end{array} . \tag{24}
$$

If the state $\mathbf{x}(t)$ of the over-all system is multiplied by the $n \times n$ matrix $\Phi(\tau)$, then the states of the discrete dynamic elements and sample-and-hold elements remain the same and the states of the continuous dynamic elements undergo a continuous transition as a function of τ. The argument of $\Phi(\tau)$ may have any value in the interval $0 < \tau \leqq t_{k+1} - t_k$, where t_k is the last sampling instant and t_{k+1} is the next sampling instant.

The system transition matrix, for any interval of time, is formed by multiplying together the individual matrices S_i, D_{ij}... and $\Phi(\tau)$ as required by the various sampling operations taking place in the system.

Effect of System Input

We now remove the restriction $r(t) = 0$. For simplicity, we assume that $r(t)$ can affect the system only through a sample-and-hold element. This will exclude minor complications due to the convolution integral in (21). In accordance with the previous generalized notation, we now define the n-vectors

$$\mathbf{d}_i = \begin{bmatrix} 0 \\ \hline 0 \\ \vdots \\ d_{i,n+1} \\ \vdots \\ 0 \\ \hline 0 \end{bmatrix} \qquad \mathbf{s}_i = \begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \\ \vdots \\ s_{i,n+1} \\ \vdots \\ 0 \end{bmatrix} \tag{25}$$

which correspond to the coefficients of $r(t)$ on the right-hand sides of (7-i) and (8-i). These vectors show the effect of $r(t)$ on the state transitions.

Example 1. System Defined by Its Transition Equations

The over-all transition equations of a given system have the form:

$$\mathbf{x}(t_{k+1}) = \Phi(T_B)S_{10}\Phi(T_A)D_{789}S_{11}\mathbf{x}(t_k) + \Phi(T_B)S_{10}\Phi(T_A)(\mathbf{d}_7 + \mathbf{s}_{11})r(t_k). \tag{26}$$

The question is: What kind of a system is the one whose transition equation is given by (26)? The answer can be obtained by systematically examining the various terms in (26).

(a) There are eleven state variables. The first six state variables belong to continuous dynamic elements, the next three to discrete dynamic elements, the last two to sample-and-hold elements.

(*b*) Each sampling operation has the same period $T = T_A + T_B$; the sampling operations in connection with the discrete dynamic element and the second sample-and-hold element (whose output is x_{11}) occur simultaneously at t_k; the sampling operation in connection with the first sample-and-hold element (whose output is x_{10}) occurs at $t_k' = t_k + T_A$. Thus the sampling is of the nonsynchronous, fixed-period type.

(*c*) All information regarding the input $r(t)$ is obtained at t_k. The first sample-and-hold element samples only the state variables of the continuous and discrete dynamic elements. This may happen, for instance, in order to utilize the computations performed by the discrete dynamic element (which may be assumed to require less time than T_A for completion) before the next sampling point t_{k+1}.

It is easy to see that (26) may represent the operation of a system such as Fig. 1.

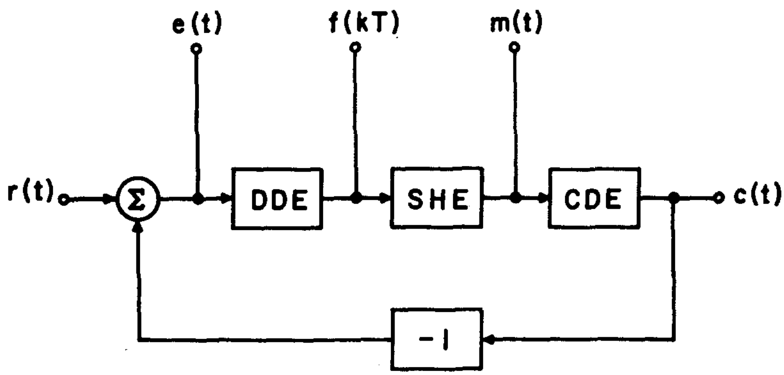

FIG. 4. Example 2.

SOME ILLUSTRATIVE EXAMPLES

To show how the formalism of the preceding sections is applied to the derivation of transition equations of systems containing various types of sampling operations, one example for each type of sampling will be given in this section.

In each example, the derivation of the transition equations requires essentially four steps:

Step (*i*). Selection of the state variables.

Step (*ii*). Calculation of the various matrices and vectors defined in the two previous sections.

Step (*iii*). Derivation of the transition equations for the various sampling operations and sample-free time-intervals.

Step (*iv*). Combination and simplification of the transition equations.

Example 2. Conventional Sampling System

A typical system of this class is shown in Fig. 4. The system contains a discrete dynamic element (for improving the stability of the system). The sampling operations in the discrete dynamic element and the sample-and-hold element are performed synchronously. (This is not always necessary or even desirable in practice. See Kranc (6).) Thus sampling occurs at the instants:

Discrete Dynamic Element: $0, T, \cdots, kT, \cdots$
Sample-and-Hold Element: $0, T, \cdots, kT, \cdots$

(i) The continuous dynamic element in Fig. 4 is described by the differential equation

$$\frac{d^2c}{dt^2} + \frac{dc}{dt} = m(t) \tag{27}$$

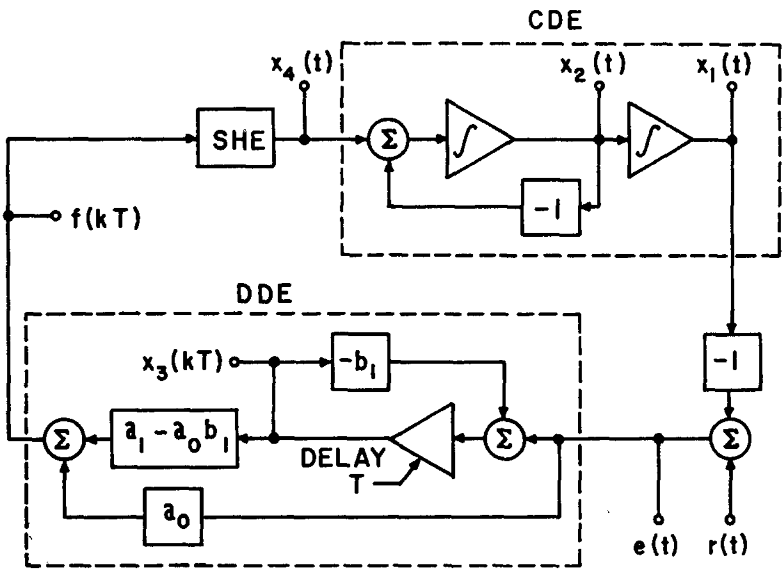

FIG. 5. Example 2 (details).

which relates the output $c(t)$ of the system to the control signal $m(t)$. The discrete dynamic element is described by the difference equation

$$f(kT) = a_0e(kT) + a_1e((k-1)T) - b_1f((k-1)T). \tag{28}$$

The input signal $e(t)$ to the discrete dynamic element is continuous. This signal is sampled and only its samples affect the state transitions in the discrete dynamic element. The output $f(kT)$ of the discrete dynamic element is a discrete signal, that is, it is available only at time kT.

The state variables for the continuous dynamic element are selected as follows:

$$\mathbf{x}^c(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c(t) \\ dc(t)/dt \end{bmatrix}. \tag{29}$$

With this choice of the state variables, the block diagram for the continuous dynamic element governed by (27) is shown in Fig. 5. This block diagram shows also how Eqs. 27 could be simulated by means of an analog computer.

To define the state variables for the discrete dynamic element, (28) is written in the form

$$f(kT) = a_0 e(kT) + (a_1 - a_0 b_1) y((k-1)T) \tag{30}$$

$$y(kT) = e(kT) - b_1 y((k-1)T). \tag{31}$$

By eliminating $y(kT)$ and $y((k-1)T)$ from (30) and (31), it follows easily that Eqs. 30 and 31 are equivalent to (28). From this, it is clear that the state of the discrete dynamic element at time kT is given by $y((k-1)T)$. Hence let

$$\mathbf{x}^d(kT) = [x_3(kT)] = [y((k-1)T)]. \tag{32}$$

Finally, the state variable of the sample-and-hold element is evidently given by:

$$\mathbf{x}^s(kT) = [x_4(kT)] = [m(kT)] = [f(kT)]. \tag{33}$$

(*ii*) The transition matrix for the continuous dynamic element is calculated according to the discussion given on p. 419; the end result is, using (24),

$$\Phi(\tau) = \left[\begin{array}{cc|c|c} 1 & 1 - e^{-\tau} & 0 & \tau - 1 + e^{-\tau} \\ 0 & e^{-\tau} & 0 & 1 - e^{-\tau} \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{34}$$

Using the fact that $e(t) = r(t) - x_1(t)$ and the notation of (32), we can write (30) and (31) in the form

$$x_3((k+1)T) = -x_1(kT) - b_1 x_3(kT) + r(kT)$$
$$f(kT) = (a_1 - a_0 b_1) x_3(kT) + a_0[r(kT) - x_1(kT)].$$

Hence, from (23) and (25)

$$D_3 = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 0 & -b_1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{35}$$

$$\mathbf{d}_3 = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \\ \hline 0 \end{array}\right]. \tag{36}$$

If Eq. 30 is changed similarly, using the notation of (33), we get

$$x_4(kT^+) = -a_0x_1(kT) + (a_1 - a_0b_1)x_3(kT) + a_0r(kT).$$

Hence, from (22) and (25),

$$S_4 = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline -a_0 & 0 & a_1 - a_0b_1 & 0 \end{array}\right]. \tag{37}$$

$$\mathbf{s}_4 = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ \hline a_0 \end{array}\right]. \tag{38}$$

(iii–iv) To obtain the transition equations for the over-all system, we must examine the sequence of the various transitions. Consider the time $t = kT$. First, the new value of the control signal $x_4(kT)$ is computed and we have

$$\mathbf{x}(kT^+) = S_4\mathbf{x}(kT) + \mathbf{s}_4r(kT).$$

Next, the data for the computation of the state transition of the discrete dynamic element are collected and the new state x_3 is calculated.

$$\begin{aligned} \mathbf{x}(kT^{++}) &= D_3\mathbf{x}(kT^+) + \mathbf{d}_3r(kT) \\ &= D_3S_4\mathbf{x}(kT) + (\mathbf{d}_3 + \mathbf{s}_4)r(kT). \end{aligned} \tag{39}$$

Actually, the new value of x_3 may not be available at kT^{++} but only somewhat later; this is immaterial, however, as far as the transition equations are concerned.

During the interval $kT < t \leqq (k+1)T$ the only transitions taking place are those of the state variables of the continuous dynamic element. Hence,

$$\mathbf{x}(kT+\tau) = \Phi(\tau)[D_3S_4\mathbf{x}(kT) + (\mathbf{d}_3 + \mathbf{s}_4)r(kT)], \quad 0 < \tau \leqq T \quad (40)$$

$$= \Psi(\tau)\mathbf{x}(kT) + \mathbf{v}(\tau)r(kT), \quad 0 < \tau \leqq T \quad (41)$$

where

$$\Psi(\tau) = \Phi(\tau)D_3S_4 =$$

$$= \left[\begin{array}{cc|c|c} 1-a_0(\tau-1+e^{-\tau}) & 1-e^{-\tau} & (\tau-1+e^{-\tau})(a_1-a_0b_1) & 0 \\ -a_0(1-e^{-\tau}) & e^{-\tau} & (1-e^{-\tau})(a_1-a_0b_1) & 0 \\ \hline -1 & 0 & -b_1 & 0 \\ \hline -a_0 & 0 & a_1-a_0b_1 & 0 \end{array}\right] \quad (42\text{-}A)$$

and

$$\mathbf{v}(\tau) = \Phi(\tau)(\mathbf{d}_3 + \mathbf{s}_4) = \left[\begin{array}{c} a_0(\tau - 1 + e^{-\tau}) \\ a_0(1-e^{-\tau}) \\ \hline 1 \\ \hline a_0 \end{array}\right] \quad (42\text{-}B)$$

Equation 41 describes the behavior of the system at any instant of time.

Remark b. By letting $\tau = T$ in (41), we have a *stationary* (constant-coefficient) difference equation

$$\mathbf{x}((k+1)T) = \Psi(T)\mathbf{x}(kT) + \mathbf{v}(T)r(kT) \quad (43)$$

relating the state variables at succeeding sampling instants. This is equivalent to the result usually obtained by means of the z-transform method.

Remark c. The last column of the matrix $\Psi(\tau)$ contains only zeros. This shows that the value of x_4 need not be known at the k^{th} sampling instant in order to find the state vector at the $(k+1)^{\text{st}}$ sampling instant. Thus x_4 may be dropped in Eqs. 41 or 43. If x_4 had been dropped before obtaining the final result, however, one could not have expressed $\Psi(\tau)$ as a simple product of matrices.

Remark d. In deriving (41) and (43), no real use was made of the fact that $T_k = T = \text{const.}$ Hence these equations are valid also when the sampling period is not a constant. In that case the transitions between successive sampling instants are governed by the *nonstationary* (varying-coefficient) difference equation

$$\mathbf{x}(t_{k+1}) = \Psi(T_k)\mathbf{x}(t_k) + \mathbf{v}(T_k)r(t_k) \tag{44}$$

where t_k denotes the k^{th} sampling instant and $T_k = t_{k+1} - t_k$.

Example 3. Multiple-Order Sampling

We consider the same system as in Example 2. We assume again that the sampling operations are synchronous, but now they take place at the instants

$$t_k = 0, \quad T_0, \quad T_0 + T_1, \; \cdots, \; T_0 + \cdots + T_{q-1} = T, \; \cdots, \; kT, \quad kT + T_0, \; \cdots.$$

In other words, the sampling period T_k is a periodic function of k with period q.

(i–iii) This example is evidently a special case of the nonstationary system (44) derived in connection with the previous example.

(iv) Because T_k is periodic, we may relate $x(kT)$ to $x((k+1)T)$ by means of the stationary difference equation:

$$\begin{aligned} \mathbf{x}((k+1)T) = \Psi(T_0, \cdots, T_{q-1})\mathbf{x}(kT) & \\ + \mathbf{v}_1(T_0, \cdots, T_{q-1})r(kT) & \\ + \mathbf{v}_2(T_1, \cdots, T_{q-1})r(kT + T_0) + \cdots & \\ + \mathbf{v}_q(T_{q-1})r(kT + T_0 + \cdots + T_{q-2}) & \end{aligned} \tag{45}$$

where

$$\begin{aligned} \Psi(T_0, \cdots, T_{q-1}) &= \Psi(T_{q-1})\Psi(T_{q-2}) \cdots \Psi(T_0) \\ \mathbf{v}_1(T_0, \cdots, T_{q-1}) &= \Psi(T_{q-1})\Psi(T_{q-2}) \cdots \mathbf{v}(T_0) \\ \mathbf{v}_2(T_1, \cdots, T_{q-1}) &= \Psi(T_{q-1})\Psi(T_{q-2}) \cdots \mathbf{v}(T_1) \\ &\cdots\cdots\cdots\cdots \\ \mathbf{v}_q(T_{q-1}) &= \mathbf{v}(T_{q-1}), \end{aligned} \tag{46}$$

where $\Psi(\tau)$, $\mathbf{v}(\tau)$ are given by (42-*A*-*B*).

Example 4. Nonsynchronous Sampling

The system is shown in Fig. 6. The sampling operations occur at the following instants of time:

Sample-and-Hold Element 1: $0, \quad T, \cdots, \quad kT, \cdots$

Sample-and-Hold Element 2: $U, \; T + U, \cdots, kT + U, \cdots$

For simplicity, it is assumed that $T > U > 0$.

(i) The definition of the state vector is apparent from Fig. 6.

(ii) The transition matrices for the two sample-and-hold elements are given by

$$S_3 = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad S_4 = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right]$$

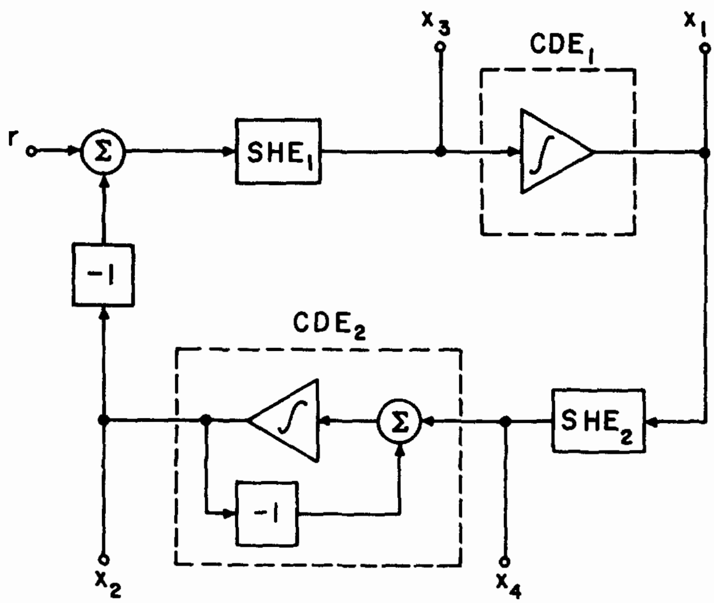

FIG. 6. Example 4.

while the transition matrix of the continuous element is

$$\Phi(\tau) = \left[\begin{array}{cc|c|c} 1 & 0 & \tau & 0 \\ 0 & e^{-\tau} & 0 & 1 - e^{-\tau} \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

From Fig. 6 we have also

$$\mathbf{s}_3 = \begin{bmatrix} 0 \\ 0 \\ \hline 1 \\ \hline 0 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_4 = 0.$$

(iii) At time $t = kT$, there is a discontinuous change in the value of x_3 which is expressed by

$$\mathbf{x}(kT^+) = S_3\mathbf{x}(kT) + \mathbf{s}_3 r(kT). \tag{47}$$

During the time-interval $kT < t \leqq kT + U$ no sampling operations occur and the state transitions are continuous. Thus

$$\mathbf{x}(kT + \tau) = \Phi(\tau)\mathbf{x}(kT^+), \quad 0 < \tau \leqq U. \tag{48}$$

At time $t = kT + U$ there is another discontinuous state transition due to a change in the value of x_4:

$$\mathbf{x}(kT + U^+) = S_4\mathbf{x}(kT + U). \tag{49}$$

During the time-interval $kT + U < t \leqq (k + 1)T$, the state transitions are again continuous:

$$\mathbf{x}(kT + U + \tau) = \Phi(\tau)\mathbf{x}(kT + U^+), \quad 0 < \tau \leqq T - U. \tag{50}$$

(iv) Equations 47–50 describe the behavior of the system at every instant of the time interval $kT < t \leqq (k + 1)T$. Confining attention to the transition from $\mathbf{x}(kT)$ to $\mathbf{x}((k + 1)T)$, we get the following stationary difference equation by combining (47–50)

$$\mathbf{x}((k + 1)T) = \Psi(T, U)\mathbf{x}(kT) + \mathbf{v}(T)r(kT) \tag{51}$$

where

$$\Psi(T, U) = \Phi(T - U)S_4\Phi(U)S_3 =$$

$$= \left[\begin{array}{cc|cc} 1 & -T & 0 & 0 \\ 1 - e^{-(T-U)} & e^{-T} - U(1 - e^{-(T-U)}) & 0 & e^{-(T-U)} - e^{-T} \\ \hline 0 & -1 & 0 & 0 \\ 1 & -U & 0 & 0 \end{array}\right] \tag{52}$$

and

$$\mathbf{v}(T, U) = \Phi(T - U)S_4\Phi(U)\mathbf{s}_3 = \left[\begin{array}{c} T \\ U(1 - e^{-(T-U)}) \\ \hdashline 1 \\ U \end{array}\right].$$

Remark e. Notice that in (51) the state variable x_3 (but not x_4) is superfluous.

Remark f. As in Example 3, the derivation of the above equation does not make use of the assumption that T, U are constants. Letting these quantities be a function of k we get the nonstationary difference equation

$$\mathbf{x}(t_{k+1}) = \Psi(T_k, U_k)\mathbf{x}(t_k) + \mathbf{v}(T_k, U_k)r(t_k) \tag{53}$$

where t_k denotes the k^{th} sampling instant of SHE_1, and

$$t_{k+1} - t_k = T_k > U_k > 0.$$

Example 5. Multi-Rate Sampling

Consider again the system shown in Fig. 6. In this example, the sampling operations have periods $T_A = T/3$ and $T_B = T/2$, respectively. Thus sampling occurs at the instants

Sample-and-Hold Element 1:

$$kT, \quad kT + T/3, \quad kT + 2T/3, \quad (k + 1)T, \cdots$$

Sample-and-Hold Element 2:

$$kT, \quad kT + T/2, \quad (k + 1)T, \cdots$$

(i–ii) The transition matrices of the two sampling operations and of the continuous dynamic element are the same as in Example 4.

(iii) At time $t = kT$, both sample-and-hold elements have a discontinuous state transition:

$$\mathbf{x}(kT^+) = S_3S_4\mathbf{x}(kT) + \mathbf{s}_3r(kT).$$

During the interval $kT < t \leqq kT + T/3$ the state transitions are continuous

$$\mathbf{x}(kT + \tau) = \Phi(\tau)\mathbf{x}(kT^+), \quad 0 < \tau \leqq T/3.$$

At time $k = kT + T/3$,

$$\mathbf{x}(kT + T/3^+) = S_3\mathbf{x}(kT + T/3) + \mathbf{s}_3r(kT + T/3).$$

During the interval $kT + T/3 < t \leqq kT + T/3$,

$$\mathbf{x}(kT + T/3 + \tau) = \Phi(\tau)\mathbf{x}(kT + T/3^+), \quad 0 < \tau \leqq T/6.$$

At time $t = kT + T/2$,

$$\mathbf{x}(kT + T/2^+) = S_4\mathbf{x}(kT + T/2).$$

During the interval $kT + T/2 < \tau \leqq kT + 2T/3$,

$$\mathbf{x}(kT + T/2 + \tau) = \Phi(\tau)\mathbf{x}(kT + T/2^+), \quad 0 < \tau \leqq T/6.$$

At time $t = kT + 2T/3$,

$$\mathbf{x}(kT + 2T/3^+) = S_3\mathbf{x}(kT + 2T/3) + \mathbf{s}_3r(kT + 2T/3).$$

During the interval $kT + 2T/3 < t \leqq (k + 1)T$,

$$\mathbf{x}(kT + 2T/3 + \tau) = \Phi(\tau)\mathbf{x}(kT + 2T/3^+), \quad 0 < \tau \leqq T/3.$$

(iv) Since the pattern of sampling operations repeats with a common period T, by combining the preceding equations we can obtain a stationary difference equation relating $\mathbf{x}((k + 1)T)$ to $\mathbf{x}(kT)$:

$$\mathbf{x}((k + 1)T) = \Psi\mathbf{x}(kT) + \mathbf{v}_1r(kT) + \mathbf{v}_2r(kT + T/3) + \mathbf{v}_3r(kT + 2T/3) \tag{54}$$

where

$$\begin{aligned} \Psi &= \Phi(T/3)S_3\Phi(T/6)S_4\Phi(T/6)S_3\Phi(T/3)S_3S_4 \\ \mathbf{v}_1 &= \Phi(T/3)S_3\Phi(T/6)S_4\Phi(T/6)S_3\Phi(T/3)\mathbf{s}_3 \\ \mathbf{v}_2 &= \Phi(T/3)S_3\Phi(T/6)S_4\Phi(T/6)\mathbf{s}_3 \\ \mathbf{v}_3 &= \Phi(T/3)\mathbf{s}_3. \end{aligned} \tag{55}$$

Remark g. It should be carefully noted that the derivation of the stationary difference Eq. 54 was possible only because the ratio of sampling periods was rational. In fact, if $T_A/T_B = p/q$ (where p, q are relatively prime integers), then the sampling patterns have a least common period $qT_A = pT_B = T$ and therefore it is possible to relate the state $\mathbf{x}(kT)$ to $\mathbf{x}((k + 1)T)$ by means of a stationary difference equation. When T_A/T_B is irrational or, what is practically the same thing, the integers p, q are large, then a very large number of intervals have to be considered in order to write down the over-all stationary transition Eq. 54.

Remark h. Alternately, it is possible to write down a nonstationary difference equation of the type

$$\mathbf{x}((k+1)T) = \Psi(k)\mathbf{x}(kT) + \mathbf{v}(k; r(t)) \tag{56}$$

where T is *not* the least common period of T_A and T_B. In this case, the matrix $\Psi(k)$ and the vector $\mathbf{v}(k; r(t))$ must be calculated separately for each interval $kT < t \leqq (k+1)T$ by considering the various sampling instants and the various sampling-free subintervals as was done above. When T_A/T_B is irrational, that is, when the least common period of T_A and T_B is infinite, the matrix $\Psi(k)$ will have coefficients which vary in a quasi-random (nonperiodic) fashion with k. Therefore under such circumstances one may expect the sampling system to exhibit a behavior which would be intuitively interpreted as "random."

Example 6. Noninstantaneous Sampling

This system is shown in Fig. 7. The sampling operation is as defined by (2–*B*.)

(i) The definition of the state variables is apparent from Fig. 7.

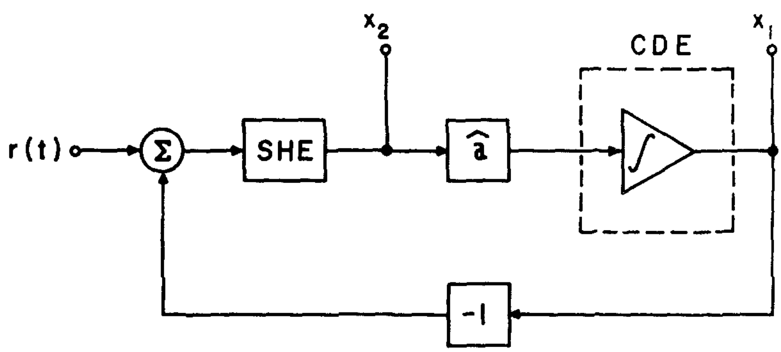

FIG. 7. Example 6.

(ii–iii) There are two transition matrices for the continuous dynamic element. During the time interval $kT < t \leqq kT + U$, the sample-and-hold element transmits all signals without modification. Thus

$$x_2(t) = r(t) - x_1(t). \tag{57}$$

During this time interval, the continuous dynamic element is governed by the differential equation

$$dx_1/dt = \bar{a}x_1 + r(t). \tag{58}$$

Integrating (58) and combining it with (57), we obtain the transition equation

$$\mathbf{x}(kT + \tau) = \bar{\Phi}(\tau)\mathbf{x}(kT) + \int_0^\tau \bar{\Phi}(\tau - \sigma)\bar{\mathbf{c}}_1 r(\sigma)d\sigma + \bar{\mathbf{s}}_2 r(kT + \tau), \quad 0 < \tau \leqq U \quad (59)$$

where

$$\bar{\Phi}(\tau) = \left[\begin{array}{c|c} e^{\bar{a}\tau} & 0 \\ \hline -e^{\bar{a}\tau} & 0 \end{array}\right]$$

$$\bar{\mathbf{c}}_1 = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] \qquad \bar{\mathbf{s}}_2 = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]. \quad (60)$$

During the time interval $kT + U \leqq t \leqq (k + 1)T$, the output of the sample-and-hold element remains constant and equal to $x_2(kT + U)$; the continuous dynamic element is governed by the differential equation

$$dx_1/dt = \bar{a}x_2(kT + U).$$

From this we find by inspection the transition equation

$$\mathbf{x}(kT + U + \tau) = \Phi(\tau)\mathbf{x}(kT + U) \quad 0 \leqq \tau \leqq T - U \quad (61)$$

where

$$\Phi(\tau) = \left[\begin{array}{c|c} 1 & \bar{a}\tau \\ \hline 0 & 1 \end{array}\right]. \quad (62)$$

(iv) Combining these results, we obtain an expression for $\mathbf{x}((k+1)T)$ in terms of $\mathbf{x}(kT)$ as follows:

$$\mathbf{x}((k + 1)T) = \Psi(T, U)\mathbf{x}(kT) + \mathbf{v}(T, U, r(t)) \quad (63)$$

where

$$\Psi(T, U) = \Phi(T - U)\bar{\Phi}(U) = \left[\begin{array}{c|c} e^{\bar{a}U}(1 - \bar{a}(T - U)) & 0 \\ \hline -e^{\bar{a}U} & 0 \end{array}\right] \quad (64)$$

and

$$\mathbf{v}(T, U, r(t)) = \Phi(T - U)\left[\mathbf{s}_2 r(kT + U) + \int_0^U \bar{\Phi}(U - \sigma)\bar{\mathbf{c}}_1 r(\sigma)d\sigma\right].$$

It should be observed that (63) is not a true difference equation (but rather a difference-integral equation) because of the complicated fashion in which the system input $r(t)$ affects the state transitions.

Example 7. Other Applications

There are many other problems in the analysis of nonstationary linear dynamic systems which can be handled by our methods. We mention two further examples:

(1) In discussing the solutions of Mathieu's equation with a square-wave parametric excitation, that is, the equation

$$d^2x/dt^2 + adx/dt + b(1 + c \operatorname{sgn}(\sin wt)) = 0$$

where a, b, c, w are positive constants and $\operatorname{sgn}(0) = 0$, $\operatorname{sgn} x = x/|x|$ if $x \neq 0$, Pipes (15) has used a method which is conceptually identical with ours.

(2) In a recent paper, Desoer (16) has discussed a dynamic behavior of networks which contain periodically operated switches. Desoer derived the equivalent of formula (12), apparently regarding this as a new result. In applying our method to Desoer's problem, very considerable conceptual and analytic simplifications may be obtained. We omit the details.

STABILITY

Probably the most important task of system theory is to answer the question, "Will a given dynamic system be stable?" The first significant result of the theory of (conventional) sampling systems was the derivation of a stationary difference equation governing the behavior of the system between successive sampling instants; the question of stability can be answered directly by examination of the difference equation.

Following a suggestion of Zadeh (17), the concept of stability is formulated by the following

> *Definition*—A linear system is said to be *stable* if and only if every bounded system input signal produces bounded variations in all state variables.

This definition applies to linear systems regardless of whether they are governed by differential or difference equations or whether they are stationary or nonstationary.

In a linear system, stability depends only on the transition matrix of the system; stability cannot be brought about or destroyed by a particular choice of the initial state or the system input signal. It is not difficult to show that the above definition implies.

> *Theorem 1. A stationary linear system is stable if and only if every element of* Ψ^{N} *tends to* 0 *as* $\mathrm{N} \to \infty$ [*every element of* $\Psi(\mathrm{k})\Psi(\mathrm{k}+1)\cdots\Psi(\mathrm{k}+\mathrm{N})$ *tends to* 0 *uniformly*[4] *in* k *as* $\mathrm{N} \to \infty$].

[4] It can easily be shown that if $\Psi(k)\Psi(k+1)\cdots\Psi(k+N)$ does not approach 0 ***uniformly*** as $N \to \infty$, then for some bounded system input signal some state variable will become unbounded.

In the above theorem, $\Psi(k)$ is the transition matrix of the system over some well-defined, constant interval of time.

One can "experimentally" verify from Theorem 1 the stability of a given system by multiplying together the transition matrix a large number of times. This is admittedly a brute-force procedure, but, because of the simplicity of matrix multiplication, very well suited for machine computation. Moreover, in the case of a nonstationary system, this method may be the only way of checking stability since no explicit necessary *and* sufficient stability conditions for nonstationary linear systems are known at present. (There are, of course, many sufficient and many necessary conditions.)

It is fortunate that in many sampling systems the transition matrices over sufficiently long intervals of time are constant; see Eqs. 43, 45, 51, 54 and 63. (For cases where the transition matrix is not constant, see Eqs. 44, 53, 56 and Remark *g*). For a constant transition matrix, the explicit stability conditions are as follows:

Theorem 2. A stationary linear system is stable if all n (possibly complex) zeros, z_i, *of the polynomial* $\det(\Psi - zI)$ *satisfy*

$$|z_i| < 1 \quad (i = 1, \cdots, n). \tag{65}$$

If, moreover, $|z_i| \neq 1$ *for any* i, *then condition (65) is not only sufficient but also necessary.*

The proof of this theorem follows readily by reducing Ψ to its Jordan canonic form.

Condition (65) is identical with the result that the poles z_i of the z-transform of the input-output relations of the system must lie within the unit circle (1). Of course, our result is much more general than the z-transform result because Ψ can be computed in many cases where the z-transform method is not applicable.

Finally, it should be remarked that a stationary transition matrix can be obtained in those cases and in those only where the pattern of the sampling operations repeats in a periodic fashion. This observation generalizes Floquet's theorem concerning the nonstationary differential equation

$$d\mathbf{x}/dt = A(t)\mathbf{x} \tag{66}$$

where $A(t) = A(t + T)$, that is, the system is periodic with period T. According to this theorem (14) the transition matrix $\Psi(t + T; t)$ of (66) for an interval of length T,

$$\mathbf{x}(t + T) = \Psi(t + T; t)\mathbf{x}(t)$$

is given by

$$\Psi(t + T; t) = \exp BT$$

where B is a constant matrix. This shows that the stability problem in a nonstationary but periodic linear system is essentially the same as in a stationary linear system provided that attention is focused on the transition matrix corresponding to one period.

CONCLUSIONS

The development of the theory and engineering application of sampling systems has been hampered by the fact that hitherto standard methods of analysis (the z-transform method and its various modifications) are not readily applicable unless all sampling operations in the system are synchronous and of constant sampling period. This paper shows that such restrictions are not at all basic; sampling operations of any presently conceivable kind can be readily studied within the unified framework presented here.

Sampling systems are a special class of nonstationary dynamic systems. As a result of the nonstationarity, whose complexity is directly dependent on the complexity of the sampling operations, the description of such systems is necessarily somewhat involved. In many cases the behavior of sampling systems is still stationary if attention is focused not on the instantaneous state transitions but on the transitions over a certain time-interval. The transition matrix for these intervals can then be used to answer the most basic question of system theory, namely that of stability.

In fact, any other information concerning the system behavior can be derived from the transition equations. Thus, the classical results concerning the stability of sampling systems can be extended to cases where the sampling operations are not restricted to the synchronous, constant-sampling-period case.

REFERENCES

(1) J. R. RAGAZZINI AND G. FRANKLIN, "Sampled-Data Control Systems," New York, McGraw-Hill Book Co., Inc., 1958.

(2) W. K. LINVILL, "Sampled-Data Control Systems Studied Through Comparison of Sampling with Impulse Modulation," *Trans. AIEE*, Vol. 70, pp. 1779–99 (1951).

(3) J. R. RAGAZZINI AND L. A. ZADEH, "The Analysis of Sampled-Data Systems," *Trans. AIEE*, Vol. 71, Part II, pp. 225–234 (1952).

(4) R. H. BARKER, "The Pulse Transfer Function and its Application to Sampling Servo Systems," *Proc. IEE*, Vol. 99, Part IV, pp. 302–317 (1952).

(5) J. G. TRUXAL, "Automatic Feedback Control System Synthesis," New York, McGraw-Hill Book Co., Inc., 1955, Chap. 9.

(6) G. KRANC, "Compensation of an Error-Sampled System by a Multi-Rate Controller," *Trans. AIEE*, Vol. 76, Part II, pp. 149–159 (1957).

(7) G. KRANC, "Additional Techniques for Sampled-Data Feedback Problems," IRE WESCON Convention Record, Vol. 1, Part IV, pp. 157–165, 1957.

(8) G. FARMANFARMA, "General Analysis and Stability Study of Finite Pulsed Feedback Systems," *Trans. AIEE*, Vol. 77, Part II, pp. 148–162 (1958).

(9) R. E. KALMAN AND J. E. BERTRAM, "General Synthesis Procedure for Computer Control of Single and Multi-Loop Linear Systems," *Trans. AIEE*, Vol. 77, Part II, pp. 602–609 (1958).

(10) R. E. KALMAN, "Optimal Nonlinear Control of Saturating Systems by Intermittent Action," IRE WESCON Convention Record, Vol. 1, Part IV, pp. 130–135 (1957).

(11) R. E. KALMAN AND R. W. KOEPCKE, "Optimal Synthesis of Linear Sampling Control Systems Using Generalized Performance Indexes," *Trans. ASME*, Vol. 81, pp. 1820–1826 (1959).

(12) J. E. BERTRAM, "The Effect of Quantization in Sampled-Feedback Systems," *Trans. AIEE*, Vol. 77, Part II, pp. 177–182 (1958).

(13) R. E. KALMAN, "Analysis and Design Principles of Second and Higher Order Saturating Servomechanisms," *Trans. AIEE*, Vol. 74, Part II, pp. 294–310 (1955).

(14) E. A. CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations," New York, McGraw-Hill Book Co., Inc., 1955, Chap. 3.

(15) L. A. PIPES, "Matrix Solution of Equations of the Mathieu-Hill Type," *J. Appl. Phys.*, Vol. 24, pp. 902–910 (1953).

(16) C. A. DESOER, "A Network Containing a Periodically Operated Switch Solved by Successive Approximations," *Bell System Tech. J.*, Vol. 36, pp. 1403–1428 (1957).

(17) L. A. ZADEH, "A General Theory of Linear Signal Transmission Systems," JOUR. FRANKLIN INST., Vol. 253, pp. 293–312 (1952).

(18) R. E. KALMAN, "Analysis and Synthesis of Linear Systems Operating on Randomly Sampled Data," doctoral thesis, Dept. of Electrical Engineering, Columbia University, New York, N. Y., 1957.

(19) R. E. KALMAN, "On a New Approach to Filtering and Prediction Problems," *ASME* paper 59–IRD–11; to appear in *J. Basic Engineering*, Vol. 1, 1959.

(20) FAN CHUN VUI, "On Pulsed Servo Systems Containing Two Pulse Elements with Unequal Pulse Repetition," (in Russian), *Automatika i Telemekhanika*, Vol. 19, pp. 917–930 (1958).

(21) YA. Z. TSYPKIN, "Theory of Pulse Systems," (in Russian), Moscow, State Press for Physico-Mathematical Literature, 1958.

(22) G. P. TARTAKOVSKIĬ, "On Pulse Systems with Variable Elements," *Elektroviaz* 1956; *Radiotekhnika i Elektronika*, Vol. 2, pp. 15–22 (1957).