INT. J. CONTROL, 1991, VOL. 54, NO. 5, 1097–1117

Computation of time-optimal controls applied to rigid manipulators with friction

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We present a numerical procedure to compute non-singular, time-optimal solutions for non-linear systems that are linear in the control and have fixed initial and final states and bounded control. Part of our procedure is a new numerically computable test that determines whether bang–bang solutions satisfy Pontryagin's minimum principle. This test reveals the new important fact that, for non-linear systems, with linear control and dimension n, the probability that a bang–bang solution with more than $n-1$ switches satisfies Pontryagin's Minimum Principle is almost zero. Using a parameter optimization procedure we search for bang–bang solutions with up to $n-1$ switches which transfer the system from the initial to the final state. If no controls with up to $n-1$ switches can be found to satisfy Pontryagin's Minimum Principle the problem is very likely singular. We apply our procedure to the time-optimal control problem for rigid manipulators where friction may be included in the dynamics. We will demonstrate that some solutions mentioned in the literature to satisfy Pontryagin's minimum principle do not. A class of time-optimal control problems turns out to be singular. To solve these problems we propose and demonstrate the method of control parametrization.

1. Introduction

Necessary conditions for the solution of time-optimal control problems involving a non-linear system that is linear in the control, with fixed initial and final states and bounded control, are very well known (Lewis 1986, Sage and White 1977, Bryson and Ho 1975). They can be derived from what is known as Pontryagin's minimum principle. If the time-optimal control problem is non-singular, these necessary conditions imply that the solution is of a bang–bang type, i.e. a solution where the control variables have extreme values at all times, except for the switch times where they may switch from one extreme value to the other. Note that this does not imply that a bang–bang solution, which transfers the system from the initial to the final state, necessarily satisfies the necessary conditions for the solution of the non-singular time-optimal control problem, in other words Pontryagin's minimum principle.

The necessary conditions make up a non-linear two point boundary value problem (TPBVP) (Lewis 1986, Sage and White 1977, Bryson and Ho 1975). This problem is generally very difficult to solve numerically since little information concerning the co-state involved is available while, dependent on the system dynamics, the final state is very sensitive to changes in the co-state evolution. Therefore, a general approach to solve these problems numerically is to make assumptions with regard to the number of switch times, and the initial value of each control variable. Using a parameter optimization procedure where the parameters

Received 10 September 1990. Revised 28 January 1991.

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are the switch times and the final time, with a penalty for deviations from the fixed final state, one tries to find a bang–bang control which transfers the system from the initial to the final state. Having found such a control it is assumed that it satisfies Pontryagin's minimum principle (Kahn and Roth 1971, Weinreb and Bryson 1985, Wen and Desrocher 1985). In this paper we present for the first time a numerically computable test which verifies if bang–bang solutions satisfy Pontryagin's minimum principle. Furthermore, this test reveals the previously unknown fact that the probability for a bang–bang solution, with more than $n-1$ switch times, to satisfy Pontryagin's minimum principle is almost zero, where n is the dimension of the system. Therefore, in practice, the search for bang–bang solutions can be restricted to solutions with up to $n-1$ switch times.

We apply a parameter optimization procedure in conjunction with our numerical test to compute bang–bang solutions for the IBM 7535 B 04 robot which was earlier considered by Geering *et al.* (1986). We may include viscous and Coulomb friction as well as gravity in the robot dynamics, which were not considered by Geering *et al.* (1986). Robotic manipulators constitute non-linear systems that are linear in the control, the control being bounded (Van Willigenburg 1990 a). We demonstrate that we find exactly the same bang–bang solutions as Geering *et al.* (1986) when transferring the system from the initial to the final state. Although Geering *et al.* (1986) state that they all satisfy Pontryagin's minimum principle, our numerical test reveals that some of them do not. Finally we demonstrate that a method based on control parametrization (Goh and Teo 1988, Teo *et al.* 1989) generates solutions for non-singular problems involving the IBM 7535 B 04 robot which are very close to the optimum. The method based on control parametrization can be applied to both singular and non-singular problems. For robotic manipulators it is, in both cases, expected to generate solutions that are very close to the optimum. The method explicitly considers the control to be piecewise constant. This is a realistic assumption since robotic manipulators are controlled by digital computers. Furthermore, it allows for the inclusion of bounds on the individual link velocities, which should also be considered in practical situations (Van Willigenburg 1991).

An assembly task performed by robotic manipulators generally involves the transportation of an object or a tool from one location to another. This operation is called a 'point-to-point motion' and is characterized by prescribed initial and final positions and velocities of the robot links. The link positions and velocities may be considered as the state variables of the robotic manipulator. To maximize productivity the objective is to perform the 'point-to-point motion' in minimum time. The problem of performing a 'point-to-point motion' in minimum time constitutes a time-optimal control problem with fixed initial and final states, and is of great importance.

For an extensive review of earlier work on the time-optimal control problem for robotic manipulators we refer the reader to Chernousko *et al.* (1989) and references cited there. We may roughly divide the work into three categories. One category uses linear models from which the solution to the time-optimal control problem may be computed. In this category some references approximate non-linear robot dynamics that are linear in the control by a linear model (Kahn and Roth 1971, Kim and Shin 1985, Wen and Desrocher 1986, Nijmeyer *et al.* 1988), while others use feedback to compensate for non-linear terms (Freund 1975, Katupitiya 1986) in order to arrive at a linear model. The latter is possible since robotic manipulators constitute so-called feedback linearizable systems. The disadvantage of this is that due to the feedback the bounds on the control variables become state-dependent and have to be approximated by constants.

A second category (Sahar and Hollerbach 1985, Rajan 1985, Shiller and Dubowsky 1989) uses the solution of the time-optimal control problem along a prescribed path (Bobrow, *et al.* 1985, Shin and McKay 1985, Van Willigenburg 1991). Using some optimization techniques a path that connects the initial and final states is sought, which possesses the smallest minimum travelling time. A disadvantage of this method is that an assumption has to be made concerning the shape of the path, while the solution to the problem turns out to be very sensitive to this path shape.

The third category, like the second, considers the 'true' non-linear dynamics but, except for Van Willigenburg (1990 a), neglects friction (Ailon and Langholtz 1985, Sontag and Sussmann 1985, 1986, Wen 1986, Geering *et al.* 1986, Chen 1989). In this category no assumptions concerning the solution have to be made. Pontryagin's minimum principle, which states necessary conditions for a time-optimal control, is used to investigate the time-optimal control problem. However, except for Geering *et al.* (1986), no procedures to compute time-optimal solutions have been presented within this category. The publications only state results concerning the form of the solution. Sontag and Sussmann (1985, 1986) demonstrate that the problem may be singular. As we demonstrate in this paper, some solutions presented by Geering *et al.* (1986) are *not* time-optimal, i.e. they do not satisfy Pontryagin's minimum principle.

2. Dynamics of rigid manipulators with friction

The dynamics of a rigid N-link manipulator with friction can be written as (Asada and Slotine 1986)

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + F(\theta, \dot{\theta}) \tag{1 a}$$

where

$$\theta = (\theta_1 \quad \theta_2 \quad \dots \quad \theta_N)^{\mathrm{T}} \tag{1 b}$$

is an $N \times 1$ vector containing the joint angles of the links and

$$\tau = (\tau_1 \quad \tau_2 \quad \dots \quad \tau_N)^{\mathrm{T}} \tag{1 c}$$

is an $N \times 1$ vector containing the actuation torques, which are considered to be the control variables. $M(\theta)$ is an $N \times 1$ positive-definite inertia matrix, $V(\theta, \dot{\theta})$ is an $N \times 1$ vector representing centrifugal and Coriolis forces, $G(\theta)$ is an $N \times 1$ vector of forces due to gravity and $F(\theta, \dot{\theta})$ is an $N \times 1$ vector of friction forces.

To investigate the time-optimal control problem, we write the non-linear system (1) in state-space form, the state and control vector being $(\theta^{\mathrm{T}} \ \dot{\theta}^{\mathrm{T}})^{\mathrm{T}}$ and τ, respectively. Since $M(\theta)$ is positive definite we obtain from (1)

$$\ddot{\theta} = M^{-1}(\theta)[\tau - V(\theta, \dot{\theta}) - G(\theta) - F(\theta, \dot{\theta})] \tag{2}$$

Introducing

$$x_1 = \theta \tag{3 a}$$

$$x_2 = \dot{\theta} \tag{3 b}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3 c}$$

$$u = \tau \tag{3 d}$$

$$T = V + G + F \tag{3 e}$$

(2) can be written in state-space form using (3)

$$\dot{x}_1 = x_2 \tag{4 a}$$

$$\dot{x}_2 = -M^{-1}(x_1)T(x) + M^{-1}(x_1)u \tag{4 b}$$

Observe that the dynamics given in (4) are linear in the control. In the following, the index i refers to the ith component if it is associated with a row or column vector and to the ith column in the case of matrices.

Each component of the control vector u is assumed to be bounded.

$$|u_i| \leqslant b_i, \quad i = 1, \ldots, N \tag{5}$$

If, for instance, the manipulator is actuated by current controlled DC-motors, the torque is proportional to the motor current, which is limited in the case of DC-motors.

3. Time-optimal control problem

Consider a non-linear time-optimal control problem that is linear in the control. Given the system

$$\dot{x} = f(x) + B(x)u \tag{6 a}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $r \leqslant n$, with fixed initial state

$$x(t_0) = x_0 \tag{6b}$$

and bounded control

$$|u_i| \leqslant a_i, \quad i = 1, \ldots, r \tag{7}$$

It is possible to minimize the cost criterion

$$J(t_0) = \int_{t_0}^{t_f} (1)\, dt \tag{8}$$

subject to the final state constraint

$$x(t_f) = x_f \tag{9}$$

where x_f is fixed and t_f is free.

The hamiltonian for system (6 a) and the cost criterion (8) is given by

$$H(x, u, \lambda) = 1 + \lambda^{\mathrm{T}}[f(x) + B(x)u] \tag{10}$$

where λ is the co-state of the system (6 a). The co-state variables satisfy the adjoint differential equation

$$-\dot{\lambda} = \frac{\delta H}{\delta x} \tag{11}$$

Pontryagin's minimum principle states that a necessary condition for an optimal control is that is minimizes the hamiltonian for optimal values of the state and co-state, i.e. (Lewis 1986)

$$H(x^*, u^*, \lambda^*) \leqslant H(x^*, u, \lambda^*) \quad \text{for all admissible } u \tag{12}$$

where the superscript * denotes an optimal quantity. Since (10) is linear in the control, we obtain from (7), (10) and (12) the following control law

$$u_i^*(t) = \begin{cases} +a_i & \text{if } [\lambda^{\mathrm{T}}(t)B]_i < 0 \\ -a_i & \text{if } [\lambda^{\mathrm{T}}(t)B]_i > 0, \end{cases} \quad t_0 \leqslant t \leqslant t_f \tag{13}$$

where, for obvious reasons, $[\lambda^{\mathrm{T}}(t)B]_i$, $t_0 \leqslant t \leqslant t_f$, is called the switching function corresponding to the control variable $u_i(t)$. If one or several of the switching functions are equal to zero over some time interval the control law (13) does not determine a solution. The time optimal control problem is called singular in this case. If the switching functions equal zero at isolated times only, (13) determines a solution and the problem is called non-singular. For the moment, we only consider non-singular time optimal control problems. In the case of a non-singular time optimal control problem we observe from (13) that almost everywhere each control variable takes on an extreme value. This type of control is called bang–bang control.

The solution of the time-optimal control problem (6)–(9) is determined by (6), (9), (11) and (13) and constitutes a two point boundary value problem (TPBVP) where the boundary conditions are given by (6 *b*) and (9) and, in addition, we have the boundary condition (Lewis 1986)

$$H(t_f) = 0 \tag{14}$$

Since the system (6 *a*) and the integrand of the cost criterion (8) do not explicitly depend on time, the hamiltonian (10) is not an explicit function of time. So we have (Lewis 1986)

$$\dot{H} = 0 \tag{15}$$

and together with (14)

$$H(t) = 0 \quad t_0 \leqslant t \leqslant t_f \tag{16}$$

The TPBVP (6), (9), (11), (13) and (14), is very difficult to solve numerically since, except (16), no information concerning the values of the co-state is available. In addition, we have found that the solution is very sensitive to changes in the co-state. The usual approach is to search for bang–bang controls which transfer the system from the initial to the final state and to assume that the one with the smallest transition time satisfies Pontryagin's minimum principle. We demonstrate that this assumption may not be correct. In § 4 we present a numerical test to verify whether bang–bang solutions satisfy Pontryagin's minimum principle.

As an introduction to § 4 let us finally look at the time optimal control problem from a different point of view. Given an initial co-state

$$\lambda(t_0) = \lambda_0 \tag{17}$$

which according to (16) must satisfy

$$H(t_0) = 0 \tag{18}$$

the system (6 *a*), the adjoint system (11) and the control law (13), when integrated from the initial conditions (6 *b*) and (17), generate time optimal solutions satisfying Pontryagin's minimum principle. The final state in this case depends on the initial co-state (17) and the time at which the integration is stopped. So, given a fixed final state (9) the time optimal control problem may be regarded as an initial value problem for the co-state. Again however, this initial value problem is very difficult to solve numerically since, except for (18), we have no information concerning the initial co-state and, in addition, we have found the problem to be very sensitive to changes in the initial co-state. Since, furthermore, the final time is unknown, during integration we constantly have to check whether we come across the fixed final state. In § 4, however, we demonstrate that given a bang–bang control we may compute whether or not an initial co-state exists which generates this bang-bang control. If it exists the solution satisfies Pontryagin's minimum principle, otherwise it does not. When the initial co-state exists we are able to compute it and thereby compute the evolution of the co-state and the switching functions corresponding to the time optimal control and state trajectory.

4. Numerical test to verify whether bang–bang solutions satisfy Pontryagin's minimum principle

The state and co-state equations for the time-optimal control problems are given by (6 *a*) and (11). Pontryagin's minimum principle states that a necessary condition for a time-optimal solution is

$$u_i(t) = -a_i \operatorname{sgn}[\lambda^{\mathrm{T}}(t)B]_i \quad t_0 \leqslant t \leqslant t_f \tag{19}$$

In the following, a time-optimal solution will denote a solution satisfying Pontryagin's minimum principle, i.e. (19). Assume we have a bang–bang control

$$u_b(t) \quad t_0 \leqslant t \leqslant t_f \tag{20}$$

which transfers the system from the initial to the final state, i.e.

$$x_b(t_0) = x_0 \tag{21 a}$$

$$x_b(t_f) = x_f \tag{21 b}$$

where x_b is the state trajectory of the system (6 *a*) generated by the control (20). The question of whether this control is time-optimal comes down to the question of whether an initial co-state vector $\lambda(t_0)$, satisfying (18), exists which by (6 *a*), (11) and (19) generates the bang–bang control. Equation (19) demands that at each switching instant the corresponding switching function is equal to zero. We demonstrate that this condition can be transformed into p computable linear relationships between the n components of the initial co-state vector, where p is the number of switches.

The solution to the linear adjoint differential equation (11) is given by

$$\lambda(t) = \Phi(t, t_0)\lambda(t_0) \tag{22}$$

where Φ is the fundamental matrix associated with (11). Note that since both x_b and u_b are known $\Phi(t, t_0)$ can be computed. If the ith control variable switches from one extreme value to the other at time t_s, the corresponding switching function must be equal to zero. So we must have

$$[\lambda^{\mathrm{T}}(t_s)B(t_s)]_i = 0 \tag{23}$$

From (22) and (23) we must therefore have

$$\lambda^{\mathrm{T}}(t_0)\Xi_i(t_s)=0 \tag{24 a}$$

where

$$\Xi_i(t_s)=\Phi^{\mathrm{T}}(t_s, t_0)B_i(t_s) \tag{24 b}$$

Equations (24 *a* and *b*) constitute a linear relationship between the n components of the initial co-state vector. With p switching times we obtain p linear relationships between the components of the initial co-state vector. According to (10) and (18), the initial co-state must also satisfy

$$\lambda^{\mathrm{T}}(t_0)[f(x_0)+B(x_0)u(t_0)]=-1 \tag{25}$$

Equations (24 *a*) and (24 *b*) (which hold at each switching instant) and (25) define a non-homogeneous system of $p+1$ linear equations with n unknowns. When $p+1=n$, the system has a unique solution if these $p+1$ equations are linearly independent. If $p+1>n$, then at least $p+1-n$ equations must be linearly dependent for the system to have a solution, otherwise the system has no solution. If $p+1<n$, a set of solutions exists.

The non-homogeneous system of $p+1$ equations constitutes the necessary conditions for the initial co-state. If the system has no solution, the bang-bang control is not time-optimal. If the system has one or several solutions, we have to check if (19) is satisfied for all $t\in[t_0, t_f]$. This can be done by numerical integration of (6 *a*), (11) and (19), given $x(t_0)$ and $\lambda(t_0)$.

5. Time-optimal control problem for a two-link robotic manipulator

5.1. *Introduction*

In order to demonstrate our approach we investigate the industrial IBM 7535 B 04 robot, treated by Geering *et al.* (1986). This is one of few papers in which actual numerical calculation of the time-optimal controls is performed. Since Geering *et al.* also use Pontryagin's minimum principle to investigate the time-optimal control problem, this paper serves as a reference for our approach. We show that some solutions that were presented in that paper as time-optimal, are not.

5.2. *Dynamic model of the IBM* 7535 *B* 04 *robot*

The industrial IBM 7535 B 04 robot is schematically shown in Fig. 1. This robot consists of two links, which move in a horizontal plane. A third link which allows for vertical translations is mounted at the end of the second link. To perform various tasks a gripper, which may hold a load during operations, is mounted at the end of the third link. The vertical motion is completely decoupled from the horizontal motion of the first two links and is not treated here. The dynamics of this robotic manipulator as presented by Geering *et al.* (1986) are quite inaccessible and do not allow for easy extensions to include gravity and friction terms. As we demonstrate in Appendix A, this robot can be fully described by the closed-form dynamics of a two-link robotic manipulator (Asada and Slotine 1986). The third link, the gripper and the load are then considered as integral parts of the second link. The numerical values for the parameters of the IBM 7535 B 04 robot given by Geering *et al.* (1986) can still be used after appropriate transformation. We now present the closed form dynamics of the two-link robotic manipulator and give the

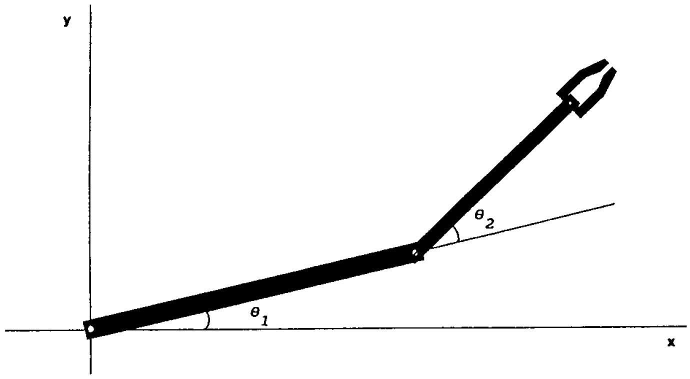


Figure 1. Geometry of the IBM 7535 B 04 robot.

actual values of the parameters. For a detailed description of the matching of the closed form dynamics with the model and the data presented by Geering *et al.* (1986) refer to Appendix A.

Consider a common two-link robotic manipulator that suffers from viscous and Coulomb friction. This is a robot with a geometry as shown in Fig. 1 without a third link. The robot motion may be considered either in a horizontal or in a vertical plane. Let m_1 and l_1 be the mass and the length of the first link and m_2 and l_2 the mass and length of the second link. The moments of inertia about the centroids are given by I_1 and I_2. The angular rotation θ_2 of the second link is measured relative to the first link. The distances between the centroids of links and the joint axes are denoted by l_{C1} adn l_{C2}.

The closed form dynamics are given by (Asada and Slotine 1986)

$$\tau_1 = M_{11}\ddot{\theta}_1 + M_{12}\ddot{\theta}_2 - h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 + G_1 + F_1 \qquad (26\,a)$$

$$\tau_2 = M_{12}\ddot{\theta}_1 + M_{22}\ddot{\theta}_2 - h\dot{\theta}_1^2 + G_2 + F_2 \qquad (26\,b)$$

where

$$M_{11} = m_1 l_{C1}^2 + I_1 + m_2[l_1^2 + l_{C2}^2 + 2l_1 l_{C2}\cos(\theta_2)] + I_2 \qquad (27\,a)$$

$$M_{12} = m_2 l_1 l_2 \cos(\theta_2) + m_2 l_{C2}^2 + I_2 \qquad (27\,b)$$

$$M_{22} = m_2 l_{C2}^2 + I_2 \qquad (27\,c)$$

$$h = m_2 l_1 l_{C2}\sin(\theta_2) \qquad (27\,d)$$

$$G_1 = m_1 l_{C1} g\cos(\theta_1) + m_2 g\{l_{C2}\cos(\theta_1+\theta_2) + l_1\cos(\theta_1)\} \qquad (27\,e)$$

$$G_2 = m_2 l_{C2} g\cos(\theta_1+\theta_2) \qquad (27\,f)$$

$$F_1 = c_1\,\mathrm{sgn}(\dot{\theta}_1) + v_1\dot{\theta}_1 \qquad (27\,g)$$

$$F_2 = c_2\,\mathrm{sgn}(\dot{\theta}_2) + v_2\dot{\theta}_2 \qquad (27\,h)$$

The terms G_1 and G_2 account for the effect of gravity, while the terms F_1 and F_2 which are not considered by Asada and Slotine (1986) represent both Coulomb and viscous friction.

If we consider the robot to move in a horizontal plane, we must exclude gravity terms, because this force is orthogonal to the robot motion. If we consider the robot motion in a vertical plane, the gravity terms (27 *e*) and (27 *f*) play a major role in dynamics.

The values of the parameters of the IBM 7535 B 04 robot, computed from the parameter values presented by Geering *et al.* (1986) are given below.

$$
\begin{aligned}
& l_1 = 0{\cdot}4\ \mathrm{m} \quad l_2 = 0{\cdot}25\ \mathrm{m} \quad l_{C2} = 0{\cdot}161\ \mathrm{m} \\
& m_2 = 21\ \mathrm{kg} \quad I_1 + m_1 l_{C1}^2 = 1{\cdot}6\ \mathrm{m^2\,kg} \quad I_2 = 0{\cdot}273\ \mathrm{m^2\,kg} \\
& c_1 = 0{\cdot}05\ \mathrm{N\,m} \quad v_1 = 0{\cdot}025\ \mathrm{N\,m\,s^{-1}} \quad c_2 = 0{\cdot}15\ \mathrm{N\,m} \quad v_2 = 0{\cdot}005\ \mathrm{N\,m\,s^{-1}} \\
& b_1 = 25\ \mathrm{N\,m} \quad b_2 = 9\ \mathrm{N\,m}
\end{aligned} \tag{28}
$$

For the computation of these parameter values, refer to Appendix A.

5.3. *Time-optimal solutions*

The hamiltonian (10) is affine in the controls u_i. Pontryagin's minimum principle then yields that the controls are of bang–bang type or may be singular. In the case where we have a bang–bang control that transfers the manipulator from the initial state to the final state, we are able to check whether or not this bang–bang solution satisfies Pontryagin's minimum principle. In order to find such a bang–bang control, we assume an initial control vector and we assume that the number of switching times equals p. Then we use a parameter optimization method to optimize the p switching times and the final time, using a penalty for deviations from the final state to force the final state to be reached.

Geering *et al.* (1986) treat several types of solutions for the special case in which the links are stretched in both the initial and final configurations. We find exactly the same bang–bang solutions as Geering *et al.* (1986) proving that we are concerning ourselves with exactly the same robot dynamics. To demonstrate this, we give our computations of some of the robot motions presented by Geering *et al.* (1986). Furthermore, we check whether or not the computed solutions satisfy Pontryagin's minimum principle. Some solutions presented by Geering *et al.* (1986) as time-optimal turn out not to be.

In § 4 we noted that for a bang–bang solution, with p switching times, to satisfy Pontryagin's minimum principle we have to consider a non-homogeneous system of $p+1$ linear equations with n unknowns. Therefore, it seems natural to look at first for a bang–bang solution with $n-1$ switching times, for then we have to solve a system of n linear equations with n unknowns; n being 4 in the case of a two-link manipulator. Geering *et al.* (1986) also found time-optimal solutions with three switching times, designated as type A_0.

The actuation torque u_2 of the second link acts on the first link too. If the sign of u_2 is the opposite of the sign of u_1 at the beginning of the robot motion, u_2 increases the accelerating effect of u_1. It seems natural for the initial control vector to have the first component positive and the second component negative. However, we try other initial control vectors as well.

For a robot motion with initial state

$$x_0 = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{29 a}$$

and final state

$$x_f = [0{\cdot}975 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{29 b}$$

assuming three switch times, parameter optimization, performed by the routine BCPOL from the IMSL library, yields the bang–bang control shown in Fig. 2, which transfers the system from the initial to the final state as shown in Fig. 3. Now we have to check whether this bang–bang control satisfies Pontryagin's minimum principle. This is done by calculating an initial co-state following the method given in § 4. Then we numerically integrate the system (6 a), (11) and (19), given $x(t_0)$ and $\lambda(t_0)$ from the initial time to the final time to check whether (19) is satisfied for all $t \in [t_0, t_f]$.

For the robot motion (29) the bang–bang solution satisfies Pontryagin's minimum principle, as can be seen from Fig. 3. In this figure both the optimized bang–bang control and the bang–bang control generated by the switching functions are sketched: they are identical in this case.

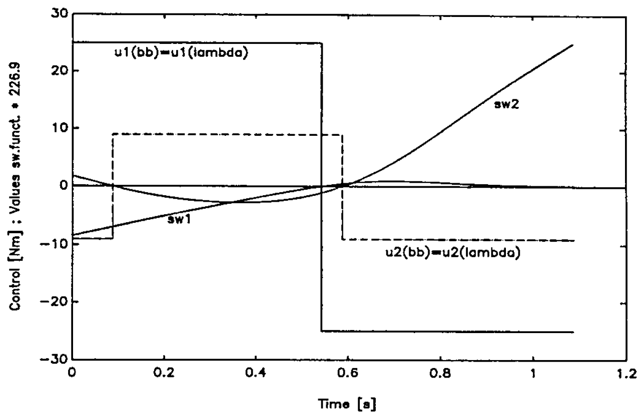


Figure 2. Time-optimal control for the robot motion in Fig. 2; u_1 switches at 0·5423 s, u_2 switches at 0·088 s and 0·588 s and the final time $t_f = 1{\cdot}085$ s. The bang–bang control and the control generated by the switching functions match. The switching functions are scaled as indicated.

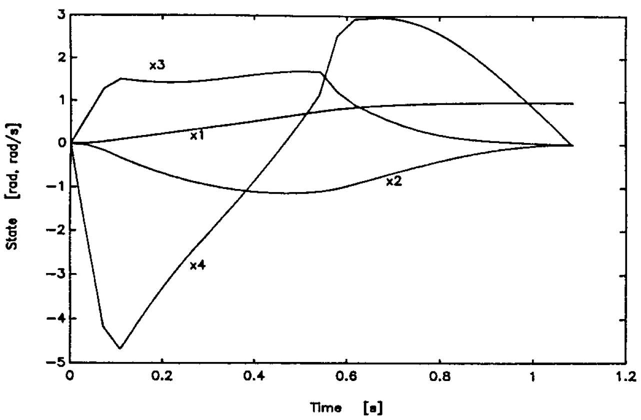


Figure 3. Time-optimal state trajectory for a robot motion with stretched initial and final configuration; $\theta_1(t_f) = 0{\cdot}975$ rad, $\theta_2(t_f) = 0$ rad.

Next we present, in Fig. 4, our results for robot motion with initial state

$$x_0 = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{30 a}$$

and final state

$$x_f = [1{\cdot}5 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{30 b}$$

the control again having three switches. Obviously the computed control and the control generated by the switching function do not match, so this control is not time optimal. The initial co-state, which constitutes the unique solution to the necessary conditions (24) and (25) does not yield the desired bang–bang control.

For $\theta_1(t_f) > 0{\cdot}98$, bang–bang controls with three switches do not satisfy Pontryagin's minimum principle. Next we assume four switching times in order to try and find time-optimal controls which transfer the robot to a configuration with $\theta_1(t_f) > 0{\cdot}98$. Geering *et al.* (1986) found this type of solution and denoted it as type A_1. We now demonstrate that this bang–bang control with two switches for each torque does not satisfy Pontryagin's minimum principle, i.e. these solutions are *not* time-optimal!

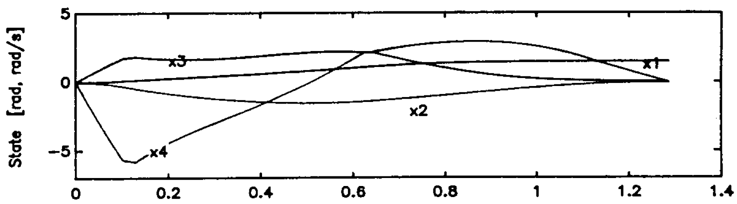
The results for a robot motion with the control switching four times given the initial state

$$x_0 = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{31 a}$$

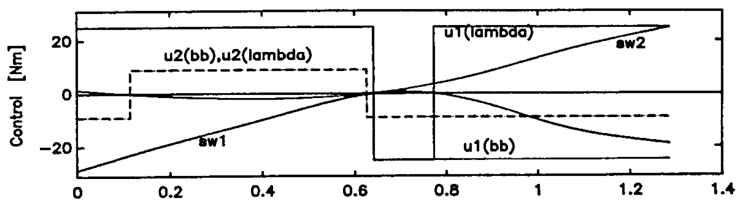
and the final state

$$x_f = [1{\cdot}0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{31 b}$$

are shown in Fig. 5. An initial co-state is calculated from the first three switching times, which determine a unique solution for the initial co-state. Integrating the system yields that the calculated bang–bang solution does not satisfy Pontryagin's minimum principle, as can be seen from Fig. 5(*b*).

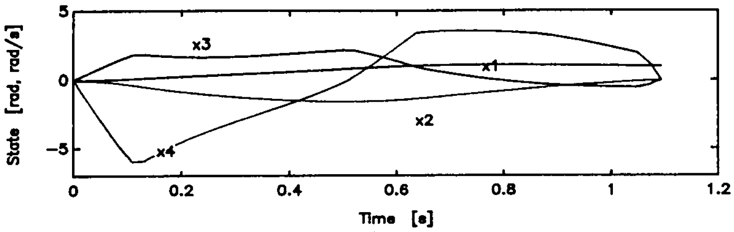


(*a*)

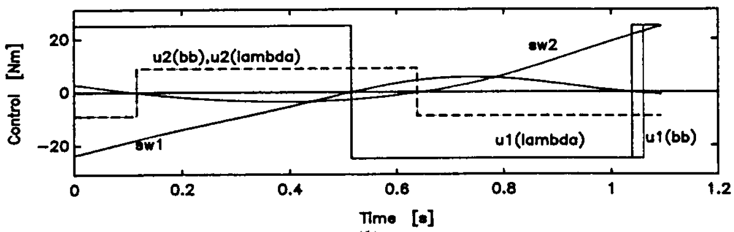


(*b*)

Figure 4. (*a*) State trajectory for a robot motion with stretched initial and final configuration; $\theta_1(t_f) = 1{\cdot}5$ rad, $\theta_2(t_f) = 0$ rad and $t_f = 1{\cdot}28$ s; (*b*) the bang–bang control with three switches and the control generated by the switching functions do not match. The first switching function is scaled by 140·8 and the second switching function by 802·6.



(a)



(b)

Figure 5. (a) State trajectory for a robot motion with stretched initial and final configuration; $\theta_1(t_f) = 1{\cdot}0$ rad, $\theta_2(t_f) = 0$ rad and $t_f = 1{\cdot}09$ s; (b) the bang–bang control with four switches and the control generated by the switching functions do not match. The first switching function is scaled by 225·3 and the second switching function by 675·9.

Next we compute a time-optimal bang–bang control with four switching times. In § 3 we noted that in this case there must be a linear dependency in the non-homogeneous system which determines the initial co-state. We consider a solution with the second link swinging through, u_1 switching three times and u_2 once. Geering *et al.* (1986) denoted this type of solution as B_0. The results are depicted in Fig. 6, for the robot motion from the initial state

$$x_0 = [0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{32 a}$$

to the final state

$$x_f = [0{\cdot}76 \quad -2\pi \quad 0 \quad 0]^{\mathrm{T}} \tag{32 b}$$

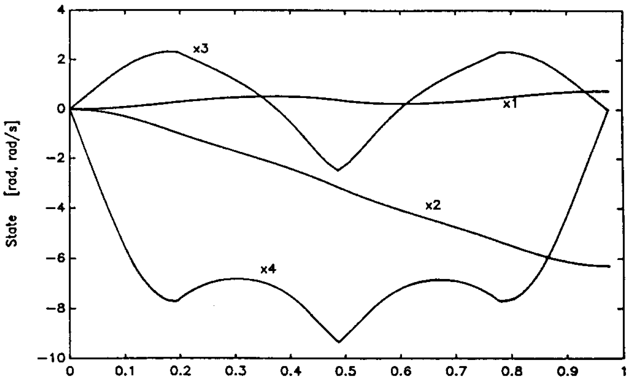


Figure 6. Time-optimal state trajectory for a robot motion with stretched initial and final configuration, the second link swinging through; $\theta_1(t_f) = 0{\cdot}76$ rad, $\theta_2(t_f) = -2\pi$ rad and $t_f = 0{\cdot}975$ s.

As can be seen from Fig. 7, although the solution has more than $n-1$ switching times, the computed bang–bang control with four switching times satisfies Pontryagin's minimum principle! In general the probability that a solution with more than $n-1$ switching times satisfies Pontryagin's minimum principle is almost zero, since the probability of the necessary conditions (24) and (25), for the initial co-state, being linearly dependent is almost zero. Therefore, there must be an explanation of why we find a complete class of time-optimal control problems of type B_0 having time-optimal solutions with n switching times. This can be explained by the symmetry of the switching functions, which causes linear dependence in the necessary conditions (24) and (25) for the initial co-state. The symmetry of the switching functions for time-optimal control problems of type B_0 is explicitly contained in the robot dynamics. For a detailed analysis refer to Appendix B.

Once, however, we introduce friction into the robot dynamics, this symmetry immediately vanishes. We show this by evaluating the same type B_0 robot motion as above, only now with the addition of small Coulomb and viscous friction terms as described in equations (27 g) and (27 h), with parameter values given by equation (28). The results are presented in Fig. 8. In this case the switching functions are only nearly symmetric. The symmetry, which causes the linear dependence in conditions (24) and (25) for the initial co-state, is therefore lost and the solution does not satisfy Pontryagin's minimum principle. Also, if we consider the robot motion in a vertical plane the influence of gravity destroys the symmetry. We wish to make the point here that the solutions of type B_0, when friction and gravity are disregarded, constitute a very special class of time-optimal control problems for which a solution with more than $n-1$ switches satisfies Pontryagin's minimum principle. In other words, if we consider an arbitrary initial and final state, then the probability that the time-optimal control consists of a bang–bang control with more than $n-1$ switches is almost zero.

Therefore, to solve the time-optimal control problem for a given initial and final state, we search for a bang–bang control with no more than $n-1$ switches which

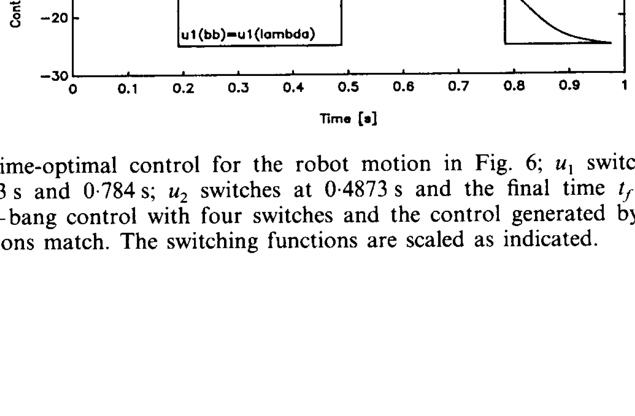
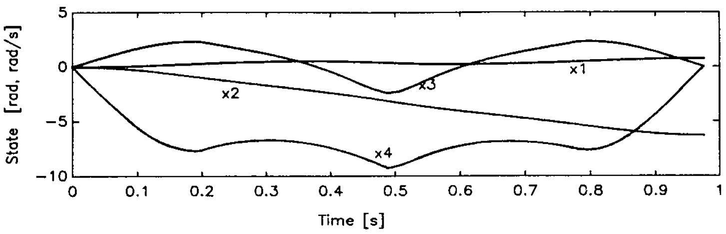
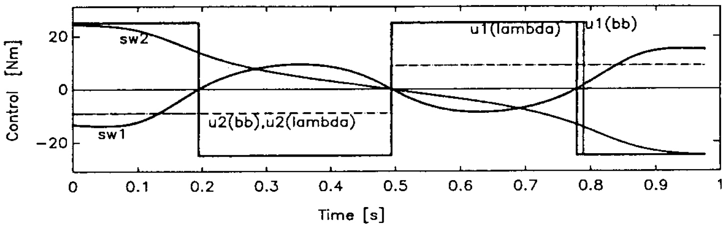


Figure 7. Time-optimal control for the robot motion in Fig. 6; u_1 switches at 0·191 s, 0·4873 s and 0·784 s; u_2 switches at 0·4873 s and the final time $t_f = 0{\cdot}975$ s. The bang–bang control with four switches and the control generated by the switching functions match. The switching functions are scaled as indicated.



(a)



(b)

Figure 8. (a) State trajectory for a robot motion with stretched initial and final configuration with friction, the second link swinging through; $\theta_1(t_f) = 0{\cdot}76$ rad, $\theta_2(t_f) = -2\pi$ rad and $t_f = 0{\cdot}976$ s; (b) the bang–bang control with four switches and the control generated by the switching functions do not match. The first switching function is scaled by 271·6 and the second switching function by 1765·4.

satisfies Pontryagin's minimum principle. If such a bang–bang control cannot be found, the time-optimal control problem is very likely singular.

The procedure described above can be applied to robot motions in vertical planes as well. Although we computed several, we do not actually include examples in which gravity is contained.

6. Sub-time-optimal solutions computed by control parametrization

From § 5 we observe that the time-optimal control problem is non-singular only for a limited class of initial and final states. For singular problems, Pontryagin's minimum principle does not yield an optimal control. In this section we demonstrate that the method of control parametrization can be used to compute sub-optimal controls. The control parametrization will be based on the assumption that the control is of a piecewise constant nature, which is a realistic assumption when using a digital controller. We show that for a non-singular time-optimal control problem, solutions computed by means of control parametrization transfer the manipulator in near minimum time from the initial to the final state. Since, for singular time-optimal control problems, the optimum is very flat, solutions found by control parametrization are expected to be near time-optimal as well.

Consider the system (6) and the cost function (8). The control parametrization is given by

$$u_i(t) = u_i(t_k), \quad t \in [t_k, t_{k+1}), \quad i = 1, \dots, r, \quad k = 0, 1, \dots, N \tag{33}$$

where, although not necessary, we assume t_k are equidistant time instants and $t_{N+1} = t_f$. The controls are assumed to be bounded

$$|u_i(t_k)| \leqslant A_i \quad i = 1, \dots, r, \quad k = 0, 1, N \tag{34}$$

The control parametrization (33) and (34) constitutes a bounded piecewise constant control. The time-optimal control problem now is to find a control $u(t)$ satisfying (33) and (34) which transfers the manipulator from the initial to the final state, such that the cost functional (8) is minimized. As can be observed from (33) the control variable u_i is a function of $r(N+1)$ parameters, the parameters being the amplitudes of the control variable J. By a parameter optimization method, i.e. the routine BCPOL from the IMSL library, the amplitudes are varied in order to find a control which drives the system from the initial to the final state, using a penalty for deviations from the final state in order to force the final state to be reached. A final time is assumed, and when a solution is found the final time is decreased; otherwise it is increased and the process is repeated until changes in the final time become insignificant.

For non-singular time-optimal control problems we are able to compute time-optimal solutions. Therefore the minimum transition time is known. We demonstrate that the method of control parameterization yields near time-optimal controls which differ significantly from the time-optimal bang–bang controls, demonstrating that the cost functional has a weak minimum.

For the robot motion with initial and final states given by (29), we have computed a time-optimal solution with a minimum transition time of 1·085 s, as shown in Figs 2 and 3. We have applied control parametrization using 20 equidistant time-intervals. We find a piecewise constant control which transfers the robot from the initial to the final state in 1·095 s, as shown in Figs 9 and 10.

Next, we applied control parametrization to the singular problem presented in Fig. 4, i.e. robot motion with initial and final states given by (30). The results are depicted in Figs 11 and 12. We observe that the final time found by control parametrization is equal to 1·225 s. Previously we had obtained a bang–bang solution with three switches and a final time of 1·28 s which we proved was not time optimal. Based on type A_1 of Geering *et al.* (1986), a solution type we proved is not time-optimal either, we computed a bang–bang control with four switches yielding a final time of 1·235 s. Control parametrization obviously gives the smallest transition time for this problem.

Summarizing the method of control parametrization seems to be well suited to solve both non-singular and singular problems.

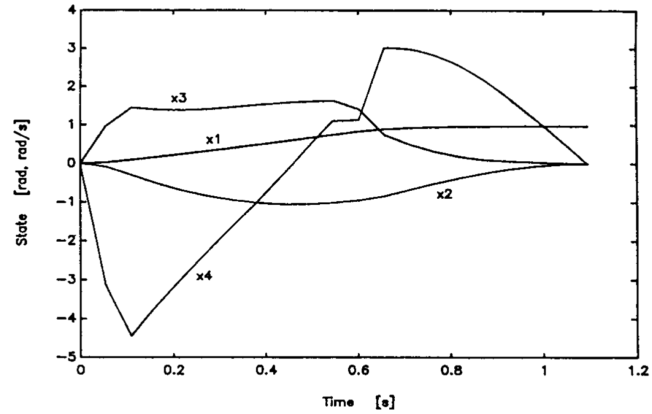


Figure 9. State trajectory for a robot motion with stretched initial and final configuration; $\theta_1(t_f) = 0{\cdot}975$ rad, $\theta_2(t_f) = 0$ rad.

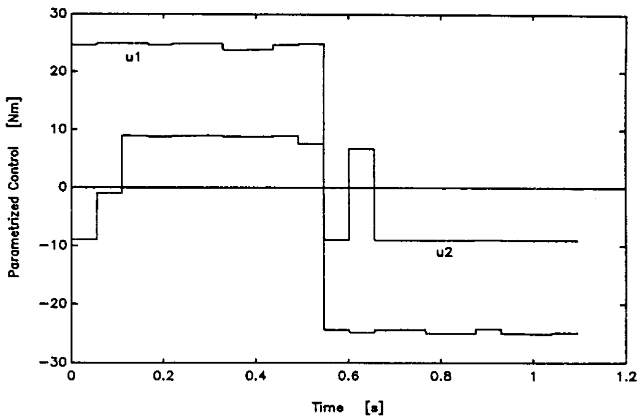


Figure 10. Parametrized control for the robot motion in Fig. 9; the number of time intervals is 20 and the final time $t_f = 1{\cdot}095$ s.

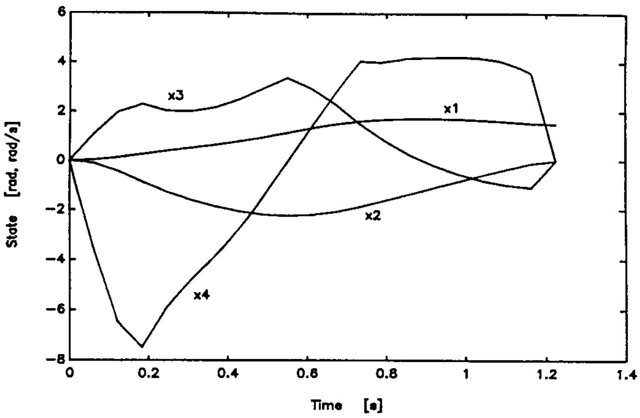


Figure 11. State trajectory for robot motion with stretched initial and final configuration; $\theta_1(t_f) = 1{\cdot}5$ rad, $\theta_2(t_f) = 0$ rad.

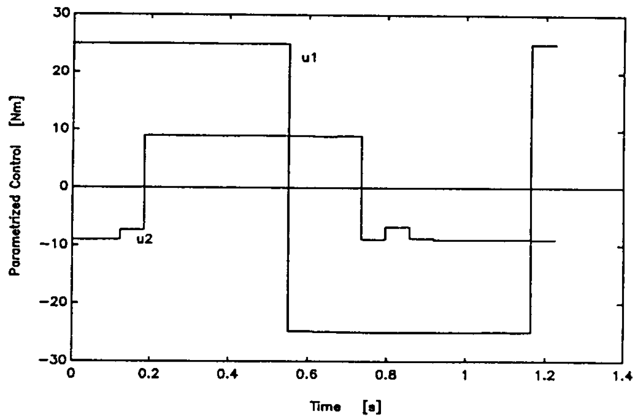


Figure 12. Parametrized control for the robot motion in Fig. 11; the number of time intervals is 20 and the final time $t_f = 1{\cdot}225$ s.

7. Conclusions

For non-linear systems, with linear control, we have presented a new, numerically computable test which determines whether bang–bang solutions are time-optimal, i.e. satisfy Pontryagin's minimum principle. The test reveals the new important fact that if we consider such a system with dimension n, the probability that a certain bang–bang solution 'with more than $n-1$ switches' satisfies Pontryagin's minimum principle is almost zero. Given arbitrary initial and final states we therefore search for bang–bang solutions with up to $n-1$ switches, which transfer the system from the initial to the final state. The search constitutes a parameter optimization procedure in which the parameters are the switch times and the final time. A penalty for deviations from the final state is used to force the final state to be reached. The numerical test is applied to verify whether or not these solutions satisfy Pontryagin's minimum principle. If no solutions with up to $n-1$ switches can be found transferring the system from the initial to the final state while satisfying Pontryagin's minimum principle, the problem is very likely singular.

Our method can be applied to rigid robotic manipulators, where both friction and the effect of gravity may be included in the robot dynamics. We computed time-optimal solutions for an IBM 7535 B 04 robot, which can be modelled as a two-link manipulator. We demonstrated that for this robot some numerical solutions mentioned in the literature to satisfy Pontryagin's minimum principle do not. Furthermore, we demonstrated that for non-singular problems a method based on control parametrization generates sub-optimal solutions that are very close to the optimum for this robot. Since for singular problems the minimum is very flat, in these cases we also expect the method based on control parametrization to generate sub-optimal solutions that are very close to the optimum. Summarizing—the method of control parametrization seems to be very well suited to solve general time-optimal control problems for rigid manipulators. The method explicitly assumes the control to be piecewise constant, which is a realistic assumption, since robots are controlled by digital computers. Bounds on the individual link velocities, which have to be considered in practice as well, are also easily included in the problem. The solutions are in open-loop form but, when conservative bounds on the control variables are used, the solutions can be implemented in conjunction with a perturbation controller, the result yielding a time-optimal feedback controller (Van Willigenburg 1990 b).

We did not treat all the details involved in the computation of the numerical test to verify whether bang–bang solutions satisfy Pontryagin's minimum principle. In a future paper we plan to treat the numerical computation of the test in detail, together with the influence of numerical errors. For instance, the numerical determination of whether or not a non-homogeneous set of equations is linearly dependent presents a problem. Furthermore, the errors introduced by numerical integration, which plays a crucial role in the computation of the numerical test, have to be considered. Questions concerning numerical errors are related to questions concerning the sub-optimality of solutions. Questions concerning sub-optimality are interesting since they may answer the question of to what extend solutions generated by control parametrization are optimal. Furthermore, they are of interest since, in practice the application of bang–bang controls increases wear. One generally prefers a more smooth control. The question concerning the sub-optimality of solutions will also be a subject of future research.

Appendix A

We can easily include the effects of the third link, the gripper and the load in the closed form dynamics of the two-link manipulator by integrating the third link, the gripper and the load in the second link. Since the third link moves perpendicular to the second link the centroid of the third link, the gripper and the load may be located at the end of the second link. Assume the third link, the gripper and the load have a total mass m_3 and a moment of inertia I_3. The second link as a whole will have a mass m'_2, a centroid l'_{C2} and a moment of inertia I'_2, which can all be calculated from the masses and moments of inertia of the second and third links together with the gripper and the load by the application of Steiner's translation theorem.

Application of Steiner's theorem yields

$$m'_2 = m_2 + m_3 \tag{A 1}$$

$$l'_{C2} = \frac{m_2 l_{C2} + m_3 l_3}{m_2 + m_3} \tag{A 2}$$

$$I'_2 = I_2 + I_3 + \frac{m_2 m_3}{m_2 + m_3}(l_2 - l_{C2})^2 \tag{A 3}$$

Equations (26) and (27) still hold in the case of a third link with a gripper and a load at the end of the second link, but must be transformed by setting $m_2 = m'_2$, $l_{C2} = l'_{C2}$ and $I_2 = I'_2$.

Geering *et al.* (1986) use the moment of inertia ξ with respect to the joint axes, while I in our equations denotes the moment of inertia about the centroid of the link. These descriptions can be related using Steiner's theorem. For example, we have for the first link

$$\xi_1 = I_1 + m_1 l_{C1}^2 \tag{A 4}$$

As can be seen in (26) and (27), the parameter m_1 does not occur in the closed-form dynamics when we consider the robot motion in a horizontal plane and we use (A 4). If, however, we consider the robot motion in a vertical plane, m_1 should be known since it occurs in the gravity term.

Geering *et al.* (1986) located the centre of gravity of the second link in the middle of the second link, so

$$l_{C2.} = \tfrac{1}{2}(l_2) \tag{A 5}$$

The above yields the following numerical values for the IBM 7535 B 04 robot in terms of the closed-form dynamics of a two-link robot.

$$\begin{array}{llll} l_1 = 0{\cdot}4\ \mathrm{m} & l_2 = 0{\cdot}25\ \mathrm{m} & l'_{C2} = 0{\cdot}161\ \mathrm{m} & \\ m'_2 = 21\ \mathrm{kg} & \xi_1 = 1{\cdot}6\ \mathrm{m^2\,kg} & I'_2 = 0{\cdot}273\ \mathrm{m^2\,kg} & \\ c_1 = 0{\cdot}05\ \mathrm{N\,m} & v_1 = 0{\cdot}025\ \mathrm{N\,m\,s^{-1}} & c_2 = 0{\cdot}15\ \mathrm{N\,m} & v_2 = 0{\cdot}005\ \mathrm{N\,m\,s^{-1}} \\ b_1 = 25\ \mathrm{N\,m} & b_2 = 9\ \mathrm{N\,m} & & \end{array} \tag{A 6}$$

Friction will play only a minor role in the dynamics, as can be seen from the values in (A 6). In modern robots, in which significant gearing is typical, friction forces can be actually quite large, up to 25% of the torque required to move the manipulator (Craig 1986).

Appendix B

In this appendix we demonstrate that the symmetry and anti-symmetry in Figs 6 and 7 are explicitly contained in the robot dynamics for the robot motion of type B_0, where gravity and friction terms are not taken into account.

From the closed form dynamics (26) we observe

$$\ddot{\theta}_1 = f_1(\theta_2, \dot{\theta}_1, \dot{\theta}_2, \tau) \tag{B 1}$$

$$\ddot{\theta}_2 = f_2(\theta_2, \dot{\theta}_1, \dot{\theta}_2, \tau) \tag{B 2}$$

By inspection of the closed form dynamics we have

$$f_1(-\pi + \theta_2, \dot{\theta}_1, \dot{\theta}_2, \tau) = -f_1(-\pi - \theta_2, \dot{\theta}_1, \dot{\theta}_2, -\tau) \tag{B 3}$$

$$f_2(-\pi + \theta_2, \dot{\theta}_1, \dot{\theta}_2, \tau) = -f_2(-\pi - \theta_2, \dot{\theta}_1, \dot{\theta}_2, -\tau) \tag{B 4}$$

Starting at a configuration where $\theta_2 = -\pi$, (B 1)–(B 4) imply that the future behaviour of $\dot{\theta}_1$ and $\dot{\theta}_2$ equal the past behaviour, if the controls have opposite signs. This explains the symmetry of $\dot{\theta}_1$ and $\dot{\theta}_2$ and the anti-symmetry of θ_1 and θ_2 in Figs 6 and 7.

From the co-state equation (11), using (B 1) and (B 2), we obtain the following equations for the co-state.

$$\left.\begin{aligned} \dot{\lambda}_1 &= 0 \\ -\dot{\lambda}_2 &= \frac{\delta f_1}{\delta \theta_2}\lambda_3 + \frac{\delta f_2}{\delta \theta_2}\lambda_4 \\ -\dot{\lambda}_3 &= \lambda_1 + \frac{\delta f_1}{\delta \dot{\theta}_1}\lambda_3 + \frac{\delta f_2}{\delta \dot{\theta}_1}\lambda_4 \\ -\dot{\lambda}_4 &= \lambda_2 + \frac{\delta f_1}{\delta \dot{\theta}_2}\lambda_3 + \frac{\delta f_2}{\delta \dot{\theta}_2}\lambda_4 \end{aligned}\right\} \tag{B 5}$$

The symmetry of $\dot{\theta}_1$ and $\dot{\theta}_2$ and the anti-symmetry of f_1, f_2 and θ_2 with respect to $\theta_2 = -\pi$ imply the symmetry of λ_1 and λ_2 and the anti-symmetry of λ_3 and λ_4, assuming λ_3 and λ_4 are equal to zero when $\theta_2 = -\pi$. When $\theta_2 = -\pi$ both control variables switch, which implies that both switching functions must be equal to zero. Given (4), (10) and (13) this implies that

$$[\lambda_3 \quad \lambda_4]M^{-1} = 0 \tag{B 6}$$

Since M^{-1} is a positive definite matrix this implies that both $\lambda_3 = 0$ and $\lambda_4 = 0$ when $\theta_2 = -\pi$.

Finally from the closed form dynamics (26) we observe that

$$M(-\pi + \theta_2) = M(-\pi - \theta_2) \tag{B 7}$$

So M is symmetric with respect to $\theta_2 = -\pi$ and therefore M^{-1} is also. The symmetry of M^{-1} and the anti-symmetry of λ_3 and λ_4 imply that the switching functions are anti-symmetric. Given this anti-symmetry, a switching point on one side of $\theta_2 = -\pi$ automatically implies a switching point on the other side.

Summarizing—the linear dependency of the non-homogeneous system of $p + 1$ equations for the initial co-state is explicitly contained in the robot dynamics.

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