International Journal of Systems Science
Vol. 39, No. 10, October 2008, 969–983

How non-zero initial conditions affect the minimality of linear discrete-time systems

L.G. Van Willigenburg[a]* and W.L. De Koning[b]

[a]*Systems and Control Group, Wageningen University, Wageningen, The Netherlands;* [b]*Delft University of Technology, Delft, The Netherlands*

(*Received 13 July 2005; final version received 21 January 2008*)

From the state-space approach to linear systems, promoted by Kalman, we learned that minimality is equivalent with reachability together with observability. Our past research on optimal reduced-order LQG controller synthesis revealed that if the initial conditions are non-zero, minimality is no longer equivalent with reachability together with observability. In the behavioural approach to linear systems promoted by Willems, that consider systems as exclusion laws, minimality is equivalent with observability. This article describes and explains in detail these apparently fundamental differences. Out of the discussion, the system properties *weak reachability* or *excitability*, and the dual property *weak observability* emerge. Weak reachability is weaker than reachability and becomes identical only if the initial conditions are empty or zero. Weak reachability together with observability is equivalent with minimality. Taking the behavioural systems point of view, minimality becomes equivalent with observability when the linear system is time invariant. This article also reveals the precise influence of a possibly stochastic initial state on the dimension of a minimal realisation. The issues raised in this article become especially apparent if linear time-varying systems (controllers) with time-varying dimensions are considered. Systems with time-varying dimensions play a major role in the realisation theory of computer algorithms. Moreover, they provide minimal realisations with smaller dimensions. Therefore, the results of this article are of practical importance for the minimal realisation of discrete-time (digital) controllers and computer algorithms with non-zero initial conditions. Theoretically, the results of this article generalise the minimality property to linear systems with time-varying dimensions and non-zero initial conditions.

Keywords: realisation theory; minimal realisation; free response; discrete-time systems; time-varying dimensions

1. Introduction

It was of course Kalman who already observed that to build a proper realisation theory for linear *time-varying* systems one has to consider systems with *time-varying dimensions* (Kalman et al. 1969). On the other hand systems are often conceived as descriptions of physical systems the dimensions of which, in general, cannot or do not change. Probably therefore, the realisation theory of linear time-varying discrete-time systems that was initially developed assumes that the system dimensions must be constant (Evans 1972; Weiss 1972). When investigating the synthesis of discrete-time (digital) optimal reduced-order LQG controllers over a finite horizon however, we automatically ran into Kalman's observation (Van Willigenburg and De Koning 1999, 2002). In this case, minimal LQG controllers with time-varying dimensions appear naturally. These controllers also appear naturally when the sampling is performed asynchronously (Van Willigenburg and De Koning 2001a). Furthermore, given a certain input-output behaviour, systems with time-varying dimensions, in general, allow for *minimal realisations with smaller dimensions* than systems with constant dimensions.

Some 15 years ago people started to apply systems theory to obtain efficient computer implementations of algorithms (efficient realisations of computational networks). An algorithm is a sequence of computations which can be described by a discrete-time system (Gohberg, Kaashoek and Lerer 1992; Van der Veen and De Wilde 1992). Roughly speaking, at each computational stage the variables computed by the algorithm that have to be stored in computer memory to continue the remaining sequence of computations are the state variables. The inputs are the other variables that enter each computational stage and the outputs are the variables to be computed by the algorithm at each computational stage. Finally, each computational stage is a discrete-time instant. Clearly in this case the dimension of the state, the input and the output, in general, vary with the discrete time. Also, in general, the horizon is finite. To the best knowledge of the authors, in the associated literature the influence of non-zero initial conditions on minimality has not been considered while a major reason to apply systems

*Corresponding author. Email: gerard.vanwilligenburg@wur.nl

ISSN 0020–7721 print/ISSN 1464–5319 online

DOI: 10.1080/00207720801979950
http://www.informaworld.com

theory is to obtain computer implementations of algorithms, which make minimal use of computer memory and computations. In other words, the central issue is: minimal realisation of discrete-time systems with time-varying dimensions. To further reduce the amount of computations and memory needed for algorithm implementation, model reduction techniques may be applied. In this case the model is no longer an exact but only an approximate description of the algorithm. Also, in this case minimal realisation is of primary importance.

A major contribution of this article is to extend the minimal realisation of discrete-time systems with time-varying dimensions, and a possibly finite horizon, to the case where the initial conditions are non-zero. Moreover, the case of stochastic initial conditions and stochastic inputs will be considered. Our starting point to obtain a minimal realisation is a linear discrete-time state-space system description including a possibly non-zero initial state. So we will *not* be concerned with the identification problem of how to obtain such a system description from input–output data or an input-output map.

Discrete-time (digital) optimal reduced-order LQG controller synthesis merges the two applications of systems theory mentioned above. On the one hand, the reduced-order discrete-time (digital) controller is a discrete-time system, which bears strong connections to the controlled system that has fixed dimensions and in general a non-zero initial state. On the other hand, the discrete-time (digital) controller is an algorithm that has to be implemented in the control computer.

Being unaware of Gohberg et al. (1992), Van der Veen and de Wilde (1992) in Van Willigenburg and De Koning (1999, 2001a, 2002) finite-horizon discrete-time compensators with time-varying dimensions were considered. These appeared naturally as the solution to the finite-horizon optimal reduced-order LQG problem. Now interestingly the results in Van Willigenburg and De Koning (2002) are slightly more general than those by Van der Veen and De Wilde (1992) and Gohberg et al. (1992) because compensators with a non-zero initial state were considered. If the initial state is non-zero, so is the associated free response which contributes to the output behaviour. Therefore, a non-zero initial state generalises the definition of minimality, which then is no longer equivalent with reachability together with observability (Van Willigenburg and De Koning, 2001a,b). Furthermore in Van Willigenburg and De Koning, (1999, 2001a,b, 2002) the horizon is finite as opposed to Evans (1972), Weiss (1972), Van der Veen and De Wilde (1992) and Gohberg et al. (1992). Having both an upper and lower bound for time has an effect on the system properties reachability, observability and minimality because these depend on the past and the future, which are both bounded if the horizon is finite.

The fact that minimality is not necessarily equivalent with reachability together with observability also follows from Sontag (1979) and Willlems and Polderman (1998). In Sontag (1979), linear time-invariant discrete-time systems that are minimal, denoted by span-canonical in this article, are both so-called span-reachable and observable. Willems and Polderman (1998) consider system descriptions as *exclusion laws*. Then for linear time-invariant systems minimality is equivalent with observability. As demonstrated in this article both results also follow from our analysis, which applies to the more complicated class of linear time-varying discrete-time systems with time-varying dimensions and a finite horizon. Moreover, the case of stochastic inputs and stochastic initial conditions will also be considered in this article.

Another major contribution of this article is to describe and explain in detail the loss of the equivalence between minimality and reachability together with observability due to a non-zero initial state. Furthermore, this article reveals the precise influence of a non-zero initial state on the dimension of a minimal realisation of a system. In Van Willigenburg and De Koning (2002), minimality is defined based on the so-called *modified* reachability grammian introduced in that paper. Associated to this grammian is a canonical form, which was introduced and exploited in the paper by Van Willigenburg and De Koning (2001b), to further reduce the storage needed for minimal LQG controllers. These results clearly suggest the system property modified reachability. The analysis presented in this article proposes to change some of the previous definitions in Van Willigenburg and De Koning (2001b, 2002) slightly, by making use of an empty matrix concept so that the suggested system property modified reachability turns in what may now be called *weak reachability*. After applying similar changes to the minimality property, an important result of this article is that *minimality is equivalent with weak reachability together with observability*. If the system is deterministic, linear, time-invariant and if the horizon is infinite this result complies with Sontag (1979) and Van den Hof (1998). In Van den Hof (1998), which considers linear continuous time-invariant systems, the term weak reachability seems to be introduced. In Sontag (1979), which considers time-invariant linear discrete-time systems, it is called span-reachability. Furthermore, the precise influence of the initial state on the dimension of a minimal realisation of a system is not further investigated in these papers.

Finally this article introduces what we call the weak reachability unit and double staircase form and the weak observability double staircase form of time-varying linear discrete-time systems with time-varying dimensions. These double staircase forms illustrate precisely the difference between reachability and weak reachability and dually the difference between observability and weak observability and the influence of the initial state on this and the dimension of a minimal realisation. Based on these staircase forms, we can easily compute minimal realisations. Finally, based on the double staircase form, we give an interpretation of the system property weak observability, which is dual to weak reachability.

2. An illustrative example

To illustrate some of the key issues of this article consider the following example.

Example 1: Suppose we have a linear discrete-time system,

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, y_i = C_i x_i, \quad i = 0, 1, \ldots \tag{1}$$

where x_i is the state, u_i is the input and y_i is the output. Let,

$$x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2}$$

Then we obtain,

$$\begin{aligned} x_1 = \Phi_0 x_0 + \Gamma_0 u_0 &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_0 \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_0 \end{aligned} \tag{3}$$

From Equation (3), the state space at time $i=1$, which can be reached by the system (1), through variation of the control u_0, is the line drawn in Figure 1.

This line does *not* contain the origin. Due to the latter there does not exist a *standard* basis transformation that transforms this line into one of the axis of the new state basis. If this would be so we could drop the other axis and represent the reachable space of dimension one at time $i=1$ by just one state variable. In our case however *to represent the reachable space of dimension one, which does not contain the origin, we need two state variables*. Also observe from Equation (3) that *the reason why the reachable space at time* $i=1$ *does not contain the origin is that* $\Phi_0 x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq 0$ *and is not a point on the line* $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u_0$, *which represents the reachable space from* $x_0 = 0$ *at time* $i=1$.

Next consider the following standard basis transformation at $i=0$. At $i=0$, we take as new basis vectors respectively,

$$p_0^1 = x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad p_0^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

Then,

$$x_0' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5}$$

is the initial state obtained after this basis transformation at time $i=0$ and

$$\Phi_0' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{6}$$

the new state transition matrix while Γ_0 remains unchanged. Now let us drop the zero component of x_0' in (5) and the associated part of Φ_0' in (6) to obtain,

$$x_0' = 1, \quad \Phi_0' = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{7}$$

As we already explained the reachable space at time $i=1$, although it has dimension one, needs two state variables to be represented. Therefore, at time $i=1$ we cannot further reduce the dimension. Observe that with (7), *we have obtained a realisation with time-varying dimensions, which has smaller dimensions than the original one that has fixed dimensions*. Also observe that, irrespective of the dimension n of the system, selecting $p_0^1 = x_0$ results in an $x_0' \in R^n$ the first component of which equals one while all the others are zero and can therefore be dropped. Therefore, *a minimal realisation with time-varying dimensions of any discrete-time system has no more than one state variable at the initial time zero*.

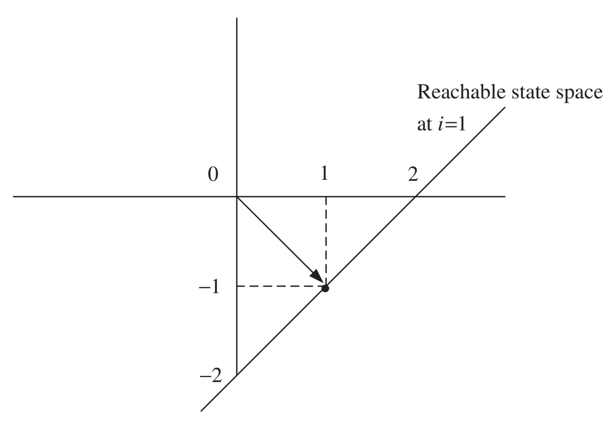


Figure 1. The reachable state space at time $i=1$.

The term $\Phi_0 x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$in (3) shifts the line $\Gamma_0 u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_0$ that can be reached from $x_0 = 0$ away form the origin as indicated by the arrow in Figure 1. We can apply a *non-standard* basis transformation at $i = 1$ by taking a new basis the origin of which is shifted towards $\Phi_0 x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If in addition to this shift we select,

$$p_1^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8}$$

as the first basis vector the reachable space is the line along this basis vector and so can be described by just one state variable. Applying this non-standard basis transformation, however, introduces an *additional realisation problem*, namely the realisation of the shift $\Phi_0 x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Instead of solving separately two interconnected realisation problems, obtained from realising a discrete-time system with non-zero initial conditions, as demonstrated in this article, it is preferable to solve this realisation problem directly by extending the realisation theory to discrete-time systems with non-zero initial conditions. On the other hand one may argue that the shift $\Phi_0 x_0$ of the origin at $i = 1$, and in general shifts of the origins at each time i, do not *fundamentally* change system properties. But they do change the output into $y_1' = y_1 - C_1 \Phi_0 x_0$, and therefore they do change the input–output behaviour.

The example in this section demonstrated that (1) allowing for time-varying dimensions results in minimal realisations with smaller dimensions, (2) a non-zero initial condition affects the output behaviour and (3) a non-zero initial condition can be eliminated, but only by applying non-standard basis transformations at each discrete-time instant. Disregarding the non-standard basis transformations, theoretically one may conclude that a non-zero initial condition does not change system properties. In practice, when having to realise or build the system (e.g. in a computer), one cannot disregard the non-standard basis transformations; they have to be realised as well. Then the generalisation of minimality and the generalisation and introduction of several related system properties, to accommodate for non-zero initial conditions, as presented in this article become vital.

3. Minimality, reachability, weak reachability and observability

3.1 *Introduction and motivation*

To obtain one of the main results of this article, namely that minimality of a linear discrete-time system is equivalent with weak reachability together with observability, in Section 3.2 these system properties will be defined. The definitions apply to time-varying linear discrete-time systems having time-varying dimensions. Moreover, these systems are defined over a *finite* time horizon. Lower bounding time is necessary to consider the influence of initial conditions. Lower and upper bounding time introduces additional difficulties. As we will explain in Remarks 3 and 4 in Section 3.3, these difficulties disappear if the upper and lower bound on time are removed. Then conventional results are obtained. The definitions in Section 3.2 differ from those in the previous papers (Van Willigenburg and De Koning 2001b, 2002) but only at the initial and terminal time. In the previous papers the need for an empty matrix concept was deliberately circumvented. Section 3.2 demonstrates that it is preferable to use an empty matrix concept to be able to redefine reachability at the initial time, and observability at the terminal time. As a result, the modified reachability grammian (Van Willigenburg and De Koning 2001b, 2002) turns into what may now be called the weak reachability grammian. The advantages of this will be clearly stated and discussed. To clearly illustrate the similarities and differences with the 'conventional theory', the system properties are introduced starting from basic definitions. In Section 4 we generalise the system properties even further to allow for stochastic initial conditions and stochastic inputs.

When reading this section, the reader should be especially aware of three things. First, it is very tentative to read this section with the presumption that minimality is equivalent with reachability together with observability. However, a major consequence of (Van Willigenburg and De Koning 2001b, 2002) and this article is that this equivalence fails to hold. To see this, the implications of the definitions presented in this section will be and have to be investigated separately. Secondly, the reader should carefully read the ranges of each time index. Specifically note whether a range starts at the initial time 0 or at 1. Similarly, whether it ends at the final time N or $N-1$. Thirdly, at first sight, the definitions may appear to be strange when time is bounded from above and below. This is because the definitions imply that some dimensions are zero and some input and output sequences are empty. This, however, is necessary to arrive at our most important very practical results that match very closely the conventional results. The zero dimensions require an empty matrix concept. The following empty matrix concept, which is implemented in Matlab (Nett and Haddad 1993) is adopted,

$$A = [\,] \in R^{n\times 0}, \quad B = [\,] \in R^{0\times m} \Rightarrow AB = 0 \in R^{n\times m} \tag{9}$$

In Equation (9), R denotes the real numbers and n, m denote non-negative row and column dimensions, respectively. Moreover, the empty matrix,

$$BB^T = [\,] \in R^{0\times 0} \tag{10}$$

is considered to be *symmetric* and *full rank*. Finally, the following equivalence of notations applies,

$$A = [\,] \in R^{n\times m} \Leftrightarrow A = 0 \in R^{n\times m}, \quad n = 0 \text{ and/or } m = 0 \tag{11}$$

3.2 *Definitions and main results*

Consider the following deterministic time-varying system defined over a finite horizon,

$$\begin{aligned} x_{i+1} &= \Phi_i x_i + \Gamma_i u_i, \quad i = 0, 1, \dots, N-1, \\ x_j &\in R^{n_j\times 1}, \quad j = 0, 1, \dots, N, \\ u_j &\in R^{m_j\times 1}, \quad j = 0, 1, \dots, N-1. \end{aligned} \tag{12}$$

The dimensions n_j, $j = 0, 1, \dots, N$ and m_j, $j = 0, 1, \dots, N-1$ of the system state x_j and the system input u_j respectively, may vary with time. Denote this system by (x_0, Φ^N, Γ^N), where $\Phi^N = \{\Phi_0, \Phi_1, \dots, \Phi_{N-1}\}$ and where $\Gamma^N = \{\Gamma_0, \Gamma_1, \dots, \Gamma_{N-1}\}$. The reason why x_0 is part of the system notation is that some properties of the system, such as the states that can be reached at each time $0 \le i \le N$, depend on the initial state x_0. This was illustrated by Example 1.

For the system (12) we have,

$$x_i = \Phi_{i,0} x_0 + \sum_{k=0}^{i-1} \Phi_{i,k+1} \Gamma_k u_k, \quad i = 1, 2, \dots, N, \tag{13}$$

where,

$$\Phi_{l,m} = \Phi_{l-1}\Phi_{l-2}\cdots\Phi_m, \quad l > m, \quad \Phi_{l,m} = I_{n_l}, \quad l = m. \tag{14}$$

Definition 1: The system (x_0, Φ^N, Γ^N) is called *reachable* if $\forall i = 0, 1, 2, \dots, N$, $\forall x \in R^{n_i}, x_i = x$ can be reached through an appropriate choice of the input sequence $U_{0,i} = \{u_0, u_1, \dots, u_{i-1}.\}$

Note that Definition 1 is identical to the one in Weiss (1972). Also note (see Remark 3 and Sontag (1979)), that weaker definitions occur in the control literature. As opposed to Weiss (1972), the time for which the system is defined is bounded from above and below and furthermore the system is allowed to have time-varying dimensions. The only difference between Definition 1 and the associated previous definition (Van Willigenburg and De Koning 2001b, 2002) is the inclusion of time zero in the sequence $\forall i = 0, 1, 2, \dots, N$ (in the previous definitions this sequence read $\forall i = 0, 1, 2, \dots, N$). Since there are no controls available before time zero (the input sequence $U_{0,0}$ is empty), the only state that can be reached at time zero is the initial state x_0. If this state has a dimension larger than zero, then its associated state space is not reachable. Therefore, according to Equation (11), Definition 1 implies that for a system to be reachable,

$$n_0 = 0 \tag{15}$$

must hold. The previous definitions in Van Willigenburg and De Koning (2001b, 2002) did not impose (15) and thereby circumvented the need for an empty matrix concept.

According to Definition 1, Equation (13) determines the states that can be reached. In the conventional case $x_0 = 0$ and the first term $\Phi_{i,0},x_0$ on the right in Equation (13) is zero. The second term, through the variation of $u_0, u_1, \dots, u_{i-1}$ either represents the full system state-space at time i, i.e. R^{n_i}, or it represents *a linear subspace*, i.e. a hyperplane that contains the origin, with dimension $n_i^r < n_i$. This hyperplane is spanned by n_i^r basis vectors. If $x_0 \neq 0$, the first term $\Phi_{i,0},x_0$ in Equation (13) may be unequal to zero and may not be part of this hyperplane. Then it shifts the hyperplane away from the origin as in Example 1. To represent the hyperplane in this case, instead of n_i^r, now $n_i^r + 1$ basis vectors are needed! Adding $\Phi_{i,0},x_0$ as a basis vector, a linear subspace of dimension $n_i^r + 1$ is spanned that *contains* the reachable space.

Definition 2: Given the system (x_0, Φ^N, Γ^N). Suppose that at time i, $\Phi_{i,0},x_0$ is not part of the linear subspace of dimension n_i^r that is spanned by the states that can be reached from $x_0 = 0$. Then the linear subspace of dimension $n_i^r + 1$ spanned by adding $\Phi_{i,0},x_0$ as an additional basis vector is called *the weakly reachable subspace* of the system (x_0, Φ^N, Γ^N) at time i.

To derive the main results in this article, the reachable subspace and the associated property reachablity that are commonly associated with minimality have to be replaced with the weakly reachable subspace and the associated property weak reachability. After making these substitutions in the standard theory for linear systems with time-varying dimensions (Gohberg et al. 1992; Van der Veen and De Wilde 1992) this theory also applies to systems with non-zero initial conditions. The property weak reachability and the associated weak reachability grammian will be introduced shortly. We first introduce observability and minimality.

Consider the following deterministic time-varying system defined over a finite horizon

$$\begin{aligned} x_{i+1} &= \Phi_i x_i, \quad y_i = C_i x_i, \quad i = 0, 1, \ldots, N-1, \\ x_j &\in R^{n_j \times 1}, \quad j = 0, 1, \ldots, N, \\ y_j &\in R^{l_j \times 1}, \quad j = 0, 1, \ldots, N-1. \end{aligned} \tag{16}$$

Note that the dimension l_j of the system output y_j, $j = 0, 1, \ldots, N-1$ may vary with time. Denote this system by (Φ^N, C^N) where $C^N = \{C_0, C_1, \ldots, C_{N-1}\}$. One may wonder why y_N is not part of the system description (16) while x_N is. This will be explained below, immediately after Equation (17).

Definition 3: The system (Φ^N, C^N) is called *observable* if $\forall i = 0, 1, 2, \ldots, N$ the output sequence $Y_{i,N} = \{y_i, y_{i+1}, \ldots, y_{N-1}\}$ being entirely zero implies $x_i = 0$.

Dual to Definition 1, Definition 3 differs from the associated one presented in the previous papers (Van Willigenburg and De Koning 2001b, 2002) by the fact that the final time N is included in the sequence $\forall i = 0, 1, \ldots, N$. Since there are no measurements available at and after time N (the output sequence $Y_{N,N}$ is empty), according to Equation (11), Definition 3 implies that for a system to be observable,

$$n_N = 0 \tag{17}$$

must hold. This result is dual to (15). Observe from (17) that if the system (16) is observable, effectively, the final time is not N but $N-1$. The dual result (17) is only obtained if we do consider x_N and leave out y_N in the system definition (16).

Consider the system

$$\begin{aligned} x_{i+1} &= \Phi_i x_i + \Gamma_i u_i, \quad y_i = C_i x_i, \quad i = 0, 1, \ldots, N-1, \\ x_j &\in R^{n_j \times 1}, \quad j = 0, 1, \ldots, N, \\ u_j &\in R^{m_j \times 1}, \quad j = 0, 1, \ldots, N-1, \\ y_j &\in R^{l_j \times 1}, \quad j = 0, 1, \ldots, N-1. \end{aligned} \tag{18}$$

Denote this system by $(x_0, \Phi^N, \Gamma^N, C^N)$.

Definition 4: The *input–output map* $M(x_0, \Phi^N, \Gamma^N, C^N) : U_{0,N} \to Y_{0,N}$ of the system $(x_0, \Phi^N, \Gamma^N, C^N)$ maps any input sequence $U_{0,N} = \{u_0, u_1, \ldots, u_{N-1}\}$ to the associated output sequence $Y_{0,N} = \{y_0, y_1, \ldots, y_{N-1}\}$ as determined by the system Equations (18) and the initial state x_0.

The *input–output behaviour* of the system $(x_0, \Phi^N, \Gamma^N, C^N)$ is fully determined by its input–output map $M(x_0, \Phi^N, \Gamma^N, C^N) : U_{0,N} \to Y_{0,N}$.

Definition 5: A system $(x_0, \Phi^N, \Gamma^N, C^N)$ with state dimensions n_i, $i = 0, 1, \ldots, N$ is called *minimal* if no system $(x'_0, \Phi'^N, \Gamma'^N, C'^N)$ with state dimensions $n'_i \le n_i$, $i = 0, 1, \ldots, N$ and $n'_j < n_j$ for at least one $0 \le j \le N$ exists that has the same input–output map. Then, $(x_0, \Phi^N, \Gamma^N, C^N)$ is also called a *minimal realisation* of its input–output map and of any other system that has the same input–output map.

Definition 5 is stated in terms of the input–output map and not in terms of grammians as in Van Willigenburg and De Koning (2001b, 2002). Therefore, Definition 5 appears to be more fundamental.

As to the input–output map of the system $(x_0, \Phi^N, \Gamma^N, C^N)$ note from Equation (18) that x_N is irrelevant. Therefore, it follows from Definition 5 that for a system to be minimal,

$$n_N = 0 \tag{19}$$

must hold. Observe from (19) that if the system (18) is minimal, effectively, the final time is not N but $N-1$. This complies with Equation (17). Moreover, from Example 1 in Section 2 we concluded that for a system to be minimal,

$$n_0 = 1 \tag{20}$$

must hold. Applying the empty matrix concept (9)–(11), without affecting the input–output behaviour of the system, we may perform the following replacement in Equation (18),

$$x_i = 0 \in R^{n_i \times 1} \to x_i = [\,] \in R^{0 \times 1} \tag{21}$$

where $\to$ denotes replacement. From Equation (21) it follows that Equation (20), which holds for a minimal system, now changes into,

$$x_0 \ne 0 \Rightarrow n_0 = 1; \quad x_0 = [\,] \in R^{0 \times 1} \Rightarrow n_0 = 0 \tag{22}$$

The advantage of applying the replacement (21) is that we can associate the rank of the observability grammian and the weak reachability grammian, to be introduced shortly, directly to the dimensions of a minimal realisation of the system, also if the rank is zero. Furthermore, according to Definition 5, $n_i = 0$, which is obtained after the replacement (21), is to be preferred over $n_i = 1$, which is obtained from Definition 5 if the replacement (21) is not made. On the other hand $x_i = [\,] \in R^{0\times 1}$ suggests that the state space, which is both reachable and observable at time i, is empty, while it is actually the origin.

Corollary 1: *Without affecting the input–output behaviour of the system, throughout this article, the empty matrix concept* (9)–(11) *and the associated replacement* (21) *may be applied. To obtain a minimal realisation, according to Definition* 5, *these replacements have to be applied.*

Let $W_{0,i} \in R^{n_i\times n_i}$ denote the reachability grammian of the system (Φ^N, Γ^N) associated to the state transition $x_0 = 0 \in R^{n_0\times 1}$ to $x_i = x,\ i \in [0, N]$, i.e.

$$W_{0,i} = \sum_{k=0}^{i-1} \Phi_{i,k+1}\Gamma_k\Gamma_k^T\Phi_{i,k+1}^T, \quad i = 1, 2, \ldots, N, \quad W_{0,0} = 0 \in R^{n_0\times n_0} \tag{23}$$

Based on Equation (13) define,

$$W'_{0,i} = \Phi_{i,0}x_0x_0^T\Phi_{i,0}^T + \sum_{k=0}^{i-1} \Phi_{i,k+1}\Gamma_k\Gamma_k^T\Phi_{i,k+1}^T, \quad i = 1, 2, \ldots, N-1, \quad W'_{0,0} = x_0x_0^T \in R^{n_0\times n_0} \tag{24}$$

Equation (24) is a grammian associated to the system state transition from x_0 to $x_i = x,\ i \in [0, N]$. It is called the *modified reachability grammian* in Van Willigenburg and De Koning (2002, 2001b) but according to Definition 2, it is called the *weak reachability grammian* in this article. Dual to the reachability grammian (23) consider the observability grammian $M_{i,N} \in R^{n_i\times n_i}$ given by,

$$M_{i,N} = \sum_{k=i}^{N-1} \Phi_{k,i}^T C_k^T C_k \Phi_{k,i}, \quad i = 0, 1, \ldots, N-1, \quad M_{N,N} = 0 \in R^{n_N\times n_N}. \tag{25}$$

Associated with the replacement (21) are the following replacements,

$$W'_{0,i} = 0 \in R^{n_i\times n_i} \to W'_{0,i} = [\,] \in R^{0\times 0}, \tag{26}$$

$$M_{i,N} = 0 \in R^{n_i\times n_i} \to M_{i,N} = [\,] \in R^{0\times 0}. \tag{27}$$

Furthermore from Equation (19), it follows that $M_{N,N} = [\,] \in R^{0\times 0}$ holds for a minimal system. Then from the empty matrix concept (9)–(11) and Equations (13), (23)–(27) the following two lemmas are immediate. Due to the modifications of Definitions 1 and 3, Lemma 1 simplifies compared to the one presented in the previous papers (Van Willigenburg and De Koning 2001b, 2002) while Lemma 2 also applies to the initial time $i = 0$ now.

Lemma 1: (x_0, Φ^N, Γ^N) *reachable* $\Leftrightarrow W_{0,i}$ *full rank* $\forall i \in [0, N]$. *Dually* (Φ^N, C^N) *observable* $\Leftrightarrow M_{i,N}$ *full rank* $\forall i \in [0, N]$.

Lemma 2:

(1) *The first term on the right in Equation* (13) *lies inside the hyperplane with dimension* $n_i^r = \mathrm{rank}(W_{0,i}) < n_i$ *determined by the second term on the right in Equation* (13) $\Rightarrow \mathrm{rank}(W'_{0,i}) = \mathrm{rank}(W_{0,i}) = n_i^r < n_i,\ i \in [0, N]$.
(2) *The first term on the right in Equation* (13) *lies outside the hyperplane with dimension* $n_i^r < n_i$ *determined by the second term on the right in Equation* (13) $\Rightarrow \mathrm{rank}(W'_{0,i}) = \mathrm{rank}(W_{0,i}) + 1 = n_i^r + 1 \le n_i, i \in [0, N]$.
(3) *The second term on the right in Equation (13) spans the full state-space* $R^{n_i} \Rightarrow \mathrm{rank}(W'_{0,i}) = \mathrm{rank}(W_{0,i}) = n_i,\ i \in [0, N]$.

From Lemma 2 and Definition 2, $\mathrm{rank}(W'_{0,i})$ represents precisely the dimension of the weakly reachable subspace at time $i \in [0, N]$. Together with Definition 2 and the second part of Lemma 1 this implies to the following theorem.

Theorem 1: $(x_0, \Phi^N, \Gamma^N, C^N)$ *is minimal* $\Leftrightarrow \forall i \in [0, N]$ $W'_{0,i}$ *full rank and* $M_{i,N}$ *full rank*.

It is well known that the reachability and observability grammian (23), (25) can be given in recursive form as follows,

$$W_{0,i+1} = \Phi_i W_{0,i}\Phi_i^T + \Gamma_i\Gamma_i^T, \quad i = 0, 1, \ldots, N-1, \quad W_{0,0} = 0 \in R^{n_0\times n_0} \tag{28}$$

$$M_{i,N} = \Phi_i^T M_{i+1,N}\Phi_i + C_i^T C_i, \quad i = 0, 1, \ldots, N-1, \quad M_{N,N} = 0 \in R^{n_N\times n_N} \tag{29}$$

Similar to (28), the recursive form of the weak reachability grammian (24) is given by,

$$W'_{0,i+1} = \Phi_i W'_{0,i}\Phi_i^T + \Gamma_i\Gamma_i^T, \quad i = 0, 1, \ldots, N-1, \quad W'_{0,0} = x_0x_0^T \in R^{n_0\times n_0} \tag{30}$$

Equations (28) and (30) are identical except for the initial value which from 0 is changed into $x_0x_0^T$. This constitutes the generalisation. Introduce

$$r_i = \min\Big(\mathrm{rank}\Big(W'_{0,i}\Big), \mathrm{rank}\Big(M_{i,N}\Big)\Big), \quad i = 0, 1, \ldots, N. \tag{31}$$

Then from Equations (29), (30) and the replacements (26), (27),

$$x_0 \neq [\,] \wedge x_0 \neq 0 \wedge M_{0,N} \neq 0 \Rightarrow r_0 = 1 \tag{32}$$

$$x_0 = [\,] \vee x_0 = 0 \vee M_{0,N} = 0 \Rightarrow r_0 = 0 \tag{33}$$

$$r_N = 0 \tag{34}$$

$$r_i - l_i \leq r_{i+1} \leq r_i + m_i, \quad i \in [0, N-1]. \tag{35}$$

From Equations (29), (30), (31) and Theorem 1, the dimensions of a minimal system satisfy,

$$n_i = r_i, \quad i = 0, 1, \ldots, N. \tag{36}$$

On the other hand if $(x_0, \Phi^N, \Gamma^N, C^N)$ has dimensions n_i satisfying (36), for certain given values r_i, $i = 0, 1, \ldots, N$ that satisfy (32)–(35), then one can always choose the system such that it is minimal. Equations (32)–(35) imply that the *change* of the dimension of the state of a minimal system, from one discrete-time instant to the next, is bounded from above and below.

Lemma 3: *For the system* $(x_0, \Phi^N, \Gamma^N, C^N)$, *let* k, $0 \leq k \leq N$ *denote the first time-instant for which* $\text{rank}(W'_{0,k}) = \text{rank}(W_{0,k})$ *holds. Then,* $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i})$, $k \leq i \leq N$.

Proof: Follows directly from Equations (28) and (30). □

From Lemmas 1–3, before time k, $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i}) + 1$ and the state $\Phi_{i,0}x_0$ is not captured by the space that is reachable from $x_0 = 0$ at time i. At and after time k in Lemma 3, $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i})$ and the state $\Phi_{i,0}x_0$ is captured by the space that is reachable from $x_0 = 0$ at time i. If k in Lemma 3 does not exist, $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i}) + 1$, $\forall i = 0, 1, \ldots, N$. Then the state $\Phi_{i,0}x_0$ is never captured by the space that is reachable from $x_0 = 0$ at time i.

Theorem 2: *For a system* $(x_0, \Phi^N, \Gamma^N, C^N)$, *minimality only implies reachability if* $x_0 = [\,]$.

Proof: For the system to be reachable $k = 0$ must hold in Lemma 3 and $x_0 = [\,]$ must hold. But from Equations (13), (23) and (24), it follows that $x_0 = [\,] \Rightarrow k = 0$. On the other hand, if $x_0 = [\,]$ and if $(x_0, \Phi^N, \Gamma^N, C^N)$ is minimal, according to Theorem 1 and Lemma 3, this implies that (x_0, Φ^N, Γ^N) is reachable. □

Definition 6: A system (x_0, Φ^N, Γ^N) is called *weakly reachable* if $W'_{0,i}$ is full rank $\forall i \in [0, N]$.

From Definition 6 and Equation (30) observe that *weak* reachability demands that the initial state x_0 is either empty or a non-zero scalar. The old definition of reachability in Van Willigenburg and De Koning (2001b, 2002) imposes *no* restriction on the dimension of x_0 whereas our new Definition 1 demands that the initial state is empty. For the rest both definitions are identical. Therefore, only when we use our new definition of reachability, we may use the term *weak* reachability in Definition 6.

Theorem 3: *A system* $(x_0, \Phi^N, \Gamma^N, C^N)$ *is minimal if and only if* (x_0, Φ^N, Γ^N) *is weakly reachable and* (Φ^N, C^N) *is observable.*

Proof: Follows immediately from Definitions 1, 3, 6 and Theorem 1. □

Corollary 2: *The differences between the definitions of reachability, observability and minimality in this article and the previous papers (Van Willigenburg and De Koning 2001b, 2002) are as follows. Due to the application of the empty matrix concept, the smallest possible dimension of the state at each discrete time instant is zero, instead of one. Reachability has been strengthened at the initial time and now demands an empty initial condition. Observability has been strenghthened at the terminal time and now demands an empty terminal condition. Minimality is slightly strengthened at the initial time because it no longer allows for a scalar zero initial condition but only for a scalar non-zero or empty initial condition. Due to these changes minimality is now equivalent with weak reachability together with observability without exceptions at the initial and terminal time as in Van Willigenburg and De Koning (2001b, 2002). Moreover, the ranks of both the weak reachability grammian and the observability grammian now determine directly the dimension of a minimal realisation at each time* $0 \leq i \leq N$.

3.3 *Canonical staircase representations and examples*

3.3.1 *State basis for canonical representation*

According to Definition 2 and Lemma 2, for each system $(x_0, \Phi^N, \Gamma^N, C^N)$ we can determine the following state basis at each time $0 \leq i \leq N$ based on $W_{0,i}$ and $W'_{0,i}$. Take the first $\text{rank}(W_{0,i})$ basis vectors to span the space, which is reachable at time i from $x_0 = 0$. If $\text{rank}(W_{0,i}) = n_i$, then the state space is reachable at time i and is spanned by these basis vectors. If $\text{rank}(W_{0,i}) < n_i$, then select $n_i - \text{rank}(W_{0,i})$ additional basis vectors as follows. If $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i}) + 1 = n_i$, then select as an additional basis vector either A) $\Phi_{i,0}x_0$ or B) a vector with one component along $\Phi_{i,0}x_0$ and the other component inside the reachable space. If $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i}) + 1 < n_i$, then next to the above mentioned basis vectors select $n_i - \text{rank}(W'_{i,0})$ additional basis vectors to span the state space at time i. The state basis and basis vectors associated to the choices (A) and (B) will be denoted by A and B, respectively.

Definition 7: When represented in the state basis described above the state variables are called *the modes of the system* $(x_0, \Phi^N, \Gamma^N, C^N)$.

Definition 7, Equations (13), (24) and Lemmas 1–3 imply the following.

Corollary 3: *If a system* $(x_0, \Phi^N, \Gamma^N, C^N)$ *is weakly reachable then its state space at each time* $0 \le i \le N$ *is either reachable or it is has a single unreachable mode associated with the basis vector A or B. The latter holds before time k in Lemma* 3 *while the former holds at and after time k in Lemma* 3. *If we select basis vector A, the unreachable mode is always equal to one, irrespective of the control. If we select basis vector B, the unreachable mode has a non-zero value. If the system is not weakly reachable then at some time instants* $0 \le i \le N$, $\text{rank}(W'_{0,i}) < n_i$. *At these time instants, the system has* $n_i - \text{rank}(W'_{0,i})$ *modes equal to zero, irrespective of the system input. The other modes of the system are all reachable except possibly for the single unreachable non-zero mode that is associated with the basis vector A or B.*

From the last part of Corollary 2, we obtain the following definition.

Definition 8: If $\text{rank}(W'_{0,i}) < n_i$, the $n_i - \text{rank}(W'_{0,i})$ zero modes of the system $(x_0, \Phi^N, \Gamma^N, C^N)$ at time i are called the *unexcitable modes* of the system at time i. If $\text{rank}(W'_{0,i}) = \text{rank}(W_{0,i}) + 1$ holds the mode of the system at time i that is associated with the basis vector A or B is called the *excited unreachable mode* of the system at time i.

If we select state basis A, then the representation of the system in this state basis is as follows,

$$x_i = \begin{bmatrix} x_i^r \\ x_i^{eur} \\ x_i^{ue} \end{bmatrix}, \quad \Phi_i = \begin{bmatrix} \times & 0 & \times \\ 0 & 1 & \times \\ 0 & 0 & \times \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} \times \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

where $x_i^r \in R^{n_i^r}$ are the *r̲eachable state variables* (*modes*), $x_i^{eur} \in R^{n_i^{eur}}$ is the *e̲xcitable u̲nr̲eachable state variable* (*mode*) of the system where $n_i^{eur} = 1$, $x_i^{eur} = 1$ unless this mode is empty. Furthermore, $x_i^{ue} \in R^{n_i^{ue}}$ are the *u̲ne̲xcitable state variables* (*modes*). The crosses in (37) indicate submatrices with compatible dimensions and arbitrary entries and the zeros indicate submatrices with compatible dimensions and entries that are all zero. Note that this staircase form splits the excitable modes into reachable modes and the excitable unreachable mode, which is always equal to one. Note that at the initial time zero there are no reachable modes and only if the initial state is non-empty and non-zero we have an excited unreachable mode i.e.

$$n_0^r = 0, \; x_0 \neq [\,] \wedge x_0 \neq 0 \Rightarrow n_0^{eur} = 1 \text{ else } n_0^{eur} = 0 \tag{38}$$

Definition 9: The representation (37) is called *the weak reachability unit staircase form.*

If we select the state basis B, then the representation of the system is as follows,

$$x_i = \begin{bmatrix} x_i^r \\ x_i^{eur} \\ x_i^{ue} \end{bmatrix}, \quad \Phi_i = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} \times \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

Definition 10: The representation (39) is called *the weak reachability double staircase form.*

In the next section, we will consider stochastic initial conditions. Then the weak reachability double staircase form still applies but the number of unreachable excited modes may be larger than one.

Remark 1: Definition 8 and the weak reachability unit and double staircase forms (37), (39) clearly illustrate the difference between reachability and weak reachability. If the system is weakly reachable, then before time k in Lemma 3 the system has a single mode, which is excited by the non-zero initial condition, but which is not reachable. Since minimality relates to input–output behaviour, which relates to the modes that are or can be excited, minimality relates to weak reachability rather than reachability. This is reflected by Theorems 2 and 3. Therefore, weak reachability might be called *excitability*.

Remark 2: To obtain a minimal realisation of the system $(x_0, \Phi^N, \Gamma^N, C^N)$, we first represent it in the weak reachability unit or double staircase form. Then at every time $i = 0, 1, \ldots, N$ we remove the unexcitable states (modes) and the associated parts of the system matrices Φ_i, Γ_i, C_i. Next a procedure dual to this one is performed to remove the unobservable modes of the system.

Example 2: Consider the discrete-time system,

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \quad i = 0, 1, \ldots \tag{40}$$

where,

$$\Phi_i = \begin{bmatrix} 1.1 & -0.8 & 0.7 & 0.6 \\ 0 & 0.9 & -0.5 & 0.6 \\ 0 & 0.9 & 0.7 & -0.4 \\ 0 & 0 & 0 & 0.8 \end{bmatrix},$$

$$\Gamma_i = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad i = 0, 1, \ldots, \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{41}$$

With the data (41), the system (40) is time invariant and has constant dimensions. Observe that the second, third and fourth state variable are unreachable and that the fourth state variable in addition is not excited by the initial condition. Then from the theory we developed it follows that the systems (40), (41) at each time $i = 1, 2, \ldots$ has three unreachable modes of which one is excited by the initial condition (not two because there is at most one unreachable excited mode!). At the initial time $i = 0$ no mode is reachable, one mode is excited and three are unexcitable.

Next we apply a randomly selected basis transformation at each time $i = 0, 1, \ldots$ characterised by,

$$T = \begin{bmatrix} -0.4326 & -1.1465 & 0.3273 & -0.5883 \\ -1.6656 & 1.1909 & 0.1746 & 2.1832 \\ 0.1253 & 1.1892 & -0.1867 & -0.1364 \\ 0.2877 & -0.0376 & 0.7258 & 0.1139 \end{bmatrix} \tag{42}$$

The columns of T represent the new basis vectors. After this basis transformation the system is described by,

$$\Phi_i = \begin{bmatrix} -0.6773 & 1.4075 & -0.7276 & 1.7723 \\ -1.0107 & 0.0266 & 0.6553 & 1.4770 \\ 0.4083 & -0.7376 & 1.2682 & -0.4507 \\ 0.7963 & 0.8894 & -0.9292 & -0.3174 \end{bmatrix},$$
$$\Gamma_i = \begin{bmatrix} -0.9936 & -1.9871 \\ 0.0907 & 0.1814 \\ 0.5319 & 1.0639 \\ -0.8500 & -1.7001 \end{bmatrix}, \quad i = 0, 1, \ldots,$$
$$x_0 = \begin{bmatrix} -1.9394 \\ 1.0241 \\ 1.0834 \\ -1.6669 \end{bmatrix} \tag{43}$$

For the systems (40), (43) we calculate state basis A for each time $i = 0, 1, 2, 3$ in the manner described before Definition 7. When represented in this state basis we obtain the following weak reachability unit staircase form of the system,

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} 0 & -3.5237 & -8.2837 & -1.3053 \\ 1 & 1.5636 & 5.1487 & 1.3265 \\ 0 & 0.2714 & 1.0816 & 0.3796 \\ 0 & 0.7240 & 0.1732 & 0.8626 \end{bmatrix},$$
$$\Gamma_0 = \begin{bmatrix} 1.4145 & 2.8291 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Phi_1 = \begin{bmatrix} 1.1000 & 0 & -12.1088 & -5.4646 \\ 0 & 1 & 4.4661 & 2.0827 \\ 0 & 0 & -5.5552 & -4.8298 \\ 0 & 0 & 7.6803 & 7.0041 \end{bmatrix},$$
$$\Gamma_1 = \begin{bmatrix} 1.4145 & 2.8291 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Phi_2 = \begin{bmatrix} 1.1000 & 0 & -5.1253 & -2.4732 \\ 0 & 1 & 1.6895 & 0.8006 \\ 0 & 0 & -1.3576 & -1.0771 \\ 0 & 0 & 0.7754 & 0.6228 \end{bmatrix},$$
$$\Gamma_2 = \begin{bmatrix} 1.4145 & 2.8291 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{44}$$

From this weak reachability unit staircase form observe that at $i = 1, 2, 3$, the system has indeed a single reachable, a single excited unreachable and two unexcitable modes. Whatever the system, at the initial time zero no mode is reachable. Therefore the structure of Φ_0 corresponds to the final two columns of the block structure of Φ_i in (37). Because the initial state is non-zero at the initial time $i = 0$, the system has one excited unreachable and three unexcitable modes. Also, note that the possibly non-zero numbers in the weak reachability unit staircase form are not unique because the choice of the state basis as described before Definition 7 is not unique. Finally, observe that although we started with a time-invariant system the associated weak reachability unit staircase form is time varying. This reflects the fact that the weak reachability unit and double staircase form do not preserve time invariance.

Remark 3: From Example 2, we can see how to translate the results of this article to linear systems with constant dimensions. From Evans (1972), Weiss (1972) and the extension for non-zero initial conditions presented in this article, observe that a system with constant dimensions is called minimal if there is *at least one time instant* where all the system modes are both excitable (weakly reachable) and observable. Similarly, for systems with constant dimensions we may weaken the definitions of reachabilty, weak reachability and observability. E.g. weak reachability then means that there is at least one time instant where all the system modes are excitable (weakly reachable). Using these weakened definitions, if the discrete-time linear system is time invariant and if the horizon is infinite, we obtain the equivalence of minimality with weak reachability together with observability as presented in Sontag (1979). Note that in Sontag (1979) span-canonical corresponds to minimal and span-reachable to weakly reachable.

Remark 4: To check whether all modes are both excitable (weakly reachable) and observable at some time $0 \leq i \leq N$, it suffices to check whether the weak reachability grammian $W'_{0,i}$ and the observability grammian $M_{i,N}$ are both full rank. If the systems time horizon N instead of finite is infinite, then instead of checking the rank of $M_{i,N}$ we must check whether there is a time $j > i$ such that $M_{i,j}$ is full rank. If time extends to $-\infty$, initial conditions play no role and instead of $W'_{0,i}$ we must check whether there is a $j < i$ such that $W'_{j,i}$ is full rank.

Remark 5: Example 2 demonstrates that, even within the class of systems with constant dimensions, the systems (40), (41) are not minimal because at *every* time instant, $i = 0, 1, 2, \ldots$ the system has at least two unexcitable modes. Also, observe that if we remove these two unexcitable modes at every time instant $i = 0, 1, 2, \ldots$ we are left with a minimal realisation with constant dimensions according to Remarks 3 and 4. However, this minimal realisation has one excited unreachable mode at every time instant so it is not minimal in the conventional sense. If we also remove this mode, the system does become minimal in the conventional sense but at the expense of eliminating the contribution of the free response to the output.

4. Weak reachability, weak observability and stochastic boundary conditions and inputs

This section explains why Theorem 3 and the weak reachability double staircase form (39) also apply to systems with stochastic initial conditions and stochastic inputs. This result is of practical importance for discrete-time (digital) optimal reduced-order LQG compensation of non-linear systems (Van Willigenburg and De Koning, 2001b, 2002). Furthermore, the equivalence of minimality with observability is explained when we take the behavioural systems point of view.

If the initial state is stochastic, the linear system becomes stochastic and we have to consider first and second moments of the state, the output and possibly the input. As in LQG control system design, higher moments will not be considered. The concept of excitable (weakly reachable) modes developed in the previous section is especially suitable to be generalised to the stochastic case considered in this section. This is reflected by the following generalised definition and its interpretation where an overbar denotes expectation.

Definition 11: A system (x_0, Φ^N, Γ^N) with a stochastic initial condition x_0 is called *excitable* (*weakly reachable*) if it satisfies Definition 6 and the associated Equations (24), (30) with $x_0 x_0^T$ replaced by $\overline{x_0 x_0^T}$.

First, note that replacing $x_0 x_0^T$ by $\overline{x_0 x_0^T}$ generalises the case where x_0 is deterministic. The initial value of the weak reachability grammian is now by definition equal to the *second moment* of the stochastic initial state. Observe that the first term in Equation (24) now describes precisely the propagation of the second moment of the system when the inputs are zero. Also now $\mathrm{rank}(W'_{0,0})$ is no longer necessarily equal to zero or one. As a result $\mathrm{rank}(W'_{0,i}) - \mathrm{rank}(W_{0,i}) > 1$ may now hold in Lemmas 2 and 3 where $\mathrm{rank}(W'_{0,i}) - \mathrm{rank}(W_{0,i})$ is now *non-increasing* with i. Then from Definition 7, the single unreachable excited mode associated to either basis vector A or B must now be replaced with $\mathrm{rank}(W'_{0,i}) - \mathrm{rank}(W_{0,i})$ unreachable system modes that are excited by the non-zero stochastic initial condition. Due to the stochastic nature of the system, a system mode that is excited now is a mode that is *non-zero with probability one*.

Theorem 4: *If the initial state, and possibly the inputs, are stochastic* Theorem 3, Lemma 3 *and the weak reachability double staircase form* (39) *still apply. Only now, before time k in* Lemma 3, *the number of unreachable modes n_i^{eur} that are excited by the initial state may be larger than one and is non-increasing with i.*

Proof: Follows from Definition 11 and the results presented in Van Willigenburg and De Koning (2002). □

Example 3: Consider again the discrete-time systems (40), (41) from Example 2 but with the deterministic initial condition x_0 replaced with the stochastic initial condition,

$$\overline{x_0 x_0^T} = \begin{bmatrix} 1.2432 & 0.1658 & 0.3330 & 0 \\ 0.1658 & 1.9400 & -0.5465 & 0 \\ 0.3330 & -0.5465 & 1.0637 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{45}$$

Observe that the initial state excites the first three states of the system at each time $i = 0, 1, \ldots$. At times $i = 1, 2, \ldots$, the first state of the system is reachable and excited by the initial state, the second and third state are unreachable but excited by the initial state while the fourth state is unexcitable.

Applying again the state basis transformation characterised by (42) at each time $i = 0, 1, \ldots$ the system is described by (40), (43) but with the stochastic initial condition,

$$\overline{x_0 x_0^T} = \begin{bmatrix} 2.1391 & -0.8694 & -1.2057 & 1.9930 \\ -0.8694 & 0.8014 & -0.5465 & -1.2948 \\ -1.2057 & -0.5465 & 0.7083 & -1.2736 \\ 1.9930 & -1.2948 & -1.2736 & 2.6542 \end{bmatrix} \tag{46}$$

For the systems (40), (43), (46), again a state basis for $i = 0, 1, 2, 3 \ldots$ is calculated in the manner described before Definition 7. Now the $\text{rank}(W'_{0,i}) - \text{rank}(W_{0,i})$ basis vectors are computed such that, together with the first $\text{rank}(W_{0,i})$ basis vectors that span the reachable space, they span the excitable space, determined by $W'_{0,i}$. The computation of these basis vectors can be performed by computing singular value decompositions of both $W_{0,i}$ and $W'_{0,i}$ (Van Willigenburg and De Koning 2002). Representing the system in this state basis we obtain the following weak reachability double staircase form of the system for $i = 0, 1, 2, 3$,

$$\overline{x_0 x_0^T} = \begin{bmatrix} 5.6680 & 0 & 0 & 0 \\ 0 & 0.5324 & 0 & 0 \\ 0 & 0 & 0.1026 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Phi_0 = \begin{bmatrix} 0.0317 & 2.1747 & -3.7253 & 3.1702 \\ 1.1742 & -2.2011 & 4.0656 & -2.9605 \\ 0.0067 & -1.0163 & -1.9048 & -0.0266 \\ 0 & 0 & 0 & -0.8000 \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} 1.4145 & 2.8291 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_1 = \begin{bmatrix} 1.1000 & 0.0220 & 0.5908 & -4.0132 \\ 0 & 1.1398 & -0.9020 & 3.7792 \\ 0 & 0.0471 & 1.6993 & -0.3450 \\ 0 & 0 & 0 & 0.8000 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 1.4145 & 2.8291 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} 1.1000 & 0.0222 & 0.2069 & -3.4118 \\ 0 & 1.0931 & -0.6315 & 3.1595 \\ 0 & 0.1046 & 1.1201 & -0.5026 \\ 0 & 0 & 0 & -0.8000 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} 1.4145 & 2.8291 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{47}$$

From the weak reachability double staircase form (47) observe that, as expected from our earlier observations, the system at times $i = 1, 2, 3$, has three modes excited by the initial condition of which two are unreachable. In addition, there is one unexcitable mode. At time $i = 0$, the system also has three modes that are excited by the non-zero stochastic initial condition. Like Example 2, even within the class of systems with constant dimensions, the system in this example is not minimal.

Remark 6: If the initial state is taken to be stochastic, it may be selected to excite *every* mode of the system at the initial time $i = 0$. Then $W'_{0,0}$ is full rank. In that case, if the system has constant dimensions, it follows from Remarks 3 and 4 that the system is minimal if $M_{0,N}$ (or $M_{0,j}$ for some $j > 0$) is full rank. If moreover, the system is time invariant and has an infinite horizon, then the latter condition is equivalent with *observability* of the system.

The behavioural approach to systems views mathematical models as *exclusion laws* (Willems and Polderman 1998). When applying this point of view to state-space models, *no* presumptions are made with respect to the initial state. In our analysis this is equivalent with stating that all the modes of the initial state are excited i.e. $W'_{0,0}$ is full rank. Similarly in this case, if the system is linear time invariant and has an infinite horizon, minimality is equivalent with observability. This complies with Willems and Polderman (1998).

Let (Φ^N, C^N, x_N) denote the system (16) with a stochastic terminal condition x_N. Then dual to the weak reachability grammian $W'_{0,i}$, given in recursive form by (30) with $x_0x_0^T$ replaced by $\overline{x_0x_0^T}$, we have the weak observability grammian $M'_{i,N}$,

$$M'_{i,N} = \Phi_i^T M'_{i+1,N}\Phi_i + C_i^T C_i, \quad i = 0, 1, \ldots, N-1,$$
$$M'_{N,N} = \overline{x_N x_N^T} \in R^{n_N \times n_N} \tag{48}$$

Then dual to weak reachability we may define weak observability.

Definition 12: (Φ^N, C^N, x_N) is called *weakly observable* if $(x_N, \Phi^{T^N}, C^{T^N})$ is weakly reachable.

Note that $(x_N, \Phi^{T^N}, C^{T^N})$ denotes a system with an initial condition equal to x_N, the terminal condition of the system (Φ^N, C^N, x_N). Dual to the weak reachability double staircase form (39), we have the weak observability double staircase form of the system (Φ^N, C^N, x_N),

$$x_i = \begin{bmatrix} x_i^o \\ x_i^{uoe} \\ x_i^{uo} \end{bmatrix}, \quad \Phi_i = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}, \quad C_i = \begin{bmatrix} \times & 0 & 0 \end{bmatrix} \tag{49}$$

All observations at and after each time i being zero implies that, at time i, $\text{rank}(M_{i,N})$ observable system modes x_i^o are zero with probability one. Furthermore, $\text{rank}(M'_{i,N}) - \text{rank}(M_{i,N})$ unobservable system modes x_i^{uoe} excite one or several of the excited terminal modes specified by the terminal condition $\overline{x_N x_N^T}$. This excitation is dual to the excitation of unreachable modes at time i by the initial condition $\overline{x_0 x_0^T}$. Finally, x_i^{uo} are the $n_i - \text{rank}(M'_{i,N})$ remaining unobservable modes that do not excite any of the excited terminal modes.

5. Conclusions

The free response associated to non-zero initial conditions contributes to the output behaviour of systems and therefore influences minimality. This article revealed this influence precisely for time-varying linear discrete-time systems. The precise influence is most apparent from two canonical system representations, introduced in this article, called the unit and double weak reachability staircase form. We considered in this article the more difficult case where the time-varying linear discrete-time system has time-varying dimensions and is defined over a finite time horizon. Moreover, the case of stochastic initial conditions and inputs has been considered. It has been shown how the results specialise to linear time-invariant systems with constant dimensions and an infinite time horizon. If also the lower bound on time is removed, then initial conditions play no role and conventional results are obtained.

System modes that are excited by the initial conditions contribute to the output behaviour but need not necessarily be reachable. Therefore, if the initial conditions are non-zero, minimality is no longer equivalent with reachability together with observability. Instead, minimality is equivalent with weak reachability (excitability) together with observability. If we take the behavioural approach to systems, *no* presumptions are made with respect to the initial state. In our analysis this is equivalent with stating that all the system modes are excited at the initial time zero. Then if the system is time invariant, has constant dimensions and an infinite horizon, minimality is equivalent with observability.

With non-zero initial conditions both the reachable modes and the modes excited by the initial condition determine the output. The modes excited by the initial condition may be partly or completely identical to the reachable modes. Therefore, realising the reachable and excitable modes separately does not result in minimal realisations. These separate realisations are obtained by applying the conventional realisation theory, which disregards the free response, in combination with a realisation of the shifts of the origins of the state basis at each discrete time i, to account for the free response, as suggested in Section 2.

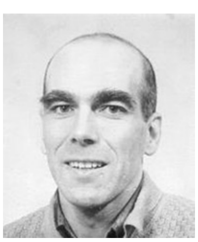
Although the influence of the initial conditions on minimality is most apparent for time-varying linear discrete-time systems with time-varying dimensions, Example 2 and Remark 4 in this article demonstrated that even for some time-invariant systems with constant dimensions this influence cannot be ignored.

For time-varying linear discrete-time systems with *constant* dimensions and zero initial conditions, the problem of identifying a minimal realisation from input–output data or an input–output map has been solved (Verhaegen and Yu 1995). How to include non-zero initial conditions follows from Van den Hof (1998). Having obtained a minimal realisation with constant dimensions, we may apply our minimisation procedure to obtain a minimal realisation with generally smaller and variable dimensions. As such the results of this article generalise the realisation theory for linear discrete-time systems with a possibly finite horizon and variable dimensions to the case where the initial state is non-zero.

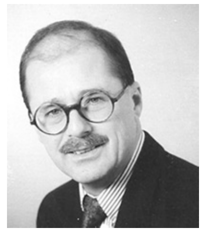
Our research into minimal realisations of finite-horizon time-varying discrete-time linear systems (controllers) in this article and in Van Willigenburg and De Koning (2001b, 2002) was initiated by rank conditions that arise in optimal reduced-order discrete-time LQG problems. In the time-invariant infinite-horizon case, the rank conditions can easily be linked to the minimality of the LQG controller (Van Willigenburg and De Koning 2000). In the finite horizon time-varying case, these rank conditions can only be linked to minimal LQG controllers if these are allowed to have time-varying dimensions (Van Willigenburg and De Koning (1999, 2002). The realisation theory in this article was already largely developed in our previous paper Van Willigenburg and De Koning (2002). In this article, using an empty matrix concept, several system properties were slightly modified at the initial and terminal time. Due to these small but subtle modifications, the theory now seems mature. It not only matches the conventional realisation theory, which does not include the free response and a finite horizon, but it also matches Sontag (1979) and Van den Hof (1998), which do consider the free response. As in these papers, minimality is equivalent with weak reachability (excitability) together with observability. The definitions of weak reachability, observability and minimality in this article and in Sontag (1979) and Van den Hof (1998) differ. This difference relates to the constant *versus* the time-varying system dimensions (Remark 2).

As to the equivalence of minimality with weak reachability together with observability the picture is not yet complete. In this article, the equivalence was established for time-varying linear discrete-time systems *with time-varying dimensions* and a finite or infinite horizon and for linear time-invariant discrete-time systems with constant dimensions and an infinite horizon. Can we obtain the same equivalence in the case of time-varying discrete-time systems *with constant dimensions* by adapting, in an acceptable manner, the definitions of minimality, weak reachability and observability? The same question applies to linear time-invariant discrete-time systems with constant dimensions *defined over a finite time horizon*. And what about continuous-time linear systems? These will be the topics of future research.

Notes on contributors



L. Gerard Van Willigenburg was born in Leiden, The Netherlands, in 1958. He received his MSc in electrical engineering and his PhD degree in control engineering at Delft University of Technology, The Netherlands, in 1983 and 1991, respectively. From 1986 to 1991 he was a research engineer at the process dynamics and control group in the department of applied physics. Since 1991, he has been an Assistant Professor of the systems and control group at Wageningen University. His research interests include digital optimal control, reduced-order control, model predictive control and adaptive dual control. The application areas range from indoor climate control (storage rooms, greenhouses, stables and buildings), robot control, automatic guidance, to process control (fermentation) and the control of economic and living systems (plants).



Willem L. De Koning was born in Leiden, The Netherlands, in 1944. He received his MSc in electrical engineering and his PhD degree in control engineering at Delft University of Technology, The Netherlands, in 1975 and 1984, respectively. From 1969 to 1975 he was a research engineer of power electronics and control at the department of electrical engineering. From 1987 to 2006, he was an Associate Professor of mathematical system theory at the department of mathematical system theory in the department of technical mathematics and informatics. He has also held a visiting position at the Florida Institute of Technology, Melbourne. His research interests include control of distributed parameter systems, robust control, adaptive control, reduced-order control and digital control. The main application areas are the process industry and mechatronics. Although he retired in 2006, he continues some of his research on a personal basis.

References

Evans, D.S. (1972), "Finite-dimensional Realizations of Discrete-time Weighting Patterns," *SIAM Journal of Applied Mathematics*, 22(1), 45–67.

Gohberg, I., Kaashoek, M.A., and Lerer, L. (1992), "Minimality and Realization of Discrete Time-varying Systems," *Operator theory: Advances and applications, Birkhauser*, 56, 261–296.

Kalman, R.E., Falb, P.L., and Arbib, M. (1969), *Topics in Mathematical Systems Theory*, New York: Mc Graw Hill.

Nett, C.N., and Haddad, W.M. (1993), "A System-theoretic Appropriate Realization of the Empty Matrix Concept," *IEEE Transactions on Automatic Control*, 38(5), 771–775.

Sontag, E.D. (1979), "Realization Theory of Discrete-time Nonlinear Systems: Part 1 – The Bounded Case," *IEEE Transactions on Circuits and Systems*, CAS-26(4), 342–356.

Van den Hof, J.M. (1998), "Structural Identifiability of Linear Compartmental Systems," *IEEE Transactions on Automatic Control*, 43(6), 800–818.

Van der Veen, A.J. and Dewilde, P. (1992), "Time-varying Systems for Computational Networks," in *Algorithms and Parallel VLSI Architectures II*, eds. P. Quinton and Y. Roberts, Dordrecht: Elsevier Science Publishers, pp. 103–127.

Van Willigenburg, L.G., and De Koning, W.L. (1999), "Optimal Reduced-order Compensators for Time-varying Discrete-time Systems with Deterministic and White Parameters," *Automatica*, 35, 129–138.

Van Willigenburg, L.G., and De Koning, W.L. (2000), "Numerical Algorithms and Issues Concerning the Discrete-time Optimal Projection Equations," *European Journal of Control*, 6(1), 93–110.

Van Willigenburg, L.G., and De Koning, W.L. (2001a), "Synthesis of Digital Optimal Reduced-order

Compensators for Asynchronously Sampled Systems," *International Journal of System Science*, 32(7), 825–835.
Van Willigenburg, L.G., and De Koning, W.L. (2001b), "Minimality, Canonical Forms and Storage of Finite-horizon Discrete-time Compensators," Preprints IFAC World Congress, Barcelona, 21–26 July, 2002, Paper 984.
Van Willigenburg, L.G., and De Koning, W.L. (2002), "Minimal and Non-minimal Optimal Fixed-order Compensators for Time-varying Discrete-time Systems," *Automatica*, 38(1), 157–165.
Verhaegen, M., and Yu, X. (1995), "A Class of Subspace Model Identification Algorithms to Identify Periodically and Arbitrarily Time-varying Systems," *Automatica*, 31(2), 201–216.
Weiss, L. (1972), "Controllability, Realization and Stability of Discrete-time Systems," *SIAM Journal on Control*, 10(2), 230–251.
Willlems, J.C., and Polderman, J.W. (1998), *Introduction to Mathematical Systems Theory*, New York Inc.: Springer-Verlag.