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First order controllability and the time optimal control problem for rigid articulated arm robots with friction

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For non-linear systems, linear in the control, a relationship between the singularity of the time optimal control problem and the differential controllability of the linearized dynamics along a time optimal solution that satisfies the minimum principle is established. Known results concerning the solution of the time optimal control problem for a two link planar articulated arm robot without friction are extended to a general rigid articulated arm robot with friction. Finally, these results are used to prove differential controllability of the linearized dynamics (called first order controllability) along any trajectory of rigid articulated arm robots with friction.

1. Introduction

The production of an assembly line is determined by the speed of operation of parts operating in the line. If a robot is part of an assembly line its speed of operation may determine the production rate. The most common operation performed by robots is 'pick and place'. Therefore performing a pick-and-place operation in the minimal time is a very important robot motion control problem. This problem is a special case of the time optimal robot motion control problem with fixed initial and final states and with bounded control, which is treated here. In recent years, a number of articles on this problem have appeared. They can be divided into two categories. In one category, solutions are based on linear models (obtained in different ways under different assumptions) approximating the non-linear robot dynamics (Khan and Roth 1971, Kim and Shin 1985, Roodhart *et al.* 1987). In the other category, the 'true' non-linear dynamics of the rigid robot are considered (neglecting friction, however). Very often results are stated for the simplest articulated arm robot, i.e. a planar arm consisting of two links (Ailon and Langholz 1985, Sontag and Sussman 1985, Geering *et al.* 1986, Wen 1986).

Here we use the 'true' non-linear dynamics of a general articulated arm robot in which friction is modelled by possibly state dependent viscous and Coulomb forces. We show by simple proofs that results found in the literature (Ailon and Langholz 1985, Sontag and Sussman 1985, Wen 1986) concerning the existence and form of the solution to the time optimal control problem also apply to this general case. The paper first presents a relationship between the singularity of the time optimal control problem and the differential controllability of the linearized dynamics about a time optimal trajectory that satisfies the minimum principle. This relationship is finally used to prove that the linearized robot dynamics along any trajectory, are differentially controllable. This is a very important result since a general approach to control a non-linear system consists of controlling the system about an off-line-determined state trajectory and open-loop control, using a so-called perturbation controller

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(Athans 1971). The design of this perturbation controller is based on the linearized dynamics about the off-line-determined solution. The main condition to apply this approach successfully is that the linearized dynamics be differentially controllable, since this guarantees that all deviations from the trajectory can be controlled to zero in an arbitrarily small-time.

2. Deterministic non-linear minimum time problem with bounded control

We first consider a deterministic non-linear optimal control problem with bounded control. Given the non-linear system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \tag{1}$$

and the cost criterion to be minimized

$$J(t_0) = \Phi(\mathbf{x}(T), T) + \int_{t_0}^{T} L(\mathbf{x}, \mathbf{u})\, dt \tag{2}$$

the final state constraint

$$\psi(\mathbf{x}(T), T) = 0 \tag{3}$$

and bounds on the control

$$a_i \leqslant u_i \leqslant b_i \quad i = 1, \ldots, M \tag{4}$$

where M is the dimension of the control $\mathbf{u}$. The hamiltonian H of this problem is by definition

$$H(\mathbf{x}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\mathrm{T}} f(\mathbf{x}, \mathbf{u}) \tag{5}$$

where $\boldsymbol{\lambda}$ is the costate. The minimum principle states that the optimal control minimizes the hamiltonian with respect to the control. We further consider the special case where the hamiltonian (5) is linear in the control, i.e. both $f(\mathbf{x}, \mathbf{u})$ and $L\ \mathbf{x}, \mathbf{u})$ are assumed to be linear in the control:

$$f(\mathbf{x}, \mathbf{u}) = f_1(\mathbf{x}) + f_2(\mathbf{x})\mathbf{u} \tag{6}$$

$$L(\mathbf{x}, \mathbf{u}) = L_1(\mathbf{x}) + L_2(\mathbf{x})\mathbf{u} \tag{7}$$

In this case, the hamiltonian equals

$$H = L_1 + \boldsymbol{\lambda}^{\mathrm{T}} f_1 + (\boldsymbol{\lambda}^{\mathrm{T}} f_2 + L_2)\mathbf{u} \tag{8}$$

To minimize the hamiltonian (8) with respect to $\mathbf{u}$ given the bounds (4) on the control, $\mathbf{u}$ should be selected as

$$u_i = a_i \quad \text{if } (\boldsymbol{\lambda}^{\mathrm{T}} f_2 + L_2)_i > 0 \tag{9}$$

$$u_i = b_i \quad \text{if } (\boldsymbol{\lambda}^{\mathrm{T}} f_2 + L_2) < 0 \tag{10}$$

The component u_i of the control vector $\mathbf{u}$ takes on either its maximum or minimum value dependent on the sign of the so-called corresponding switching function $(\boldsymbol{\lambda}^{\mathrm{T}} f_2 + L_2)_i$. If the switching function equals zero for some time t, (9) and (10) do not determine the corresponding control variable. If this happens only at isolated time instants, the corresponding control variable may switch at these instants from its maximum to its minimum value, or vice versa. the problem is called *regular* in this case. When, on the other hand, a time interval exists during which one or several switching functions equal zero, (9) and (10) no longer determine a meaningful solution. The problem is called *singular* in this case.

The problem in our case is a minimum time problem, so

$$L = 1 \tag{11}$$

and therefore

$$L_1 = 1 \tag{12}$$

$$L_2 = 0 \tag{13}$$

Given (7) and (13), the switching functions in (9) and (10) simplify to

$$(\boldsymbol{\lambda}^{\mathrm{T}} f_2)_i \tag{14}$$

Definition 2.1

A minimum time problem, linear in the control, is called *singular in u_i over* (t_1, t_2), $t_1 < t_2$, if for all $t \in (t_1, t_2)$ the corresponding switching function (14) equals zero.

The linearization of (1) along a trajectory $\mathbf{x}(t)$, $t \in [0, T]$ is given by

$$\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) \tag{15}$$

where

$$A(t) = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}(t),\,\mathbf{u}=\mathbf{u}(t)} \tag{16}$$

$$B(t) = \left.\frac{\partial f}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}(t),\,\mathbf{u}=\mathbf{u}(t)} \tag{17}$$

The system (15) represents the first order dynamics of (1) along the trajectory. The dynamics of small deviations from the trajectory can be very well approximated by these first order dynamics. Since the system (15) is fully determined by $A(t)$ and $B(t)$, one often refers to $A(t)$, $B(t)$ being the linearization of (1) about the trajectory. Note that if the system (1) is linear in the control, the linearization $A(t)$, $B(t)$ is independent of $\mathbf{u}(t)$, so it depends only on $\mathbf{x}(t)$.

Lemma 2.1

A non-linear minimum time problem, linear in the control, (6), (7), (8), (12), (13) is singular in u_i over (t_1, t_2), $t_1 < t_2$, if and only if for all $t \in (t_1, t_2)$ the following holds true:

$$-\dot{\boldsymbol{\lambda}} = A^{\mathrm{T}}(t)\,\boldsymbol{\lambda} \tag{18}$$

$$B_i^{\mathrm{T}}(t)\boldsymbol{\lambda} = 0 \tag{19}$$

where $A(t)$, $B(t)$ is the linearization and $\boldsymbol{\lambda}(t)$ the costate at time $t \in (t_1, t_2)$ for a time optimal trajectory satisfying the minimum principle. B_i is by definition the ith column of B.

Proof

Equation (18) is the costate equation of the minimum time problem which holds everywhere along a time optimal trajectory satisfying the minimum principle. Equation (19) states that for time $t \in (t_1, t_2)$ the switching function corresponding to u_i equals zero. □

3. Differential controllability and reconstructibility of linear varying systems

To establish first order controllability of a non-linear system along a trajectory, we have to introduce a special kind of controllability for time varying systems called *differential controllability* (Chen 1970), or *full controllability* (Hermes and La Salle 1969). To show the analogy and differences between complete controllability, and differential controllability we first state the well-known definitions of complete controllability each followed by analogous definitions of differential controllability. Since, for a linear time varying system, reconstructibility is dual to controllability, we follow the same approach to introduce differential reconstructibility. All definitions refer to a general, finite dimensional linear time varying system (LTVS), given by

$$-\infty < t < +\infty \tag{20}$$

$$\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) \tag{21}$$

$$\mathbf{y}(t) = C(t)\,\mathbf{x}(t) \tag{22}$$

Definition 3.1

An LTVS is called *completely controllable at time* t_1 if time t_2 exists such that $t_2 > t_1$, $t_2 - t_1$ is finite, and each state $\mathbf{x}(t_1)$ can be transformed to each $\mathbf{x}(t_2)$.

Definition 3.2

An LTVS is called *differentially controllable* at time t_1, if $t_2 - t_1$ can be made arbitrarily small, $t_2 > t_1$, and each state $\mathbf{x}(t_1)$ can be transferred to each $\mathbf{x}(t_2)$.

Definition 3.3

An LTVS is called *completely controllable* if it is completely controllable at each time t.

Definition 3.4

An LTVS is called *differentially controllable* if it is differentially controllable at each time t.

Definitions 3.1 and 3.3 constitute the well-known definition of complete controllability, which means that any state can always be transferred to any other state, or, equivalently, any state can be transferred to the zero state in finite time. Differential controllability means that any state can always be transferred to any other state in an arbitrary small amount of time. The above definitions hold for the LTVS (20), (21) and (22). However, definitions concerning differential controllability also apply to an LTVS which is defined over a finite time interval $t \in [t_s, t_f]$. In this case, t in Definition 3.4 is restricted to the interval $[t_s, t_f)$.

Lemma 3.1

If an LTVS is completely controllable but not differentially controllable, at least one time interval (t_1, t_2) exists where the LTVS is differentially controllable for no $t \in (t_1, t_2)$.

Proof

If the LTVS is completely but not differentially controllable, then, according to Definitions 3.3 and 3.4, there exists at least one time t_1 for which it takes a non-arbitrarily small, but finite, time to control all $\mathbf{x}(t_1)$ to zero. Consider the earliest time $t_2 > t_1$ for which all states $\mathbf{x}(t_1)$ can be controlled to zero, and the time interval (t_1, t_2). So there must be at least one state $\mathbf{x}'(t_1)$ that can be transferred to zero no sooner than t_2. Consider the transfer $\mathbf{x}'(t)$, $t \in [t_1, t_2]$ of $\mathbf{x}'(t_1)$ to $\mathbf{x}'(t_2) = 0$. Clearly, for all $t \in (t_1, t_2)$ $\mathbf{x}'(t) \neq 0$ and cannot be controlled to zero sooner than t_2, so not in an arbitrarily short time. □

Considering Lemma 3.1, we state the following definition.

Definition 3.5

The LTVS is called *differentially uncontrollable over* (t_1, t_2) if for all $t \in (t_1, t_2)$ the LTVS is not differentially controllable.

Lemma 3.2

For an LTVS, the possibility to control $\mathbf{x}(t_1)$ to $\mathbf{x}(t_2) = 0$ is equivalent to the possibility of controlling $\alpha \cdot \mathbf{x}(t_1)$ to $\mathbf{x}(t_2) = 0$, α real and bounded.

Proof

Consider the response of the LTVS to $\mathbf{u}(\tau)$, $\tau \in [t_1, t_2]$

$$\mathbf{x}(t_2) = \Phi(t_2, t_1)\mathbf{x}(t_1) + \int_{t_1}^{t_2} \Phi(t_2, t_1)B(\tau)\mathbf{u}(\tau)\, d\tau \tag{23}$$

Clearly the transfer $\mathbf{x}(t_1)$ to $\mathbf{x}(t_2) = 0$ being realized by $\mathbf{u}(\tau)$ is equivalent to the transfer of $\alpha \cdot \mathbf{x}(t_1)$ to $\mathbf{x}(t_2) = 0$ being realized by $\alpha \cdot \mathbf{u}(\tau)$. □

Lemma 3.3

Consider again the time interval (t_1, t_2) in Lemma 3.1. For each $t \in (t_1, t_2)$, the earliest time for which all states $x(t)$, $\|x(t)\| \leqslant \varepsilon$, $\varepsilon > 0$ and small, i.e. all states in a neighbourhood of the zero state, can be controlled to the zero state, is t_2.

Proof

Lemma 3.1, and Lemma 3.2 for small enough α, imply Lemma 3.3. □

Lemma 3.3 reveals an important property of an LTVS that is completely but not differentially controllable. For such a system there always exists a time interval (t_1, t_2) during which it is impossible to control any state in the neighbourhood of the zero state, or equivalently any other state, to that state. Loosely speaking, the LTVS is temporarily uncontrollable.

It is well known that the controllability of an LTVS is equivalent to reconstructibility of the dual system

$$\dot{\mathbf{x}}(t) = A^{\mathrm{T}}(t^* - t) + C^{\mathrm{T}}(t^* - t)\mathbf{u}(t) \tag{24}$$

$$\mathbf{y}(t) = B^{\mathrm{T}}(t^* - t)\mathbf{x}(t) \tag{25}$$

This suggests the possibility of introducing differential reconstructability dual to differential controllability.

Definition 3.6

An LTVS is called *completely reconstructible at time* t_2 if $t_1 < t_2$ exists such that $\mathbf{u}(t) = 0$, $\mathbf{y}(t) = 0$, $t_1 < t < t_2$ implies $\mathbf{x}(t_1) = 0$, and $t_2 - t_1$ is finite.

Definition 3.7

An LTVS is called *differentially reconstructible at time* t_2 if $t_2 - t_1$ can be made arbitrarily small, $t_1 < t_2$, such that $\mathbf{u}(t) = 0$, $\mathbf{y}(t) = 0$, $t_1 < t < t_2$ implies $x(t) = 0$.

Definition 3.8

An LTVS is *completely reconstructible* if it is reconstructible for all time t.

Definition 3.9

An LTVS is *differentially reconstructible* if it is differentially reconstructible for all time t.

Analogous to Definition 3.5, we introduce differential unreconstrucibility dual to differential uncontrollability.

Definition 3.10

An LTVS is *differentially unreconstructible over* (t_1, t_2) if for all $t \in (t_1, t_2)$ the LTVS is not differentially reconstructible.

Definitions 3.6 and 3.8 are based on a theorem concerning complete reconstructibility for an LTVS (Kwakernaak and Sivan 1972). Complete reconstructibility means that measurements over a finite time interval in the past always completely determine the current state. Differential reconstructibility means that measurements over an arbitrarily small time interval in the past completely determine the current state. Since aspects of controllability and reconstructibility for linear time varying systems are completely determined by the controllability and reconstructibility gramian, which for the LTVS and its dual system are the same (Johnson 1985), it can easily be seen that differential controllability is indeed dual to differential reconstructibility, as defined above.

Lemma 3.4

If for all $t \in (t_1, t_2)$, the following hold:

$$\mathbf{x}(t) \neq 0 \tag{26}$$

$$B^{\mathrm{T}}(t)\mathbf{x}(t) = 0 \tag{27}$$

$$\dot{\mathbf{x}}(t) = A^{\mathrm{T}}(t)\mathbf{x}(t) \tag{28}$$

then the system $A(t)$, $B(t)$ is differentially uncontrollable, and equivalently the dual system $B^{\mathrm{T}}(t)$, $A^{\mathrm{T}}(t)$ differentially unreconstructible over (t_1, t_2).

Proof

For all $t \in (t_1, t_2)$ we may consider an interval $(t, t+\varepsilon)$, $t+\varepsilon \leqslant t_2$ within (t_1, t_2). So, (20), (21), (22) hold over $(t, t+\varepsilon)$, which by Definitions 3.7 and 3.9 implies that, for all $t \in (t_1, t_2)$, $B^{\mathrm{T}}(t)$ and $A^{\mathrm{T}}(t)$ are not differentially reconstructible. Hence, by Definition 3.10, this implies Lemma 3.4. □

4. First order controllability of a non-linear system along a trajectory

Consider the system (6), which is linear in the control, with given initial state, and bounded control.

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{29}$$

$$\mathbf{u}(t) \in U \tag{30}$$

Define the attainable set $\Omega(\mathbf{x}_0, T)$ from $\mathbf{x}_0$ consisting of all solutions to (6), (29), (30) for some finite time T

$$\Omega(\mathbf{x}_0, T) = \{\mathbf{x}(t) \text{ solves (6), (29), (30), } 0 < t < T < \infty\} \tag{31}$$

The formula $\Omega(\mathbf{x}_0, T)$ can be considered both as the set of all possible trajectories $\mathbf{x}(t)$, $t \in [0, T]$, $\mathbf{x}(0) = \mathbf{x}_0$ and as the set of states $\mathbf{x}$ that can be reached from $\mathbf{x}_0$ within time T.

Consider the linearization (16), (17) along any trajectory from $\Omega(\mathbf{x}_0, T)$.

Definition 4.1

Any trajectory from $\Omega(\mathbf{x}_0, T)$ is called *first order controllable* if the linear time varying system $A(t)$, $B(t)$, $t \in [0, T]$ determined by the linearization about the trajectory $\mathbf{x}(t)$, is differentially controllable.

A well-known approach to control a non-linear system is to use a linear perturbation controller to control the system about an off-line determined trajectory and open-loop control. The perturbation controller has a design based on a first order approximation of the non-linear dynamics, i.e. the linearization (16), (17) about the trajectory (Athans 1971). The main condition for successfully applying this approach is that this linearization, which constitutes a time varying linear system, is differentially controllable. For this implies any deviation can be controlled to zero in an arbitrarily short time. We use the term first order controllability since controllability along a trajectory is defined using the complete non-linear dynamics (Hermes 1976). Sufficient conditions for controllability along a trajectory are presented in this paper. They coincide with what we call first order controllability. First order controllability, therefore, implies controllability along a trajectory.

Definition 4.2

Any trajectory from $\Omega(\mathbf{x}_0, T)$ is called *first order controllable from u_i* if the time varying linear single input system $A(t)$, $B_i(t)$ is differentially controllable, where $A(t)$, $B(t)$ is the linearization about the trajectory and i again refers to the ith column.

The definition refers to first order controllability only if u_i is used to control the system about the trajectory. Since u_i in general is not the only control variable, the trajectory being first order uncontrollable from u_i does not mean the trajectory is first order uncontrollable. For it can be first order controllable from another, or a combination of other, control variables. If, however, the trajectory is first order controllable from one of the control variables, it is first order controllable.

Definition 4.3

The linear time varying system $A(t)$, $B(t)$, $t \in [0, T]$ is called *differentially uncontrollable from u_i over (t_1, t_2)*, $0 < t_1 < t_2 < T$ if $A(t)$, $B_i(t)$ is differentially uncontrollable, over (t_1, t_2) where i refers to the ith column of B.

As we can see from Lemma 2.1 and Lemma 3.4, the conditions for the linearized system to be differentially uncontrollable from u_i over (t_1, t_2) are almost the same as for the minimum time problem to be singular in u_i over (t_1, t_2). There is a sign difference between (18) and (28) which is, however, unimportant, since the controllability of $A(t)$, $B(t)$ is equivalent to the controllability of $-A(t)$, $B(t)$. Furthermore it can be proved (Sage and White 1977) that, for a minimum time problem that is linear in the control with fixed initial and final states,

$$H \equiv 0 \tag{32}$$

everywhere along a time optimal trajectory that satisfies the minimum principle.

Given (5), (11) we can write the following general expression for the hamiltonian:

$$H = 1 + \boldsymbol{\lambda}^{\mathrm{T}} f \tag{33}$$

Given (32), (33) everywhere along a time optimal trajectory

$$\boldsymbol{\lambda} \neq 0 \tag{34}$$

which matches the condition $\mathbf{x}(t) \neq 0$ in Lemma 3.4.

Theorem 4.1

A minimum time problem, linear in the control, with fixed initial and final states being singular in u_i over (t_1, t_2), $t_1 < t_2$, along a time optimal trajectory satisfying the minimum principle, is equivalent to the locally linearized system about the time optimal trajectory being differentially uncontrollable from u_i over (t_1, t_2).

Theorem 4.1 states that the conditions stated in Lemma 2.1 and Lemma 3.4 are in fact equivalent, as we have shown. The only difference is that Lemma 3.4 refers to any trajectory, and so to all sets $\Omega(\mathbf{x}_0, T)$ defined by (29), (30), (31), for all $T$and $\mathbf{x}_0$. Lemma 3.4 refers only to time optimal trajectories which, within each $\Omega(\mathbf{x}_0, T)$, form a subset. This is also reflected by the fact that $\boldsymbol{\lambda}$ in Lemma 2.1 is the costate resulting from the minimum time problem, whereas $\mathbf{x}$ in Lemma 3.4 may be any state.

5. Time optimal control problem and first order controllability for articulated arm robots with friction

The dynamics of an N link articulated arm robot manipulator without friction can be written (Craig 1986) as

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + G(\boldsymbol{\theta}) \tag{35}$$

where

$$\boldsymbol{\theta} = (\theta_1 \quad \theta_2 \quad \dots \quad \theta_N)^{\mathrm{T}}, \quad = (\tau_1 \quad \tau_2 \quad \dots \quad \tau_N)^{\mathrm{T}}$$

$\theta_1, \ldots, \theta_N$ are the joint angles of the links and $\tau_1, \ldots, \tau_N$ are the actuation torques. $M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}$ represents the inertial forces where M is an $N \times N$ positive definite mass matrix dependent on link positions. $V(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is an $N \times 1$ vector dependent on link positions and speeds representing centrifugal and Coriolis forces. $G(\boldsymbol{\theta})$ is an $N \times 1$ vector depending on link positions representing gravity forces. The actuation torques are considered to be the control variables of the system. If for instance the robot is actuated by current controlled DC motors, and we neglect flexibility and play in the transmission, the motor current is proportional to the actuation torque. Equation (35) is a non-linear system with a natural choice of state vector being $(\boldsymbol{\theta}^{\mathrm{T}}, \dot{\boldsymbol{\theta}}^{\mathrm{T}})^{\mathrm{T}}$. Friction effects generate damping forces dependent on the positions and speeds of the links. This means all friction effects can be modelled in (35) by introducing a term $\mathbf{F}$ on the right-hand side dependent on $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ (Craig 1986).

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + V(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + G(\boldsymbol{\theta}) + F(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \tag{36}$$

We can represent the terms V, G, F in (36) by a single term T dependent on $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$

$$\boldsymbol{\tau} = M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + T(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \tag{37}$$

Equation (37) is merely a symbolic notation which gives information about the general form of the dynamics of a rigid articulated arm manipulator with friction. However, in the rest of the paper it proves to be sufficient to use the simple form (37) without exact knowledge of the dynamics. To analyse the problem we first write (37) in state space form. Since $M(\theta)$ is positive definite, (37) can be written as

$$\ddot{\boldsymbol{\theta}} = M^{-1}(\boldsymbol{\theta})[\boldsymbol{\tau} - T(\boldsymbol{\theta}, \ddot{\boldsymbol{\theta}})] \tag{38}$$

Introducing

$$\mathbf{x}_1 = \boldsymbol{\theta} \tag{39}$$

$$\mathbf{x}_2 = \dot{\boldsymbol{\theta}} \tag{40}$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{41}$$

$$B = M^{-1}(\theta) \tag{42}$$

$$\mathbf{u} = \boldsymbol{\tau} \tag{43}$$

we can write (38) in state space form using (39), ..., (43)

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2 \tag{44}$$

$$\dot{\mathbf{x}}_2 = B(\mathbf{x}_1)T(\mathbf{x}) + B(\mathbf{x}_1)\mathbf{u} \tag{45}$$

If we partition the costate vector according to (41), the hamiltonian for the system (44), (45) equals

$$H = 1 + \boldsymbol{\lambda}_1^{\mathrm{T}}\mathbf{x}_2 + \boldsymbol{\lambda}_2^{\mathrm{T}}[B(\mathbf{x}_1)T(\mathbf{x}) + B(\mathbf{x}_1)\mathbf{u}] \tag{46}$$

and the costate equation is partitioned as

$$-\dot{\boldsymbol{\lambda}}_1 = \partial[\boldsymbol{\lambda}_1^{\mathrm{T}}\mathbf{x}_2 + \boldsymbol{\lambda}_2^{\mathrm{T}}B(\mathbf{x}_1)T(\mathbf{x}) + \boldsymbol{\lambda}_2^{\mathrm{T}}B(\mathbf{x}_1)\mathbf{u}]/\partial\mathbf{x}_1 \tag{47}$$

$$-\dot{\boldsymbol{\lambda}}_2 = \partial[\boldsymbol{\lambda}_1^{\mathrm{T}}\mathbf{x}_2 + \boldsymbol{\lambda}_2^{\mathrm{T}}B(\mathbf{x}_1)T(\mathbf{x}) + \boldsymbol{\lambda}_2^{\mathrm{T}}B(\mathbf{x}_1)\mathbf{u}]/\partial\mathbf{x}_2 \tag{48}$$

We can rewrite (47), (48) given the facts that $\partial[\boldsymbol{\lambda}_1^{\mathrm{T}}\mathbf{x}_2]/\partial\mathbf{x}_1 = 0$, $\partial[\boldsymbol{\lambda}_2^{\mathrm{T}}B(\mathbf{x}_1)\mathbf{u}]/\partial\mathbf{x}_2 = 0$

and $\partial[\lambda_1^T \mathbf{x}_2]/\partial \mathbf{x}_2 = \lambda_1$

$$-\dot{\lambda}_1 = [\partial[B(\mathbf{x}_1)T(\mathbf{x})]/\partial \mathbf{x}_1]^T \lambda_2 \tag{49}$$

$$-\dot{\lambda}_2 = \lambda_1 + [\partial[B(\mathbf{x}_1)T(\mathbf{x})]/\partial \mathbf{x}_2]^T \lambda_2 \tag{50}$$

We now state and prove a theorem concerning the form of the solution to the time optimal robot motion control problem.

Theorem 5.1

There is no time interval where the time optimal robot motion control problem with fixed initial and final states, given a general rigid articulated arm robot with friction described by (44), (45), is singular in all control variables.

Proof

If the problem is singular in all control variables over some time interval, then the switching functions must all equal zero over this time interval so (19) holds for all *i*. This is equivalent to

$$B^T \lambda = 0 \tag{51}$$

which for the system (44), (45) by inspection of (46) means

$$B^T(\mathbf{x}_1)\lambda_2 = 0 \tag{52}$$

Since M in (36) is positive definite, given (41), $B(\mathbf{x}_1)$ is positive definite so (45) can only hold if $\lambda_2 = 0$. If $\lambda_2 = 0$ over the time interval, then $\dot{\lambda}_2 = 0$ which, given (49) and (50), implies $\lambda_1 = 0$. This contradicts (33). □

The theorem states that in any time interval the solution to the minimum time problem is almost everywhere non-singular in at least one control variable. So, in any time interval almost everywhere, at least one control variable takes on an extreme value. This result has also been obtained by Ailon and Langholz (1985), Sontag and Sussman (1985), and Wen (1986). Theorem 4.1 states that if the problem is singular in some control variable over (t_1, t_2), this is equivalent to the linearized system being differentially uncontrollable from that control variable over (t_1, t_2). We can therefore derive an equivalent result concerning the first order controllability of time optimal trajectories. From the proof of Theorem 4.1 we see that along any time optimal trajectory satisfying the minimum principle, no time intervals exist where (51) holds. However, by inspection of Lemma 3.1 this implies that any time optimal trajectory satisfying the minimum principle is first order controllable. Since, for almost all time, the minimum time problem is non-singular in at least one control variable, equivalently the linearized system about a time optimal trajectory is differentially controllable from at least one control variable. It is natural to wonder if these results can be extended to all trajectories, i.e. to all sets $\Omega(\mathbf{x}_0, T)$, defined by (20), ..., (22), for all T and x_0.

Theorem 5.2

For all rigid articulated arm robots with friction, described by (44), (45), any trajectory, i.e. any $\mathbf{x}(t)$, $t \in [0, T]$, $T < \infty$, is first order controllable. Even stronger, for each time $t \in [0, T)$ the linearized system about any trajectory is differentially controllable from at least one control variable.

Proof

In the proof of Theorem 4.1, we have not made explicit use of the fact that $\boldsymbol{\lambda}$ is the costate of a minimum time problem. In fact $\boldsymbol{\lambda}$ may be any state which, given Theorem 4.1, implies Theorem 5.2. □

Theorem 5.2 states that for each time t along any trajectory, using just one control variable it is already sufficient to control the linearized system about the trajectory to a full neighbourhood in an arbitrarily short time. We have proved for general rigid articulated arm manipulators with friction described by (44), (45), that the main condition for successfully applying a perturbation controller design, based on linearized dynamics, is fulfilled.

Until now we have assumed that a solution to the minimum time problem exists. Ailon and Langholz (1985), prove this for a two link planar robot using Roxin's theorem. Another proof, given by Wen (1986), again for a two link planar robot, is based on Fillipov's theorem (Hermes and La Salle 1969). In both cases the result is that a time optimal solution exists if, and only if, the robot is able to move from the initial to the final state in some finite time. If neither the initial or final states violate the physical constraints of the robot, any robot will be designed to be capable of doing so, and the existence of the solution is guaranteed. We also prove the result using Fillipov's theorem; however, in this case we prove it in conjunction with an examination of the kinetic energy of a general rigid articulated arm robot with friction.

Theorem 5.2 (Fillipov's theorem, Hermes and La Salle 1969)

Take the system (6), with (29), (30) holding, where $\mathbf{x}_0$ is the fixed initial state of a minimum time problem. Define the set

$$R(\mathbf{x}) = \{f(\mathbf{x}, \mathbf{u}), \mathbf{u}(t) \in U\} \tag{53}$$

for any

$$\mathbf{x} \in \Omega(\mathbf{x}_0, T) \tag{54}$$

the attainable set from $\mathbf{x}_0$ for time $T < \infty$ defined in (31). If $f(\mathbf{x}, \mathbf{u})$ is continuous in $\mathbf{x}$ and $\mathbf{u}$, $R(\mathbf{x})$ is convex for all $\mathbf{x}$, the set U given by (30) is non-empty and compact, and $\Omega(\mathbf{x}_0, T)$ is bounded, then $\Omega(\mathbf{x}_0, T)$ is compact.

It can be proved (Hermes and Lasalle 1969) that if the conditions of Fillipov's theorem are satisfied and for some T, $0 < T < \infty$

$$\mathbf{x}_f \in \Omega(\mathbf{x}_0, T) \tag{55}$$

where $\mathbf{x}_f$ is the fixed final state of the minimum time problem, then a time optimal control for this problem exists. The system (44), (45) is continuous in $\mathbf{x}$ and $\mathbf{u}$, U is determined by (4) and so it is non-empty and compact. Since the system (44), (45) is linear in the control, $R(\mathbf{x})$ is convex. We now prove that trajectories cannot finitely escape, which is a sufficient condition for $\Omega(\mathbf{x}_0, T)$ to be bounded. This, then, completes the proof of the existence of a time optimal control for a general articulated arm robot with friction given by (44), (45) if (55) holds.

Consider the general expression for the kinetic energy V of a mechanical system:

$$0{\cdot}5V = \dot{\mathbf{x}}^{\mathrm{T}} M \dot{\mathbf{x}} \tag{56}$$

where $\mathbf{x}$ consists of generalized coordinates and M is a positive definite mass matrix

(Meirovitch 1970). The joint angles of a rigid articulated arm robot are generalized coordinates. So for the system (44), (45) $\mathbf{x}_1$ are generalized coordinates and the kinetic energy of this system can be represented by

$$0{\cdot}5V = \mathbf{x}_2^{\mathrm{T}} M \mathbf{x}_2 \tag{57}$$

where M is some positive definite mass matrix. Since a mechanical system cannot contain infinite energy, for all time the kinetic energy of such a system is bounded. We prove that V in (57) being bounded is equivalent to all components of $\mathbf{x}_2$ being bounded. Since, according to (44) and (55), the components of $\mathbf{x}_1$ are finite time integrals of the components of $\mathbf{x}_2$, they are also bounded, which means that all components of the state vector $\mathbf{x}$ of the system (44) and (45) are bounded. This implies trajectories do not finitely escape.

So we are left to prove that if V is bounded, all components of $\mathbf{x}_2$ are bounded. We need the following result.

Given a general quadratic expression

$$\mathbf{x}^{\mathrm{T}} Q \mathbf{x} = \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij} x_i x_j \tag{58}$$

where Q is an N square matrix with elements q and $\mathbf{x}$ an N dimensional vector with element x, then there always exists a non-singular matrix T such that

$$\mathbf{x}^{\mathrm{T}} Q \mathbf{x} = \sum_{i=1}^{N} \alpha_i y_i^2 \tag{59}$$

with

$$\mathbf{y} = T\mathbf{x} \tag{60}$$

and furthermore, if $Q > 0$ then $\alpha_i > 0$ for all i.

Since M in (57) is positive definite

$$0{\cdot}5V = \mathbf{x}_2^{\mathrm{T}} M \mathbf{x}_2 = \sum_{i=1}^{N} \alpha_i y_i^2 \tag{61}$$

with $\alpha_i \geqslant 0$ for all i. Since the kinetic energy V is bounded, and α_i is bounded for all i, this implies y_i^2 is bounded, which implies y_i is bounded for all i. Since T in (60) is non-singular, (60) is equivalent to

$$\mathbf{x} = T^{-1}\mathbf{y} \tag{62}$$

Since the elements of T and therefore T^{-1} are bounded, this implies all elements of $\mathbf{x}_2$ are bounded.

Summarizing, we have proved that for a general rigid articulated arm robot with friction described by (44) and (45), a time optimal control exists if (55) holds. This is the case if, from the initial state, the final state can be reached in some finite time. This is always the case in practical situations. Considering the form of the time optimal solution, we have proved that for almost all time at least one control variable takes on an extreme value. Finally, we have derived the very important result that any trajectory is first order controllable.

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