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Compensatability and optimal compensation of systems with white parameters in the delta domain

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Using the delta operator, the strengthened discrete-time optimal projection equations for optimal reduced-order compensation of systems with white stochastic parameters are formulated in the delta domain. The delta domain unifies discrete time and continuous time. Moreover, when formulated in this domain, the efficiency and numerical conditioning of algorithms improves when the sampling rate is high. Exploiting the unification, important theoretical results, algorithms and compensatability tests concerning finite and infinite horizon optimal compensation of systems with white stochastic parameters are carried over from discrete time to continuous time. Among others, we consider the finite-horizon time-varying compensation problem for systems with white stochastic parameters and the property mean-square compensatability (ms-compensatability) that determines whether a system with white stochastic parameters can be stabilised by means of a compensator. In continuous time, both of these appear to be new. This also holds for the associated numerical algorithms and tests to verify ms-compensatability. They are illustrated with three numerical examples that reveal several interesting theoretical and numerical issues. A fourth example illustrates the improvement of both the efficiency and numerical conditioning of the algorithms. This is of vital practical importance for digital control system design when the sampling rate is high.

Keywords: compensatability; dynamic output feedback; optimal projection equations; white stochastic parameters; multiplicative white noise

1. Introduction

There are mainly three reasons why systems with white stochastic parameters are important. Firstly, they may appear naturally because some of the plant parameters may be white (Wagenaar and De Koning 1988). Secondly, when designing digital control systems, some of the controller parameters may be white, such as the sampling period (Immer, Yükselb, and Basar 2006; Sinopoli, Schenato, Franceschetti, Poolla, and Sastry 2008) or the controller parameters due to finite word length (Wingerden and De Koning 1984). Thirdly, parameters may be *assumed* white to design a robust controller (Bernstein and Greeley 1986; Banning and De Koning 1995).

Designing and representing control systems in the *delta domain* is important for two reasons. Firstly, it enables a unification of continuous and discrete-time control system designs. Thereby, results from one domain can be carried over to the other. Secondly, if the sampling rate of a digital control system is high, e.g. due to oversampling, control system design in the delta domain is more accurate and efficient as compared to conventional digital control system design in discrete time. This applies both to the controller computation and implementation (Middleton and Goodwin 1990; Yuza, Goodwin, Feuer, and De Doná 2005).

Using the delta operator, in this article important results concerning the design of optimal full and reduced-order compensators for different types of linear systems in discrete time are carried over to continuous time, where they are partly new. The starting point is the so-called strengthened discrete-time optimal projection equations (SDOPE) for systems with white stochastic parameters developed in Bernstein and Hyland (1986) and Van Willigenburg and De Koning (1999). As opposed to the conventional discrete-time optimal projection equations (Bernstein and Hyland 1986; Haddad and Moser 1994), the SDOPE are *equivalent* to first-order necessary optimality conditions for discrete-time optimal compensation and the condition that the compensator is minimal. Based on the SDOPE, efficient numerical algorithms have been presented to solve the SDOPE and compute the associated discrete-time optimal reduced-order compensator (Bernstein and

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Hyland 1986; Haddad and Moser 1994; Van Willigenburg and De Koning 1999; De Koning and Van Willigenburg 2000). In addition to this, two compensatability tests, one of which is actually a *measure* of compensatability, are also based on the SDOPE. Compensatability refers to the possibility to design a feedback controller that stabilises the closed-loop system. If the order of the compensator is reduced, or when the system parameters are white, compensatability becomes a property that can no longer be 'broken down' into properties, such as stabilisability and detectability (De Koning 1992).

In this article, the SDOPE and associated algorithms are formulated in the delta domain. This formulation unifies continuous time and discrete time (Middleton and Goodwin 1990). Exploiting the unification, we establish that the optimal projection equations for the finite and infinite horizon deterministic parameter continuous-time cases, presented in Hyland and Bernstein (1984) and Haddad and Tadmor (1993), coincide with the strengthened optimal projection equations in the delta domain. This proves the *equivalence* of the continuous-time optimal projection equations to first-order necessary optimality conditions together with the condition that the compensator is minimal. Observe that this equivalence was suggested in Hyland and Bernstein (1984) and Haddad and Tadmor (1993) but not proved. In discrete time, the optimal projections were modified twice before equivalence was obtained (Van Willigenburg and De Koning 2000a). These modifications revealed some subtle delicate issues related to the rank conditions that must be imposed. Imposing these numerically is the key difficulty in finding numerical solutions to the optimal projection equations. Using our formulation in the delta domain enables us to use these insights to properly impose the rank conditions in continuous time.

In the finite-horizon continuous-time deterministic parameter case, the inability to specify the optimal projection equations explicitly in the LQG problem parameters (Haddad and Tadmor 1993) is explained and resolved in this article using results from Van Willigenburg and De Koning (1999, 2008). This article extends the results presented in Van Willigenburg and De Koning (2000b) that considered linear systems with deterministic parameters to systems with white stochastic parameters. Therefore, this article also strengthens and unifies the discrete-time and continuous-time results presented in De Koning (1982), Hyland and Bernstein (1984), Bernstein and Haddad (1987) and Bernstein and Hyland (1988).

The unification includes the *finite-horizon time-varying* compensation problem for systems with white stochastic parameters in continuous time. To the best of our knowledge, this problem has only been considered in discrete time. The principal application of this problem concerns the robust compensation of nonlinear systems about (optimal) trajectories (Athans 1971). Finally, the property mean-square compensatability (ms-compensatability) and the associated numerical tests to verify this property, introduced in De Koning (1992), are carried over to continuous time where they also appear to be new. In this article, systems with deterministic parameters will be treated as a special case of systems with white stochastic parameters. Also, the infinite-horizon time-invariant compensation problem will be treated as a special case of the finite-horizon time-varying compensation problem. The algorithms and compensatability tests are illustrated with three numerical examples. One example is selected to demonstrate interesting theoretical and numerical issues. A fourth example illustrates the advantage and necessity of control system design in the delta domain if the sampling rate is high.

2. The finite-horizon optimal reduced-order compensation problem for discrete-time systems with white stochastic parameters

When considering optimal reduced-order compensation problems, minimality is a crucial property of the compensator. If the horizon is finite, minimal compensators have *time-varying dimensions* in general (Van Willigenburg and De Koning 1999, 2008). Time-varying dimensions raise a number of technicalities that we want to avoid in this article. To avoid these, in the problem formulation we have to consider *pseudo-minimal* compensators (Van Willigenburg and De Koning 2002). Defining pseudo-minimal compensators is rather involved. Therefore, we have decided not to do so in this article. We think this is justified because the only difference pseudo-minimal compensators make in our main theorem is to allow for an inequality in Equation (73), instead of an equality. Furthermore, the algorithms presented in this article and in Van Willigenburg and De Koning (1999, 2002, 2008) automatically produce this type of compensator. For details, we refer to Van Willigenburg and De Koning (2002).

Consider the following discrete-time system,

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i, \quad i = 0, 1, \ldots, N-1, \tag{1}$$

$$y_i = C_i x_i + w_i, \quad i = 0, 1, \ldots, N, \tag{2}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ is the control and $y_i \in R^l$ is the observation. The processes $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are sequences of independent random matrices with appropriate dimensions and time-varying statistics and

$\{v_i\}$, $\{w_i\}$ are sequences of independent stochastic vectors with time-varying statistics. The initial condition x_0 is stochastic with mean $\bar{x}_0$ and covariance X and is independent of $\{\Phi_i, \Gamma_i, C_i, v_i, w_i\}$. Moreover, Γ_i and C_i are independent and $\{\Phi_i\}$, $\{\Gamma_i\}$, $\{C_i\}$ are independent of $v_j, w_j, i \neq j$ and uncorrelated with v_i, w_i. The processes $\{v_i\}$, $\{w_i\}$ are zero-mean with covariances $V_i \geq 0$ and $W_i > 0$ and cross-covariance V_i'. As a controller, the following time-varying dynamic compensator is chosen,

$$\hat{x}_{i+1} = F_i\hat{x}_i + K_i y_i, \quad i = 0, 1, \ldots, N, \tag{3}$$

$$u_i = -L_i\hat{x}_i, \quad i = 0, 1, \ldots, N, \tag{4}$$

where $\hat{x}_i \in R^{n^c}$ is the compensator state. The dimension n^c of the compensator state is fixed by the design. It is assumed to be of reduced or full order, i.e. $n^c \leq n$, $i = 0, 1, \ldots, N$. $F_i \in R^{n^c \times n^c}$, $K_i \in R^{n^c \times l}$ and $L_i \in R^{m \times n^c}$ are real matrices. The initial condition $\hat{x}_0 \in R^{n^c}$ is deterministic. Compensator (3), (4) is denoted by $(\hat{x}_0, F^N, K^N, L^N)$ where $F^N = \{F_i, i = 0, 1, \ldots, N-1\}$, $K^N = \{K_i, i = 0, 1, \ldots, N-1\}$ and $L^N = \{L_i, i = 0, 1, \ldots, N-1\}$.

2.1 *Problem formulation*

Given the system (1), (2) find a pseudo-minimal compensator (3), (4) with fixed prescribed dimensions $n^c \leq n$ that minimises the criterion,

$$J_N(\hat{x}_0, F^N, K^N, L^N) = E\left\{x_N^T Z x_N + \sum_{i=0}^{N-1} \left(x_i^T Q_i x_i + 2x_i^T Q_i' u_i + u_i^T R_i u_i\right)\right\}, \tag{5}$$

$$Q_i \geq 0, \quad R_i > 0, \quad Q_i - Q_i' R_i^{-1} Q_i'^T \geq 0, \quad Z \geq 0, \tag{6}$$

and find the minimum value of J_N.

Note that the first three conditions in Equation (6) guarantee $\left[\begin{smallmatrix} Q_i & Q_i' \\ Q_i'^T & R_i \end{smallmatrix}\right] \geq 0$ (Kreindler and Jameson 1972).

3. Problem reformulation in the delta domain

The delta operator δ is defined as follows (Middleton and Goodwin 1990),

$$\delta a_i = (a_{i+1} - a_i)/T, \quad T > 0, \tag{7}$$

where a_i are vectors or matrices of equal dimension and where $T > 0$ is arbitrarily fixed. Then the discrete-time compensation problem (1)–(6) may be written in terms of the delta operator,

$$\delta x_i = \Phi_i^\delta x_i + \Gamma_i^\delta u_i + v_i^\delta, \quad i = 0, 1, \ldots, N-1, \tag{8}$$

$$y_i = C_i^\delta x_i + w_i^\delta, \quad i = 0, 1, \ldots, N, \tag{9}$$

$$\delta\hat{x}_{i+1} = F_i^\delta \hat{x}_i + K_i^\delta y_i, \quad i = 0, 1, \ldots, N-1, \tag{10}$$

$$u_i = -L_i^\delta \hat{x}_i, \quad i = 0, 1, \ldots, N-1, \tag{11}$$

$$J_N(\hat{x}_0, F^{\delta N}, K^{\delta N}, L^{\delta N}) = E\left\{x_N^T Z x_N + \sum_{i=0}^{N-1} \left(x_i^T Q_i^\delta x_i + 2x_i^T Q_i' \delta u_i + u_i^T R_i^\delta u_i\right)\right\}. \tag{12}$$

In (8), (9), $v_i^\delta = v_i/T$, $w_i^\delta = w_i$ are discrete-time zero-mean white noise processes with covariance matrices $V_i^\delta \geq 0$, $W_i^\delta > 0$ and cross-covariance matrix $V_i'^\delta$. Furthermore, $Q_i^\delta, Q_i'^\delta, R_i^\delta$ satisfy conditions similar to those mentioned in Equation (6). The correspondence between data from problem description (1)–(6), and on the other hand problem description (8)–(12), is indicated below. On the left of each arrow are data from problem description (1)–(6) and on the right from problem description (8)–(12).

$$\Phi_i \leftrightarrow T\Phi_i^\delta + I_n, \quad \Gamma_i \leftrightarrow T\Gamma_i^\delta, \quad C_i \leftrightarrow C_i^\delta,$$
$$V_i \leftrightarrow T^2 V_i^\delta, \quad V_i' \leftrightarrow TV_i'^\delta, \quad W_i \leftrightarrow W_i^\delta, \tag{13}$$

$$Q_i \leftrightarrow Q_i^\delta, \quad Q_i' \leftrightarrow Q_i'^\delta, \quad R_i \leftrightarrow R_i^\delta, \tag{14}$$

$$F_i \leftrightarrow TF_i^\delta + I_{n^c}, \quad K_i \leftrightarrow TK_i^\delta, \quad L_i \leftrightarrow L_i^\delta. \tag{15}$$

In (13)–(15), I denotes a square identity matrix with a dimension given by the subscript. Because our aim is to unify discrete-time and continuous-time results, the discrete-time compensation problem description based on the delta operator (8)–(15) is now linked to that of a continuous-time compensation problem. To do this, associate the discrete-time instants i with continuous-time instants $t = iT$. Then, under suitable continuity conditions, in the limit $T \downarrow 0$, $t = iT$ fixed, the problem (8)–(15) equals the following continuous-time compensation problem,

$$\dot{x} = A(t)x(t) + B(t)u(t) + v(t), \tag{16}$$

$$y(t) = C(t)x(t) + w(t), \quad t \in [0, t_f], \tag{17}$$

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + K(t)y(t), \tag{18}$$

$$u(t) = -L(t)\hat{x}(t), \quad t \in [0, t_f], \tag{19}$$

$$J_{t_f}(\hat{x}(0), F(\cdot), K(\cdot), L(\cdot)) = E\{x(t_f)Zx(t_f)\} + E\left\{\int_0^{t_f} x^T(t)Q(t)x(t) + x^T(t)Q'(t)u(t) + u^T(t)R(t)u(t)\mathrm{d}t\right\}, \tag{20}$$

where $v(\cdot)$ and $w(\cdot)$ are zero-mean white noise processes with intensity matrices $V(\cdot) \geq 0$ and $W(\cdot) > 0$ and cross-intensity matrix $V'(\cdot)$. Furthermore, $Q(\cdot), Q'(\cdot)$, $R(\cdot)$ satisfy conditions similar to those mentioned in Equation (6). The correspondence between data from problem description (8)–(12), and on the other hand problem description (16)–(20), is indicated below. On the left of each arrow are data from problem description (8)–(12) and on the right from problem description (16)–(20).

$$\Phi_i^\delta \leftrightarrow A(t), \quad \Gamma_i^\delta \leftrightarrow B(t), \quad C_i^\delta \leftrightarrow C(t),$$

$$V_i^\delta \leftrightarrow \frac{1}{T}V(t), \quad V_i'^\delta \leftrightarrow \frac{1}{T}V'(t), \quad W_i^\delta \leftrightarrow \frac{1}{T}W(t), \tag{21}$$

$$Q_i^\delta \leftrightarrow TQ(t), \quad Q_i'^\delta \leftrightarrow TQ'(t), \quad R_i^\delta \leftrightarrow TR(t), \tag{22}$$

$$F_i^\delta \leftrightarrow F(t), \quad K_i^\delta \leftrightarrow K(t), \quad L_i^\delta \leftrightarrow L(t). \tag{23}$$

Remark 1: The time-invariant infinite-horizon discrete-time optimal reduced-order compensation problem is obtained when all matrices in the problem formulation are time-invariant and if J_N is replaced with $\frac{1}{N}J_N$, when $N \to \infty$.

Remark 2: Equations (8)–(15) may be considered an Euler approximation with time step T of Equations (16)–(23). There is a second important interpretation of Equations (8)–(15). To see this, consider the *digital* optimal compensation of the continuous-time system (16), (17), based on the integral criterion (20), by means of a *digital compensator* with a sampling period T (Van Willigenburg and De Koning 2000c, 2002). Observe that Equations (8)–(15) are also an approximation of the so-called equivalent discrete-time compensation problem presented in Van Willigenburg and De Koning (2000c, 2002). In the limit $T \downarrow 0$, $t = iT$ fixed, both the Euler approximation (8)–(15) and the equivalent discrete-time compensation problem become identical to (16)–(23).

4. Description and data of systems with white parameters

Additional data of the compensation problem needed to describe both discrete and continuous-time systems with white stochastic parameters are presented in this section. Furthermore, these data are linked to their associated delta-operator counterparts. For discrete-time systems (1), (2), at each time i, we need to specify the first and second moments of the system matrices Φ_i, Γ_i, C_i having real, random, possibly correlated, entries. The first moments are specified by $\bar{\Phi}_i$, $\bar{\Gamma}_i$, $\bar{C}_i$ where the overbar denotes expectation. $\bar{\Phi}_i$, $\bar{\Gamma}_i$, $\bar{C}_i$ are matrices having the same dimensions as Φ_i, Γ_i, C_i. Their entries are the mean values of the associated entries of Φ_i, Γ_i, C_i. The second moment of Φ_i equals,

$$\overline{\Phi_i \otimes \Phi_i} \in R^{n^2 \times n^2}, \tag{24}$$

where $\otimes$ denotes the Kronecker product. The second moment (24) may be computed knowing the first moment $\bar{\Phi}_i$ and the variance. To define the variance of Φ_i let,

$$\tilde{\Phi}_i = \Phi_i - \bar{\Phi}_i \in R^{n \times n}. \tag{25}$$

Then the variance of Φ_i equals,

$$\overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i} \in R^{n^2 \times n^2}. \tag{26}$$

Knowing the first moment $\bar{\Phi}_i$ and the variance (26), the second moment (24) is given by,

$$\overline{\Phi_i \otimes \Phi_i} = \bar{\Phi}_i \otimes \bar{\Phi}_i + \overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i} \in R^{n^2 \times n^2}. \tag{27}$$

Two system matrices at time i may be correlated except for Γ_i and C_i. If Φ_i and Γ_i are correlated, this is described by the variance,

$$\overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i} \in R^{n^2 \times nm}, \tag{28}$$

being non-zero. For the second moment $\overline{\Phi_i \otimes \Gamma_i} \in R^{n^2 \times nm}$, the following relation that is similar to (27) holds,

$$\overline{\Phi_i \otimes \Gamma_i} = \bar{\Phi}_i \otimes \bar{\Gamma}_i + \overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i} \in R^{n^2 \times nm}. \tag{29}$$

In summary, the matrices $\bar{\Phi}_i \in R_i^{n \times n}, \bar{\Gamma}_i \in R^{n \times m}, \bar{C}_i \in R^{l \times n}$ and $\overline{\tilde{\Phi}_i \otimes \tilde{\Phi}_i} \in R^{n^2 \times n^2}$, $\overline{\tilde{\Gamma}_i \otimes \tilde{\Gamma}_i} \in R^{n^2 \times m^2}$, $\overline{\tilde{C}_i \otimes \tilde{C}_i} \in R^{l^2 \times n^2}$, $\overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i} \in R^{n^2 \times nm}$, $\overline{\tilde{\Phi}_i \otimes \tilde{C}_i} \in R^{nl \times n^2}$ are the *a priori* known data related to the white stochastic discrete-time system matrices Φ_i, Γ_i, C_i (Van Willigenburg and De Koning 1999). From these data, the second moments can be computed according to (27), (29).

In addition to Equations (13)–(15), below we state the most important relations between the discrete time and corresponding delta-operator data used in this article to describe discrete-time systems with white stochastic parameters. Their interpretation relies on (24)–(27). Below I denotes the identity matrix with compatible dimensions.

$$\overline{\Phi_i \otimes \Phi_i} = \overline{\left(T\Phi_i^\delta + I\right) \otimes \left(T\Phi_i^\delta + I\right)} = T^2\overline{\Phi_i^\delta \otimes \Phi_i^\delta} + T\overline{\Phi_i^\delta} \otimes I + TI \otimes \overline{\Phi_i^\delta} + I \otimes I, \tag{30}$$

$$\overline{\bar{\Phi}_i \otimes \bar{\Phi}_i} = T^2 \overline{\bar{\Phi}_i^\delta \otimes \bar{\Phi}_i^\delta}, \tag{31}$$

$$\overline{\Phi_i \otimes \Gamma_i} = \overline{\left(T\Phi_i^\delta + I\right) \otimes T\Gamma_i^\delta} = T^2 \overline{\Phi_i^\delta \otimes \Gamma_i^\delta} + TI \otimes \overline{\Gamma_i^\delta}, \tag{32}$$

$$\overline{\tilde{\Phi}_i \otimes \tilde{\Gamma}_i} = T^2 \overline{\tilde{\Phi}_i^\delta \otimes \tilde{\Gamma}_i^\delta}, \tag{33}$$

$$\overline{\Phi_i \otimes C_i} = \overline{\left(T\Phi_i^\delta + I\right) \otimes C_i^\delta} = T\overline{\Phi_i^\delta \otimes C_i^\delta} + I \otimes \overline{C_i^\delta}, \tag{34}$$

$$\overline{\tilde{\Phi}_i \otimes \tilde{C}_i} = T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{C}_i^\delta}, \tag{35}$$

$$\overline{\Gamma_i \otimes \Gamma_i} = T^2 \overline{\Gamma_i^\delta \otimes \Gamma_i^\delta}, \tag{36}$$

$$\overline{\tilde{\Gamma}_i \otimes \tilde{\Gamma}_i} = T^2 \overline{\tilde{\Gamma}_i^\delta \otimes \tilde{\Gamma}_i^\delta}, \tag{37}$$

$$\overline{C_i \otimes C_i} = \overline{C_i^\delta \otimes C_i^\delta}, \tag{38}$$

$$\overline{\tilde{C}_i \otimes \tilde{C}_i} = \overline{\tilde{C}_i^\delta \otimes \tilde{C}_i^\delta}. \tag{39}$$

For continuous-time systems with white stochastic parameters, relations similar to (30)–(39) are needed. Continuous-time systems have been described by Equations (16), (17). If continuous-time systems have white stochastic parameters, the following description in terms of increments, denoted by the symbol d, is appropriate (Arnold 1974; Van Willigenburg and De Koning 2000c),

$$\mathrm{d}x(t) = \mathrm{d}A(t)x(t) + \mathrm{d}B(t)u(t) + \mathrm{d}v(t), \tag{40}$$

$$\mathrm{d}y(t) = \mathrm{d}C(t)x(t) + \mathrm{d}w(t). \tag{41}$$

The processes $\{A(t), t_0 \le t \le t_f\}, \{B(t), t_0 \le t \le t_f\}$, $\{C(t), t_0 \le t \le t_f\}$ have independent increments and known time-varying first and second moments. The first moments are given by $\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$. Let $\tilde{A}(t) = A(t) - \bar{A}(t)$, $\tilde{B}(t) = B(t) - \bar{B}(t)$, $\tilde{C}(t) = C(t) - \bar{C}(t)$ and consider the decompositions,

$$A(t) = \bar{A}(t) + \tilde{A}(t), \quad \mathrm{d}A(t) = \bar{A}(t)\mathrm{d}t + \mathrm{d}\tilde{A}(t), \tag{42}$$

$$B(t) = \bar{B}(t) + \tilde{B}(t), \quad \mathrm{d}B(t) = \bar{B}(t)\mathrm{d}t + \mathrm{d}\tilde{B}(t), \tag{43}$$

$$C(t) = \bar{C}(t) + \tilde{C}(t), \quad \mathrm{d}C(t) = \bar{C}(t)\mathrm{d}t + \mathrm{d}\tilde{C}(t). \tag{44}$$

Assuming $\{B(t), t_0 \le t \le t_f\}, \{C(t), t_0 \le t \le t_f\}$ to be uncorrelated, the second moments of the processes $\{A(t), t_0 \le t \le t_f\}, \{B(t), t_0 \le t \le t_f\}, \{C(t), t_0 \le t \le t_f\}$ are defined by,

$$\overline{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{A}(t)} = V^{AA}(t)\mathrm{d}t, \tag{45}$$

$$\overline{\mathrm{d}\tilde{B}(t) \otimes \mathrm{d}\tilde{B}(t)} = V^{BB}(t)\mathrm{d}t, \tag{46}$$

$$\overline{\mathrm{d}\tilde{C}(t) \otimes \mathrm{d}\tilde{C}(t)} = V^{CC}(t)\mathrm{d}t, \tag{47}$$

$$\overline{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{B}(t)} = V^{AB}(t)\mathrm{d}t, \tag{48}$$

$$\overline{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{C}(t)} = V^{AC}(t)\mathrm{d}t. \tag{49}$$

In summary, the matrices $\bar{A}(t) \in R^{n\times n}$, $\bar{B}(t) \in R^{n\times m}$, $\bar{C}(t) \in R^{l\times n}$ and $V^{AA} \in R^{n^2\times n^2}$, $V^{BB} \in R^{n^2\times m^2}$, $V^{CC} \in R^{l^2\times n^2}$, $V^{AB} \in R^{n^2\times nm}$, $V^{AC} \in R^{nl\times n^2}$ are the *a priori* known data related to the white stochastic continuous-time system matrices $\{A(t), t_0 \le t \le t_f\}$, $\{B(t), t_0 \le t \le t_f\}$, $\{C(t), t_0 \le t \le t_f\}$ (Van Willigenburg and De Koning 2000c).

In the limit, $\mathrm{d}t = T \downarrow 0$, $t = iT$ fixed, the following equivalences related to (42)–(49) hold,

$$\mathrm{d}A(t) = \bar{A}(t)\mathrm{d}t + \mathrm{d}\tilde{A}(t) \Leftrightarrow T\Phi_i^\delta = T\bar{\Phi}_i^\delta + T\tilde{\Phi}_i^\delta, \tag{50}$$

$$\mathrm{d}B(t) = \bar{B}(t)\mathrm{d}t + \mathrm{d}\tilde{B}(t) \Leftrightarrow T\Gamma_i^\delta = T\bar{\Gamma}_i^\delta + T\tilde{\Gamma}_i^\delta, \tag{51}$$

$$\mathrm{d}C(t) = \bar{C}(t)\mathrm{d}t + \mathrm{d}\tilde{C}(t) \Leftrightarrow TC_i^\delta = T\bar{C}_i^\delta + T\tilde{C}_i^\delta, \tag{52}$$

$$\begin{aligned}\overline{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{A}(t)} = V^{AA}(t)\mathrm{d}t &\Leftrightarrow \overline{T\tilde{\Phi}_i^\delta \otimes T\tilde{\Phi}_i^\delta} \\ &= TV^{AA}(t) \Leftrightarrow T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{\Phi}_i^\delta} = V^{AA}(t),\end{aligned} \tag{53}$$

$$\begin{aligned}\overline{\mathrm{d}\tilde{B}(t) \otimes \mathrm{d}\tilde{B}(t)} = V^{BB}(t)\mathrm{d}t &\Leftrightarrow \overline{T\tilde{\Gamma}_i^\delta \otimes T\tilde{\Gamma}_i^\delta} \\ &= TV^{BB}(t) \Leftrightarrow T\overline{\tilde{\Gamma}_i^\delta \otimes \tilde{\Gamma}_i^\delta} = V^{BB}(t),\end{aligned} \tag{54}$$

$$\begin{aligned}\overline{\mathrm{d}\tilde{C}(t) \otimes \mathrm{d}\tilde{C}(t)} = V^{CC}(t)\mathrm{d}t &\Leftrightarrow \overline{T\tilde{C}_i^\delta \otimes T\tilde{C}_i^\delta} \\ &= TV^{CC}(t) \Leftrightarrow T\overline{\tilde{C}_i^\delta \otimes \tilde{C}_i^\delta} = V^{CC}(t),\end{aligned} \tag{55}$$

$$\begin{aligned}\overline{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{B}(t)} = V^{AB}(t)\mathrm{d}t &\Leftrightarrow \overline{T\tilde{\Phi}_i^\delta \otimes T\tilde{\Gamma}_i^\delta} \\ &= TV^{AB}(t) \Leftrightarrow T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{\Gamma}_i^\delta} = V^{AB}(t),\end{aligned} \tag{56}$$

$$\begin{aligned}\overline{\mathrm{d}\tilde{A}(t) \otimes \mathrm{d}\tilde{C}(t)} = V^{AC}(t)\mathrm{d}t &\Leftrightarrow \overline{T\tilde{\Phi}_i^\delta \otimes T\tilde{C}_i^\delta} \\ &= TV^{AC}(t) \Leftrightarrow T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{C}_i^\delta} = V^{AC}(t).\end{aligned} \tag{57}$$

Observe from (50), (53) and (27) that in the limit $\mathrm{d}t = T \downarrow 0$, $t = iT$ fixed,

$$\begin{aligned}T\overline{\Phi_i^\delta \otimes \Phi_i^\delta} &= T\bar{\Phi}_i^\delta \otimes \bar{\Phi}_i^\delta + T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{\Phi}_i^\delta} \\ &= T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{\Phi}_i^\delta} = V^{AA}(t).\end{aligned} \tag{58}$$

Similarly, in the limit $\mathrm{d}t = T \downarrow 0$, $t = iT$ fixed,

$$T\overline{\Gamma_i^\delta \otimes \Gamma_i^\delta} = T\overline{\tilde{\Gamma}_i^\delta \otimes \tilde{\Gamma}_i^\delta} = V^{BB}(t), \tag{59}$$

$$T\overline{C_i^\delta \otimes C_i^\delta} = T\overline{\tilde{C}_i^\delta \otimes \tilde{C}_i^\delta} = V^{CC}(t), \tag{60}$$

$$T\overline{\Phi_i^\delta \otimes \Gamma_i^\delta} = T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{\Gamma}_i^\delta} = V^{AB}(t), \tag{61}$$

$$T\overline{\Phi_i^\delta \otimes C_i^\delta} = T\overline{\tilde{\Phi}_i^\delta \otimes \tilde{C}_i^\delta} = V^{AC}(t). \tag{62}$$

Remark 3: Systems with deterministic parameters are a special case of systems with white stochastic parameters. A system has deterministic parameters if all variances associated with system matrices are zero. In this article these results are obtained by deleting all terms involving a $\sim$ and by removing all overbars that denote expectation related to system matrices.

In the next section, the main theorem of this article will be presented. In this theorem, the following two expressions for some deterministic nonnegative square matrix M and stochastic system matrices Y_1, Y_2 regularly occur,

$$\overline{Y_1 M Y_2^T}, \quad \overline{Y_1^T M Y_2}. \tag{63}$$

These expressions may be computed as,

$$\begin{aligned}\overline{Y_1 M Y_2^T} &= \mathrm{st}^{-1}\left[\left(\overline{Y_1 \otimes Y_2}\right)\mathrm{st}(M)\right],\\ \overline{Y_1^T M Y_2} &= \mathrm{st}^{-1}\left[\left(\overline{Y_1 \otimes Y_2}\right)^T \mathrm{st}(M)\right],\end{aligned} \tag{64}$$

where st denotes the stack operator and st^{-1} the inverse (un) stack operator.

5. Strengthened optimal projection equations in the delta domain

The SDOPE for finite-horizon optimal reduced-order compensation of time-varying discrete-time systems with white stochastic parameters were presented in Van Willigenburg and De Koning (1999, Theorem 1). Associated with this theorem are some technical issues related to the minimality of compensators that demand time-variable (state) dimensions. These issues are treated and discussed in Van Willigenburg and De Koning (2002, 2008). The main theorem in this section is identical to theorem 1 in Van Willigenburg and De Koning (1999) but expressed in the delta domain and generalised in the manner described in theorem 1 of Van Willigenburg and De Koning (2002). To state the main theorem the following notation is introduced.

$$\begin{aligned}K^\delta_{P_i,\hat{P}_i} &= \left(\overline{P_i C_i^{\delta T}} + T\overline{\Phi_i^\delta P_i C_i^{\delta T}} + T\overline{\tilde{\Phi}_i^\delta \hat{P}_i \tilde{C}_i^{\delta T}} + TV_i'^\delta\right)\\ &\quad\times\left(T\overline{C_i^\delta P_i C_i^{\delta T}} + T\overline{\tilde{C}_i^\delta \hat{P}_i \tilde{C}_i^{\delta T}} + TW_i^\delta\right)^{-1},\end{aligned} \tag{65}$$

$$\begin{aligned}&L^\delta_{S_{i+1},\hat{S}_{i+1}}\\ &= \left(T\overline{\Gamma_i^{\delta T} S_{i+1}\Gamma_i^\delta} + T\overline{\tilde{\Gamma}_i^{\delta T}\hat{S}_{i+1}\tilde{\Gamma}_i^\delta} + \frac{1}{T}R_i^\delta\right)^{-1}\\ &\quad\times\left(\overline{S_{i+1}\Phi_i^\delta} + T\overline{\Gamma_i^{\delta T} S_{i+1}\Phi_i^\delta} + T\overline{\tilde{\Gamma}_i^{\delta T}\hat{S}_{i+1}\tilde{\Phi}_i^\delta} + \frac{1}{T}Q_i'^{\delta T}\right),\end{aligned} \tag{66}$$

$$\begin{aligned}\Psi^{1\delta}_{i+1} &= T\left(\overline{\Phi_i^\delta} - \overline{\Gamma_i^\delta}L^\delta_{S_{i+1},\hat{S}_{i+1}}\right)^T \hat{P}_i\left(\overline{\Phi_i^\delta} - \overline{\Gamma_i^\delta}L^\delta_{S_{i+1},\hat{S}_{i+1}}\right)\\ &\quad + K^\delta_{P_i,\hat{P}_i}\left(T\overline{C_i^\delta P_i C_i^{\delta T}} + T\overline{\tilde{C}_i^\delta \hat{P}_i \tilde{C}_i^{\delta T}} + TW_i^\delta\right)K^{\delta T}_{P_i,\hat{P}_i}\\ &\quad + \left(\overline{\Phi_i^\delta} - \overline{\Gamma_i^\delta}L^\delta_{S_{i+1},\hat{S}_{i+1}}\right)\hat{P}_i + \hat{P}_i\left(\overline{\Phi_i^\delta} - \overline{\Gamma_i^\delta}L^\delta_{S_{i+1},\hat{S}_{i+1}}\right)^T,\end{aligned} \tag{67}$$

$$\begin{aligned}\Psi^{2\delta}_i &= T\left(\overline{\Phi_i^\delta} - K^\delta_{P_i,\hat{P}_i}\overline{C_i^\delta}\right)\hat{S}_{i+1}\left(\overline{\Phi_i^\delta} - K^\delta_{P_i,\hat{P}_i}\overline{C_i^\delta}\right)^T\\ &\quad + L^{\delta T}_{S_{i+1},\hat{S}_{i+1}}\left(T\overline{\Gamma_i^{\delta T} S_{i+1}\Gamma_i^\delta} + T\overline{\tilde{\Gamma}_i^{\delta T}\hat{S}_{i+1}\tilde{\Gamma}_i^\delta} + \frac{1}{T}Q_i^{\delta T}\right)\\ &\quad \times L^\delta_{S_{i+1},\hat{S}_{i+1}} + \left(\overline{\Phi_i^\delta} - K^\delta_{P_i,\hat{P}_i}\overline{C_i^\delta}\right)^T\hat{S}_{i+1}\\ &\quad + \hat{S}_{i+1}\left(\overline{\Phi_i^\delta} - K^\delta_{P_i,\hat{P}_i}\overline{C_i^\delta}\right).\end{aligned} \tag{68}$$

Furthermore, let $M^\#$ denote the group generalised or Drazin inverse of a square matrix M, i.e. $M^\# = M(M^3)^+M$ where $+$ denotes the Moore–Penrose pseudoinverse. Also, let θ denote a square zero matrix with appropriate dimensions. Finally, let $M_\perp = I - M$ denote an oblique projection associated with the square matrix M.

Theorem 1: *The compensator* $(\hat{x}_0, F^{\delta N}, K^{\delta N}, L^{\delta N})$ *satisfies the first-order necessary optimality conditions and is* pseudo-minimal, *as defined in* Van Willigenburg and De Koning (2002), if and only if *there exist nonnegative symmetric* $n \times n$ *matrices* P_i, $\hat{P}_i$, $i = 1, 2, \ldots, N$ *and* S_i, $\hat{S}_i$, $i = 0, 1, \ldots, N-1$ *that satisfy*,

$$\begin{aligned}&\frac{P_{i+1} - P_i}{T}\\ &= T\overline{\Phi_i^\delta P_i \Phi_i^{\delta T}} - K^\delta_{P_i,\hat{P}_i}\left(T\overline{C_i^\delta P_i C_i^{\delta T}} + T\overline{\tilde{C}_i^\delta \hat{P}_i \tilde{C}_i^{\delta T}} + TW_i^\delta\right)K^{\delta T}_{P_i,\hat{P}_i}\\ &\quad + T\overline{\tilde{\Phi}_i^\delta \hat{P}_i \tilde{\Phi}_i^{\delta T}} - T\overline{\tilde{\Phi}_i^\delta \hat{P}_i L^T_{S_{i+1},\hat{S}_{i+1}}\tilde{\Gamma}_i^{\delta T}} - T\overline{\tilde{\Gamma}_i^\delta L_{S_{i+1},\hat{S}_{i+1}}\hat{P}_i\tilde{\Phi}_i^{\delta T}}\\ &\quad + T\overline{\tilde{\Gamma}_i^\delta L_{S_{i+1},\hat{S}_{i+1}}\hat{P}_i L^T_{S_{i+1},\hat{S}_{i+1}}\tilde{\Gamma}_i^{\delta T}} + TV_i^\delta + \overline{\Phi_i^\delta}P_i + P_i\overline{\Phi_i^{\delta T}}\\ &\quad + \tau_{\perp_{i+1}}\Psi_i^{1\delta}\tau^T_{\perp_{i+1}} + \tau_{\perp_{i+1}}\frac{\hat{P}_i}{T}\tau^T_{\perp_{i+1}},\\ &i = 0, 1, \ldots, N-1, \quad P_0 = X,\end{aligned} \tag{69}$$

$$\begin{aligned}
&\frac{S_i - S_{i+1}}{T}\\
&\quad = T\overline{\Phi_i^{\delta^T} S_{i+1} \Phi_i^{\delta}} - L^{\delta^T}_{S_{i+1},\hat{S}_{i+1}}\\
&\qquad \times \left(T\overline{\Gamma_i^{\delta T} S_{i+1} \Gamma_i^{\delta}} + T\overline{\tilde{\Gamma}_i^{\delta T} \hat{S}_{i+1} \tilde{\Gamma}_i^{\delta}} + \frac{1}{T} R_i^{\delta} \right) L^{\delta}_{S_{i+1},\hat{S}_{i+1}}\\
&\qquad + T\overline{\tilde{\Phi}_i^{\delta^T} \hat{S}_{i+1} \tilde{\Phi}_i^{\delta}} - T\overline{\tilde{\Phi}_i^{\delta^T} \hat{S}_{i+1} K_{P_i,\hat{P}_i} \tilde{C}_i^{\delta}} - T\overline{\tilde{C}_i^{\delta^T} K^{\delta^T}_{P_i,\hat{P}_i} \hat{S}_{i+1} \tilde{\Phi}_i^{\delta}}\\
&\qquad + T\overline{\tilde{C}_i^{\delta^T} K^{\delta^T}_{P_i,\hat{P}_i} \hat{S}_{i+1} K_{P_i,\hat{P}_i} \tilde{C}_i^{\delta}} + \frac{1}{T} Q_i^{\delta} + \overline{\Phi_i^{\delta^T}} S_{i+1} + S_{i+1} \overline{\Phi_i^{\delta}}\\
&\qquad + \tau^T_{\perp_i} \Psi^{2\delta}_{i+1} \tau_{\perp_i} + \tau^T_{\perp_i} \frac{\hat{S}_{i+1}}{T} \tau_{\perp_i},\\
&\qquad i = 0, 1, \ldots, N-1, \quad S_N = Z,
\end{aligned} \tag{70}$$

$$\frac{\Psi^1_{i+1} - \hat{P}_i}{T} = \Psi^{1^{\delta}}_{i+1}, \hat{P}_{i+1} = \frac{1}{2}\left(\tau_{i+1} \Psi^1_{i+1} + \Psi^1_{i+1} \tau^T_{i+1}\right), \quad i = 0, 1, \ldots, N-1, \quad \hat{P}_0 = \bar{x}_0 \bar{x}_0^T, \tag{71}$$

$$\frac{\Psi^2_i - \hat{S}_{i+1}}{T} = \Psi_i^{2^{\delta}}, \quad \hat{S}_i = \frac{1}{2}\left(\tau_i^T \Psi_i^2 + \Psi_i^2 \tau_i\right), \quad i = 0, 1, \ldots, N-1, \quad \hat{S}_N = \theta, \tag{72}$$

$$\operatorname{rank}(\hat{P}_i) = \operatorname{rank}(\hat{S}_i) = \operatorname{rank}(\hat{P}_i \hat{S}_i) = r_i^c \le n^c, \quad i = 0, 1, \ldots, N, \tag{73}$$

$$\tau_i = \hat{P}_i \hat{S}_i (\hat{P}_i \hat{S}_i)^{\#}, \quad i = 0, 1, \ldots, N, \tag{74}$$

such that

$$F_i^{\delta} = H_{i+1}\left[\Phi_i^{\delta} - K_i^{\delta} C_i^{\delta} - \Gamma_i^{\delta} L_i^{\delta}\right] G_i^T + \left(H_{i+1} G_i^T - I_{n_i^c}\right)/T, \quad i = 0, 1, \ldots, N-1, \tag{75}$$

$$K_i^{\delta} = H_{i+1} K_i^{0\delta}, \quad i = 0, 1, 2, \ldots, N-1, \tag{76}$$

$$L_i^{\delta} = -L_i^{0\delta} G_i^T, \quad i = 0, 1, 2, \ldots, N-1, \tag{77}$$

$$\hat{x}_0 = H_0 \bar{x}_0, \tag{78}$$

where $G_i, H_i \in R^{n^c \times n}$ *are two matrices that satisfy,*

$$G_i H_i^T = \begin{bmatrix} I_{r_i^c} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n^c \times n^c}, \quad G_i^T H_i = \tau_i \in R^{n \times n}. \tag{79}$$

The costs of the compensator $(\hat{x}_0, F^{\delta N}, K^{\delta N}, L^{\delta N})$ *are,*

$$J_N = J_{N_1} = J_{N_2}, \tag{80}$$

$$J_{N_1} = \operatorname{tr}\left[Z\left(P_N + \hat{P}_N\right)\right] + T\sum_{i=0}^{N-1} \operatorname{tr}\left[\frac{1}{T} Q_i^{\delta} P_i + \left(\frac{1}{T} Q_i^{\delta} + L_i^{0\delta} T \frac{1}{T} R_i^{\delta} L_i^{0\delta} - \frac{2}{T} Q_i' L_i^{0\delta}\right) \hat{P}_i\right]. \tag{81}$$

$$J_{N_2} = \operatorname{tr}\left[X\left(S_0 + \hat{S}_0\right) + \bar{x}_0 \bar{x}_0^T S_0\right] + T\sum_{i=0}^{N-1} \operatorname{tr}\left[T V_i^{\delta} S_{i+1} + \left(T V_i^{\delta} + K_i^{0\delta} T W_i^{\delta} K_i^{0\delta} T - 2T V_i' \delta K_i^{0\delta} T\right) \hat{S}_{i+1}\right]. \tag{82}$$

Proof: Follows from theorem 1 in Van Willigenburg and De Koning (1999) and theorem 1 in Van Willigenburg and De Koning (2002) and the relations (30)–(39). The notations in these articles and this article coincide except that Ψ^1_{i+1} in this article coincides with Ψ^1_i in the other papers. Dually, Ψ^2_{i+1} in this article coincides with Ψ^2_i in the other papers.

Remark 4: There are many alternatives for representing Equations (69)–(72). We have chosen this representation, because on the one hand it reveals the delta operator formulation and on the other it closely resembles the numerical computations yet to be presented (Remark 10). In Equation (69), the term on the left of the first equality equals δP_i and in (70) it equals $-\delta S_i$. They reveal the delta operator formulation of Theorem 1. The terms on the left of the first equality in Equations (71) and (72) have been deliberately stated in a similar manner. From the first equality mentioned in Equations (71) and (72), observe that $\boldsymbol{\Psi}^1_{i+1} = T\boldsymbol{\Psi}^{1\delta}_{i+1} + \hat{P}_i$ and $\boldsymbol{\Psi}^2_i = T\boldsymbol{\Psi}^{2\delta}_i + \hat{S}_{i+1}$; $\boldsymbol{\Psi}^1_{i+1}$, $\boldsymbol{\Psi}^2_i$ are used in the second equality mentioned in Equations (71) and (72) (see also Remark 10). From Equations (114)–(119) in the Appendix, observe that alternatives for Equations (71), (72) exist with $\delta\hat{P}_i$ and $-\delta\hat{S}_i$ on the left of the first equality.

Remark 5: From (65)–(70), observe that it is most convenient to select as the compensation problem data: $\bar{\Phi}_i, \bar{\Gamma}_i, \bar{C}_i$, $\overline{T\tilde{\Phi}_i^{\delta} \otimes \tilde{\Phi}_i^{\delta}}$, $\overline{T\tilde{\Gamma}_i^{\delta} \otimes \tilde{\Gamma}_i^{\delta}}$, $\overline{T\tilde{C}_i^{\delta} \otimes \tilde{C}_i^{\delta}}$, $\overline{T\tilde{\Phi}_i^{\delta} \otimes \tilde{\Gamma}_i^{\delta}}$, $\overline{T\tilde{\Phi}_i^{\delta} \otimes \tilde{C}_i^{\delta}}$ that relate to the white stochastic system matrices, $\frac{1}{T}R_i^{\delta}$, $\frac{1}{T}Q_i^{\delta T}$, $\frac{1}{T}Q_i'^{\delta T}$ that relate to the cost function and TW_i^{δ}, TV_i^{δ}, $TV_i'^{\delta}$ that relate to the additive white measurement and system noise. From (50)–(62), (20), in the limit $T \downarrow 0$, $t = iT$ fixed, these become, respectively, $\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$, $V^{AA}(t), V^{BB}(t)$, $V^{CC}(t)$, $V^{AB}(t)$, $V^{AC}(t)$, $R(t)$, $Q(t)$, $Q'(t)$, $W(t)$, $V(t)$, $V'(t)$. The latter indicates that the delta compensation problem data $\overline{T\tilde{\Phi}_i^{\delta} \otimes \tilde{\Phi}_i^{\delta}}$, $\overline{T\tilde{\Gamma}_i^{\delta} \otimes \tilde{\Gamma}_i^{\delta}}$, $\overline{T\tilde{C}_i^{\delta} \otimes \tilde{C}_i^{\delta}}$, $\overline{T\tilde{\Phi}_i^{\delta} \otimes \tilde{\Gamma}_i^{\delta}}$, $\overline{T\tilde{\Phi}_i^{\delta} \otimes \tilde{C}_i^{\delta}}$, TW_i^{δ}, TV_i^{δ}, $TV_i'^{\delta}$ may be interpreted as *densities* with respect to T of the systems multiplicative and additive uncertainties. Similarly $\frac{1}{T}R_i^{\delta}$, $\frac{1}{T}Q_i^{\delta T}$, $\frac{1}{T}Q_i'^{\delta T}$ may be interpreted as *densities* with respect to T of the quadratic penalties that make up the cost function.

Remark 6: In Theorem 1, we may change pseudo-minimal into minimal when the inequality in (73) is changed into an equality so that $G_i H_i^T = I_{n^c}$ in Equation (79). Then the delta operator representation of theorem 1 in Van Willigenburg and De Koning (1999) is obtained.

Remark 7: Observe that Equations (69)–(72) with their associated boundary conditions and rank conditions (73) constitute a two-point boundary value problem.

Remark 8: Applying the associations (21)–(23) and (50)–(57), in the limit $T \downarrow 0$, $t = iT$ fixed, Theorem 1 applies to the finite-horizon time-varying *continuous-time* compensation problem (16)–(20). To the best of our knowledge, so far this problem has only been considered in discrete time (Van Willigenburg and De Koning 1999). Because the *strengthened* discrete-time optimal projection equations are the basis for the delta domain description, in this article, an interesting question is whether the continuous-time results that can be obtained from it are strengthened too. Observe that Hyland and Bernstein (1984), Bernstein and Hyland (1988) and Haddad and Tadmor (1993) suggest equivalence of the continuous-time optimal projection equations with first-order necessary optimality conditions and minimality of the optimal compensator. They only prove that the first ones are *implied* by the latter. In the discrete-time case, the optimal projection equations were modified twice (Bernstein and Hyland 1986; Haddad and Moser 1994; Van Willigenburg and De Koning 2000a) before equivalence was obtained. The remaining three theorems in this section state that no such modifications are needed to obtain equivalence in continuous time.

Theorem 2: *The optimal projection equations presented in* Haddad and Tadmor (1993), *that apply to the deterministic parameter finite-horizon time-varying* continuous-time *reduced-order compensation problem, are* equivalent *to first-order necessary optimality conditions together with the condition that the compensator is minimal.*

Proof: In the finite-horizon time-varying case using Van Willigenburg and De Koning (2000a),

$$\tau_{\perp_i} = I - \tau_i, \tag{83}$$

$$\hat{P}_i = \tau_i \hat{P}_i = \hat{P}_i \tau_i^T = \tau_i \hat{P}_i \tau_i^T, \tag{84}$$

$$\hat{S}_i = \tau_i^T \hat{S}_i = \hat{S}_i \tau_i = \tau_i^T \hat{S}_i \tau_i, \tag{85}$$

we obtain,

$$\tau_{\perp i+1} \hat{P}_i \tau_{\perp i+1}^T = (\tau_{i+1} - \tau_i) \hat{P}_i (\tau_{i+1} - \tau_i)^T, \tag{86}$$

$$\tau_{\perp}^T i \hat{S}_{i+1} \tau_{\perp i+1} = (\tau_i - \tau_{i+1})^T \hat{S}_{i+1} (\tau_i - \tau_{i+1}). \tag{87}$$

From (86), (87), observe that in the continuous-time case, i.e. in the limit $T \downarrow 0$, $t = iT$ fixed, the final terms in (69), (70) tend to zero because $(\tau_{i+1} - \tau_i) \to 0$. Furthermore, from Van Willigenburg and De Koning (2002, 2008), it follows that, if the compensator in Theorem 1 is minimal, $r_i^c = n^c$ in Theorem 1. Using (79) and presuming $r_i^c = n^c$, in (75) the final term,

$$\left(H_{i+1} G_i^T - I_{n^c}\right)/T = (H_{i+1} - H_i) G_i^T / T \to \dot{H} G^T. \tag{88}$$

Equation (88) corresponds with the finite-horizon continuous-time fixed-order LQG result presented in Haddad and Tadmor (1993). This constitutes the most important part of the proof. The remaining part of the proof is given in the Appendix.

Theorem 3: *The optimal projection equations presented in* Hyland and Bernstein (1984), *that apply to the deterministic parameter infinite-horizon continuous time-invariant reduced-order compensation problem, are* equivalent *to first-order necessary optimality conditions together with the condition that the compensator is minimal.*

Proof: The proof is given in the Appendix.

Theorem 4: *The optimal projection equations presented in* Bernstein and Hyland (1988), *that apply to the stochastic parameter infinite-horizon continuous time-invariant reduced-order compensation problem, are* equivalent *to first-order necessary optimality conditions together with the condition that the compensator is minimal.*

Proof: The proof is given in the Appendix.

Remark 9: Theorem 2 is not fully suitable to compute continuous-time finite-horizon optimal reduced-order compensators. This is due to the fact that continuous-time finite-horizon compensators are not minimal because they all violate the rank condition $r_i^c = n^c$ at the initial and final time 0, t_f (Van Willigenburg and De Koning 2002). This prevents stating the boundary conditions associated with the optimal projection equations explicitly in the LQG problem parameters (Haddad and Tadmor 1993). Theorem 1 in this article solves this problem by considering *pseudo-minimal* compensators. These allow for the *inequality* $r_i^c \le n^c$ in Equation (73) (Van Willigenburg and De Koning 2002). Then the boundary conditions can be stated explicitly in the LQG problem parameters, as in Theorem 1.

Remark 10: Equations (69)–(72) are easily seen to be equivalent to respectively (see also

Remark 4),

$$P_{i+1} = T\begin{pmatrix} T\overline{\Phi_i^{\delta} P_i \Phi_i^{\delta T}} - K^{\delta}_{P_i,\hat{P}_i}\Big(T\overline{C_i^{\delta} P_i C_i^{\delta T}} + T\overline{\tilde{C}_i^{\delta} \hat{P}_i \tilde{C}_i^{\delta T}} + TW_i^{\delta}\Big)K^{\delta T}_{P_i,\hat{P}_i} \\ +T\overline{\tilde{\Phi}_i^{\delta} \hat{P}_i \tilde{\Phi}_i^{\delta T}} - T\overline{\tilde{\Phi}_i^{\delta} \hat{P}_i L^T_{S_{i+1},\hat{S}_{i+1}} \tilde{\Gamma}_i^{\delta T}} - T\overline{\tilde{\Gamma}_i^{\delta} L_{S_{i+1},\hat{S}_{i+1}} \hat{P}_i \tilde{\Phi}_i^{\delta T}} \\ + T\overline{\tilde{\Gamma}_i^{\delta} L_{S_{i+1},\hat{S}_{i+1}} \hat{P}_i L^T_{S_{i+1},\hat{S}_{i+1}} \tilde{\Gamma}_i^{\delta T}} + TV_i^{\delta} + \overline{\Phi_i^{\delta}} P_i + P_i \overline{\Phi_i^{\delta T}} + \tau_{\perp_{i+1}} \Psi_i^{1\delta} \tau^T_{\perp_{i+1}} \end{pmatrix}$$
$$+ \tau_{\perp_{i+1}} \hat{P}_i \tau^T_{\perp_{i+1}} + P_i, \quad i = 0, 1, \ldots, N-1, \quad P_0 = X, \tag{89}$$

$$S_i = T\begin{pmatrix} T\overline{\Phi_i^{\delta^T} S_{i+1} \Phi_i^{\delta}} - L^{\delta^T}_{S_{i+1},\hat{S}_{i+1}}\Big(T\overline{\Gamma_i^{\delta T} S_{i+1} \Gamma_i^{\delta}} + T\overline{\tilde{\Gamma}_i^{\delta T} \hat{S}_{i+1} \tilde{\Gamma}_i^{\delta}} + \tfrac{1}{T} R_i^{\delta}\Big)L^{\delta}_{S_{i+1},\hat{S}_{i+1}} \\ + T\overline{\tilde{\Phi}_i^{\delta^T} \hat{S}_{i+1} \tilde{\Phi}_i^{\delta}} - T\overline{\tilde{\Phi}_i^{\delta^T} \hat{S}_{i+1} K_{P_i,\hat{P}_i} \tilde{C}_i^{\delta}} - T\overline{\tilde{C}_i^{\delta^T} K^{T}_{P_i,\hat{P}_i} \hat{S}_{i+1} \tilde{\Phi}_i^{\delta}} \\ + T\overline{\tilde{C}_i^{\delta^T} K^{\delta^T}_{P_i,\hat{P}_i} \hat{S}_{i+1} K_{P_i,\hat{P}_i} \tilde{C}_i^{\delta}} + \tfrac{1}{T} Q_i^{\delta} + \overline{\Phi_i^{\delta^T}} S_{i+1} + S_{i+1} \overline{\Phi_i^{\delta}} + \tau^T_{\perp_i} \Psi^{2\delta}_{i+1} \tau_{\perp_i} \end{pmatrix}$$
$$+ \tau^T_{\perp_i} \hat{S}_{i+1} \tau_{\perp_i} + S_{i+1}, \quad i = 0, 1, \ldots, N-1, \quad S_N = Z, \tag{90}$$

$$\hat{P}_{i+1} = \frac{1}{2}\big(\tau_{i+1} \Psi^1_{i+1} + \Psi^1_{i+1} \tau^T_{i+1}\big), \quad \Psi^1_{i+1} = T\Psi^{1\delta}_{i+1} + \hat{P}_i, \quad i = 0, 1, \ldots, N-1, \quad \hat{P}_0 = \bar{x}_0 \bar{x}_0^T, \tag{91}$$

$$\hat{S}_i = \frac{1}{2}\big(\tau_i^T \Psi_i^2 + \Psi_i^2 \tau_i\big), \quad \Psi_i^2 = T\Psi_i^{2\delta} + \hat{S}_{i+1}, \quad i = 0, 1, \ldots, N-1, \quad \hat{S}_N = \theta. \tag{92}$$

In this article, Equations (89)–(92) will be used to numerically compute $P_i, S_i, \hat{P}_i, \hat{S}_i$, $i = 0, 1, \ldots, N$.

Remark 11: In the finite-horizon continuous-time case, the horizon $t_f = NT$ is fixed and finite. Then, as $T \downarrow 0$, $N \to \infty$. To perform numerical computations, the number of (integration) time steps must always be finite so in practice T will be made small enough and N stays finite. In this case, Equations (89)–(92) describe exactly the forward and backward in time *Euler integration with time step* T of P, $\hat{P}$ and S, $\hat{S}$, respectively.

Remark 12: After application of the changes mentioned in Remark 1, the time-invariant infinite-horizon discrete-time optimal reduced-order compensation result in Van Willigenburg and De Koning (2000a) is obtained. The associated steady state solution of Equations (69)–(72) is represented by,

$$\begin{aligned}\theta = {} & T\overline{\Phi^{\delta} P \Phi^{\delta T}} - K^{\delta}_{P,\hat{P}}\Big(T\overline{C^{\delta} P C^{\delta T}} + T\overline{\tilde{C}^{\delta} \hat{P} \tilde{C}^{\delta T}} + TW^{\delta}\Big)K^{\delta^T}_{P,\hat{P}} \\ & + T\overline{\tilde{\Phi}^{\delta} \hat{P} \tilde{\Phi}^{\delta T}} - T\overline{\tilde{\Phi}^{\delta} \hat{P} L^T_{S,\hat{S}} \tilde{\Gamma}^{\delta T}} - T\overline{\tilde{\Gamma}^{\delta} L_{S,\hat{S}} \hat{P} \tilde{\Phi}^{\delta T}} \\ & + T\overline{\tilde{\Gamma}^{\delta} L_{S,\hat{S}} \hat{P} L^T_{S,\hat{S}} \tilde{\Gamma}^{\delta T}} + TV^{\delta} + \overline{\Phi^{\delta}} P + P\overline{\Phi^{\delta T}} + \tau_{\perp} \Psi^{1\delta} \tau^T_{\perp},\end{aligned} \tag{93}$$

$$\begin{aligned}\theta = {} & T\overline{\Phi^{\delta T} S \Phi^{\delta}} - L^{\delta^T}_{S,\hat{S}}\Big(T\overline{\Gamma^{\delta T} S \Gamma^{\delta}} + T\overline{\tilde{\Gamma}^{\delta^T} \hat{S} \tilde{\Gamma}^{\delta}} + \frac{1}{T} R^{\delta}\Big)L^{\delta}_{S,\hat{S}} \\ & + T\overline{\tilde{\Phi}^{\delta^T} \hat{S} \tilde{\Phi}^{\delta}} - T\overline{\tilde{\Phi}^{\delta^T} \hat{S} K_{P,\hat{P}} \tilde{C}^{\delta}} - T\overline{\tilde{C}^{\delta^T} K^T_{P,\hat{P}} \hat{S} \tilde{\Phi}^{\delta}} \\ & + T\overline{\tilde{C}^{\delta^T} K^{\delta^T}_{P,\hat{P}} \hat{S} K_{P,\hat{P}} \tilde{C}^{\delta}} + \frac{1}{T} Q^{\delta} + \overline{\Phi^{\delta^T}} S + S\overline{\Phi^{\delta}} + \tau^T_{\perp} \Psi^{2\delta} \tau_{\perp},\end{aligned} \tag{94}$$

$$\theta = \frac{1}{2}\Big(\tau \Psi^{1^{\delta}} + \Psi^{1^{\delta}} \tau^T\Big), \tag{95}$$

$$\theta = \frac{1}{2}\Big(\tau^T \Psi^{2^{\delta}} + \Psi^{2^{\delta}} \tau\Big). \tag{96}$$

They provide the solution to the compensation problem in Van Willigenburg and De Koning (2000a). To obtain (93)–(96), observe that for a steady state solution, the last term in Equation (69) equals zero. This is due to Equations (83), (84) and because for a steady state solution $\tau_{i+1} = \tau_i$. Dually, the last term in Equation (70) equals zero as well. If the system has deterministic parameters application of Remark 1 provides the discrete-time result in De Koning and Van Willigenburg (2000).

6. Compensatability and optimal compensation in the delta domain

Optimal projection equations of different type form the basis for algorithms to compute optimal full and reduced-order compensators. The optimal projection equations associated with the finite-horizon time-varying case generalise those that apply to the infinite-horizon time-invariant case because the latter have constant data and dimensions and no boundary conditions. On the other hand, in the infinite-horizon case, the ability to stabilise a system with white stochastic parameters by means of a full or reduced-order compensator is a crucial property. The type of stability that needs to be considered in the case of systems with white stochastic parameters is mean-square stability (ms-stability; De Koning 1982). The ability to obtain an ms-stable closed-loop system by means of a compensator is called ms-compensatability. This property was introduced in De Koning (1992) together with full-order ms-compensatability tests for discrete-time systems with white stochastic parameters. One of these tests was carried over to the reduced-order discrete-time case in Van Willigenburg and De Koning (2000a). Like the algorithms to compute optimal compensators, the tests are based on the optimal projection equations. The most interesting test appears to be the one that provides a *measure* of stabilisability, not merely a test (De Koning 1992).

Let S^n denote the space of n-dimensional real square matrices. Define,

$$X_j^N = \left\{X_i^j \in S^n, i = 0, 1, \ldots, N\right\}, \quad j = 1, 2, 3, 4, \tag{97}$$

and

$$Y_j^N = \left\{Y_i^j \in S^n, i = 0, 1, \ldots, N\right\}, \quad j = 1, 2, 3, 4. \tag{98}$$

Call $X^N = (X_1^N, X_2^N, X_3^N, X_4^N) \geq 0$ if $X_i^1, X_i^2, X_i^3, X_i^4 \geq 0$, $i = 0, 1, \ldots, N$. The transformation to be defined next is based on Equations (89)–(92) where X_i^1, X_{i+1}^2, X_i^3, X_{i+1}^4 correspond to P_i, S_{i+1}, $\hat{P}_i$, $\hat{S}_{i+1}$ and Y_{i+1}^1, Y_i^2, Y_{i+1}^3, Y_i^4 to P_{i+1}, S_i, $\hat{P}_{i+1}$, $\hat{S}_i$.

Definition 1: Define the transformation $C_v : Y^N = C_v(X^N)$ given by,

$$\begin{aligned} Y_{i+1}^1 = T&\left(\begin{array}{l} T\overline{\Phi_i^\delta Y_i^1 \Phi_i^{\delta T}} - K_{Y_i^1, Y_i^3}^\delta \left(T\overline{C_i^\delta Y_i^1 C_i^{\delta T}} + T\overline{\tilde{C}_i^\delta Y_i^3 \tilde{C}_i^{\delta T}} + TW_i^\delta \right) K_{Y_i^1, Y_i^3}^{\delta T} \\ \quad + T\overline{\tilde{\Phi}_i^\delta Y_i^3 \tilde{\Phi}_i^{\delta T}} - T\overline{\tilde{\Phi}_i^\delta Y_i^3 L_{X_{i+1}^2, X_{i+1}^4}^T \tilde{\Gamma}_i^{\delta T}} - T\overline{\tilde{\Gamma}_i^\delta L_{X_{i+1}^2, X_{i+1}^4} Y_i^3 \tilde{\Phi}_i^{\delta T}} \\ \quad + T\overline{\tilde{\Gamma}_i^\delta L_{X_{i+1}^2, X_{i+1}^4} \hat{P}_i L_{X_{i+1}^2, X_{i+1}^4}^T \tilde{\Gamma}_i^{\delta T}} + TV_i^\delta + \overline{\Phi_i^\delta} Y_i^1 + Y_i^1 \overline{\Phi_i^{\delta T}} + \tau_{\perp_{i+1}} \Psi_i^{1\delta} \tau_{\perp_{i+1}}^T \end{array} \right) \\ &+ \tau_{\perp_{i+1}} Y_i^3 \tau_{\perp_{i+1}}^T + Y_i^1, \quad i = 0, 1, \ldots, N-1, \quad Y_0^1 = X, \end{aligned} \tag{99}$$

$$\begin{aligned} Y_i^2 = T&\left(\begin{array}{l} T\overline{\Phi_i^{\delta T} Y_{i+1}^2 S_{i+1} \Phi_i^\delta} - L_{Y_{i+1}^2, Y_{i+1}^4}^{\delta T} \left(T\overline{\Gamma_i^{\delta T} Y_{i+1}^2 \Gamma_i^\delta} + T\overline{\tilde{\Gamma}_i^{\delta T} Y_{i+1}^4 \tilde{\Gamma}_i^\delta} + \tfrac{1}{T} R_i^\delta \right) L_{Y_{i+1}^2, Y_{i+1}^4}^\delta \\ \quad + T\overline{\tilde{\Phi}_i^{\delta T} Y_{i+1}^4 \tilde{\Phi}_i^\delta} - T\overline{\tilde{\Phi}_i^{\delta T} Y_{i+1}^4 K_{X_i^1, X_i^3} \tilde{C}_i^\delta} - T\overline{\tilde{C}_i^{\delta T} K_{X_i^1, X_i^3}^{\delta T} Y_{i+1}^4 \tilde{\Phi}_i^\delta} \\ \quad + T\overline{\tilde{C}_i^{\delta T} K_{X_i^1, X_i^3}^{\delta T} Y_{i+1}^4 K_{X_i^1, X_i^3} \tilde{C}_i^\delta} + \tfrac{1}{T} Q_i^\delta + \overline{\Phi_i^{\delta T}} Y_{i+1}^2 + Y_{i+1}^2 \overline{\Phi_i^\delta} + \tau_{\perp_i}^T \Psi_{i+1}^{2\delta} \tau_{\perp_i} \end{array} \right) \\ &+ \tau_{\perp_i}^T Y_{i+1}^4 \tau_{\perp_i} + Y_{i+1}^2, \quad i = 0, 1, \ldots, N-1, \quad Y_N^2 = Z, \end{aligned} \tag{100}$$

$$Y_{i+1}^3 = \frac{1}{2}\left(\tau_{i+1} \Psi_i^1 + \Psi_i^1 \tau_{i+1}^T\right), \quad \Psi_i^1 = T\Psi_i^{1\delta} + Y_i^3, \quad i = 0, 1, \ldots, N-1, \quad Y_0^3 = \bar{x}_0 \bar{x}_0^T, \tag{101}$$

$$Y_i^4 = \frac{1}{2}\left(\tau_i^T \Psi_{i+1}^2 + \Psi_{i+1}^2 \tau_i\right), \quad \Psi_{i+1}^2 = T\Psi_{i+1}^{2\delta} + Y_{i+1}^4, \quad i = 0, 1, \ldots, N-1, \quad Y_N^4 = \theta, \tag{102}$$

where Ψ_i^1, Ψ_i^2 are given by (67), (68) in which P_i, S_{i+1}, $\hat{P}_i$, $\hat{S}_{i+1}$ correspond to X_i^1, X_{i+1}^2, X_i^3, X_{i+1}^4 and P_{i+1}, S_i, $\hat{P}_{i+1}$, $\hat{S}_i$ to Y_{i+1}^1, Y_i^2, Y_{i+1}^3, Y_i^4. To enforce the rank conditions (73),

$$\tau_i = U_{Y_i^3 Y_i^4} \begin{bmatrix} I_{r_i^c} & 0 \\ 0 & 0 \end{bmatrix} U_{Y_i^3 Y_i^4}^{-1}, \tag{103}$$

where $U_{Y_i^1 Y_i^3}$ is a matrix obtained from an eigenvalue decomposition of $Y_i^3 Y_i^4$ represented by,

$$Y_i^3 Y_i^4 U_{Y_i^3 Y_i^4} = U_{Y_i^3 Y_i^4} \Lambda_{Y_i^3 Y_i^4}. \tag{104}$$

As Y_i^3, Y_i^4 are nonnegative symmetric, from Lemma 2.1 in Bernstein and Hyland (1988), $Y_i^3 Y_i^4$ is diagonalisable and therefore $\Lambda_{Y_i^1 Y_i^3}$ in Equation (104) represents a diagonal matrix with the eigenvalues of $Y_i^3 Y_i^4$

Using the SDOPE in the delta domain, in this section the algorithms and ms-compensatability will also be stated in this domain. Within this domain, by taking the limit $T \downarrow 0$, $t = iT$ fixed, the discrete-time results are carried over to continuous time. To the best of our knowledge, ms-compensatability and the associated ms-compensatability tests have never been considered in continuous time. It will be demonstrated in this section that the algorithms and compensatability tests carry over to continuous time in an almost straightforward manner. In the full-order case, the tests are necessary and sufficient. In the reduced-order case, they are only sufficient, because the associated optimal projection equations may have multiple nonnegative solutions (Van Willigenburg and De Koning 1999, 2000a).

on the diagonal. The eigenvalues are ordered in the descending order of magnitude. The columns of $U_{Y_i^1 Y_i^3}$ are the associated eigenvectors. Finally, $I_{r_i^c}$ in Equation (103) is the square identity matrix of dimension,

$$r_i^c = \min\left(\operatorname{rank}\left(Y_i^3 Y_i^4\right), n^c\right). \tag{105}$$

Observe that the transformation $Y^N = C_v(X^N)$ constitutes one forward and one backward sweep of the SDOPE in delta operator form. Therefore i consecutive forward and backward sweeps are represented by the transformation $Y^N = C_v^i(X^N)$.

Algorithm 1 (the finite-horizon optimal full-order compensator in the delta domain): Take $\tau_i = G_i = H_i = I_n$, $i = 0, 1, \ldots, N$. Then, as $i \to \infty$, $C_v^i(\theta, \theta, I, I)$ converges to a solution $Y^N \geq 0$ of Theorem 1. This solution corresponds to the finite-horizon optimal full-order compensator in Van Willigenburg and De Koning (2002). In the delta domain, this compensator is given by Equations (75)–(78) and the minimal costs are represented by Equations (81), (82).

Remark 13: In De Koning (1992), homotopy degree theory is used to prove uniqueness of the infinite-horizon optimal full-order compensator. The same line of proof suggests uniqueness of the finite-horizon optimal full-order compensator computed by Algorithm 1.

Algorithm 2 (finite-horizon optimal reduced-order compensators in the delta domain): Check if $C_v^i(\theta, \theta, I, I)$ converges to $Y^N \geq 0$ as $i \to \infty$. If so, then according to Theorem 1 and Van Willigenburg and De Koning (1999), a nonnegative solution of the SDOPE in delta operator form is obtained that corresponds to a finite-horizon local optimal reduced-order compensator given by (75)–(79), with τ_i given by (103). The minimal costs are represented by Equations (81), (82).

Remark 14: To check the convergence of Algorithms 1 and 2, the convergence of $trace(Y_0^1 + Y_2^N)$ may be checked (De Koning 1992).

Remark 15: Through the associations (13)–(15) and (30)–(39), Algorithms 1 and 2 apply to the finite-horizon full and reduced-order discrete-time compensation problem (1)–(6). Through the associations (21)–(23) and (50)–(57), in the limit $T \downarrow 0$, $t = iT$ fixed, they apply to the associated continuous-time compensation problem (16)–(20). Because $T = t_f/N$, to approximate the limit $T \downarrow 0$, N must be made sufficiently large (see also Remark 11).

In the remainder of this section, infinite-horizon time-invariant compensation problems are considered. Therefore, throughout the remainder of this section, the changes mentioned in Remark 1 apply. To state the compensatability tests and algorithms for time-invariant infinite-horizon full and reduced-order compensation, a transformation similar to the one in Definition 1 is required. Let $X = (X_1, X_2, X_3, X_4)$, $X_1, X_2, X_3, X_4 \in S^n$. Call $X \geq 0$ if $X_1, X_2, X_3, X_4 \geq 0$. Let $Y = (Y_1, Y_2, Y_3, Y_4)$, $Y_1, Y_2, Y_3, Y_4 \in S^n$. Call $Y \geq 0$ if $Y_1, Y_2, Y_3, Y_4 \geq 0$. The transformation to be defined next is again based on Equations (89)–(92) where X_1, X_2, X_3, X_4 correspond to $P_i, S_{i+1}, \hat{P}_i, \hat{S}_{i+1}$ and Y_1, Y_2, Y_3, Y_4 to $P_{i+1}, S_i, \hat{P}_{i+1}, \hat{S}_i$.

Definition 2: Consider the transformation D: $(Y_1, Y_2, Y_3, Y_4) = D(X_1, X_2, X_3, X_4)$ given by,

$$Y_1 = T\begin{pmatrix} T\overline{\Phi^\delta X_1 \Phi^{\delta^T}} - K^\delta_{X_1,X_3}\left(T\overline{C^\delta X_1 C^{\delta^T}} + T\overline{\tilde{C}^\delta X_3 \tilde{C}^{\delta^T}} + TW^\delta\right)K^{\delta^T}_{X_1,X_3} \\ + T\overline{\tilde{\Phi}^\delta X_3 \tilde{\Phi}^{\delta^T}} - T\overline{\tilde{\Phi}^\delta X_3 L^T_{X_2,X_4}\tilde{\Gamma}^{\delta^T}} - T\overline{\tilde{\Gamma}^\delta L_{X_2,X_4} X_3 \tilde{\Phi}^{\delta^T}} \\ + T\overline{\tilde{\Gamma}^\delta L_{X_2,X_4} X_3 L^T_{X_2,X_4}\tilde{\Gamma}^{\delta^T}} + TV^\delta + \overline{\Phi^\delta_i}X_1 + X_1\overline{\Phi^{\delta^T}} + \tau_\perp \Psi^{1\delta}\tau_\perp^T \end{pmatrix} + \tau_\perp X_3 \tau_\perp^T + X_1, \tag{106}$$

$$Y_2 = T\begin{pmatrix} T\overline{\Phi^{\delta^T} X_2 \Phi^\delta} - L^{\delta^T}_{X_2,X_4}\left(T\overline{\Gamma^{\delta^T} X_2 \Gamma^\delta} + T\overline{\tilde{\Gamma}^{\delta^T} X_4 \tilde{\Gamma}^\delta} + \tfrac{1}{T}R^\delta\right)L^\delta_{X_2,X_4} \\ + T\overline{\tilde{\Phi}^{\delta^T} X_4 \tilde{\Phi}^\delta} - T\overline{\tilde{\Phi}^{\delta^T} X_4 K^\delta_{X_1,X_3}\tilde{C}^\delta} - T\overline{\tilde{C}^{\delta^T} K^{\delta^T}_{X_1,X_3} X_4 \tilde{\Phi}^\delta} \\ + T\overline{\tilde{C}^{\delta^T} K^{\delta^T}_{X_1,X_3} X_4 K^\delta_{X_1,X_3}\tilde{C}^\delta} + \tfrac{1}{T}Q^\delta + \overline{\Phi^{\delta^T}}X_2 + X_2\overline{\Phi^\delta} + \tau_\perp^T \Psi^{2\delta}\tau_\perp \end{pmatrix} + \tau_\perp^T X_4 \tau_\perp + X_2, \tag{107}$$

$$Y_3 = \frac{1}{2}\left(\tau\Psi^1 + \Psi^1\tau^T\right), \quad \Psi^1 = T\Psi^{1\delta} + X_3, \tag{108}$$

$$Y_4 = \frac{1}{2}\left(\tau^T\Psi^2 + \Psi^2\tau\right), \quad \Psi^2 = T\Psi^{2\delta} + X_4, \tag{109}$$

Algorithm 3 (infinite horizon full-order compensator in the delta domain)**:** Let $Y = D^i(\theta, \theta, I, I)$. Choose $G = H = \tau = I$, $n^c = n$. Then, according to Theorem 1 and De Koning (1992), Y converges as $i \to \infty$ to the unique nonnegative solution of $X = D(X)$. The associated optimal full-order compensator is globally optimal and given by Equations (75)–(78) with the time indices deleted. The minimal costs are represented by the terms after the summation symbol in Equations (81), (82) with the time indices deleted.

Algorithm 4 (infinite horizon reduced-order compensators in the delta domain)**:** Let $Y = D^i(\theta, \theta, I, I)$. Choose $n^c < n$. Then, according to Theorem 1 and Van Willigenburg and De Koning (2000a), if Y converges and $Y \geq 0$ as $i \to \infty$, Y constitutes a nonnegative solution of $X = D(X)$. The associated reduced-order compensator is locally optimal and given by Equations (75)–(79) and τ_i by (103) with all time indices deleted. The minimal costs are represented by the terms after the summation symbol in Equations (81), (82) with all time-indices deleted.

Theorem 5 (delta domain full-order compensatability test)**:** *Let* $Y = D^i(\theta, \theta, I, I)$. *Choose* $Q = I$, $V = I$, $R = I$, $W = I$, $Q' = \theta$, $V' = \theta$, $\tau = I$, $n^c = n$ *and apply the associations* (13), (14). *The system* (1), (2) *and its delta domain equivalent* (8), (9) *are ms-compensatable* $\Leftrightarrow Y$ *converges as* $i \to \infty$.

Proof: For each value $T > 0$, there is a one-to-one correspondence, given by Equations (13)–(15), between the discrete-time compensation problem (1)–(6) and the associated one formulated by means of the delta-operator (8)–(12). Also this one-to-one correspondence leaves Y unchanged. Then, Theorem 5 follows from De Koning (1992). Furthermore from De Koning (1992), $Y \geq 0$ by definition. Therefore, $Y \geq 0$ is not mentioned explicitly in Theorem 5.

Theorem 6 (delta domain reduced-order compensatability test)**:** *Let* $Y = D^i(\theta, \theta, I, I)$. *Choose* $Q = I$, $V = I$, $R = I$, $W = I$, $Q' = \theta$, $V' = \theta$, $n^c < n$ *and apply the associations* (13), (14). *If* Y *converges and* $Y \geq 0$ *as* $i \to \infty$, *the system* (1), (2) *and its delta domain equivalent* (8), (9) *are* n^c*-ms-compensatable. If not nothing may be concluded with respect to* n^c*-ms-compensatability*.

Proof: The proof is identical to the proof of Theorem 5 with reference De Koning (1992) replaced by reference Van Willigenburg and De Koning (2000a). From Van Willigenburg and De Koning (2000a), observe that $Y \geq 0$ does not hold by definition, although in practice it generally holds. Therefore, $Y \geq 0$ is mentioned explicitly in Theorem 6.

The above-mentioned compensatability tests provide a yes/no result as to ms-compensatability and a yes/undecided result as to n^c-ms-compensatability. A second more interesting ms-compensatability test was presented in De Koning (1992). This test provides a *measure* of ms-compensatability in discrete time, not merely a test. In discrete time, the spectral radius ρ of a linear system is a nonnegative *stability measure*. The smaller the spectral radius ρ, the better stability. The compensatability measure in De Koning (1992) is the *minimal spectral radius* $\rho_{\min}$ of the closed-loop system achievable with a compensator. This measure is calculated from an associated compensation problem with $W = \theta$, $R = \theta$. These zero choices of W, R fall outside the scope of our problem formulation because they may result in singular problems. In this case, according to De Koning (1992), see also De Koning (1982), $K^{\delta}_{X_1,X_3}$ and $L^{\delta}_{X_2,X_4}$ in (106), (107) may still be computed from (65), (66) if the standard inverse is replaced by the Moore–Penrose pseudoinverse. This replacement may be needed if $W = \theta$, $R = \theta$ in (65), (66) (Example 3). The compensatability measures listed below all use this replacement.

To obtain a compensatability measure in the delta domain, a delta domain stability measure is needed. This measure is taken from the delta domain stability condition given by Goodwin, Middleton, and Poor (1992),

$$\left|1 + T\lambda_i^{\delta}\right| < 1 \Leftrightarrow \mathrm{Re}\left(\lambda_i^{\delta}\right) + \frac{T}{2}\left|\lambda_i^{\delta}\right|^2 < 0, \quad i = 1, 2, \ldots, n, \quad T > 0, \tag{110}$$

where λ_i^{δ}, $i = 1, 2, \ldots, n$ are the eigenvalues of the system matrix of an autonomous system with dimension n in the delta domain (Goodwin et al. 1992). Equation (110) may be interpreted as the condition that λ_i^{δ}, $i = 1, 2, \ldots, n$ should all lie inside the complex plane circle with radius $1/T$ and centre $-1/T$. From the right-hand side of (110), the following stability *measure* in the delta domain is immediate,

$$\rho^{\delta} = \max_i\left(\mathrm{Re}\left(\lambda_i^{\delta}\right) + \frac{T}{2}\left|\lambda_i^{\delta}\right|^2\right), \quad i = 1, 2, \ldots, n. \tag{111}$$

The more negative stability measure (111), the larger stability. Maximum stability is achieved when $\rho^{\delta} = -1/T$ since this is the minimum value (111) can achieve. Observe that this stability measure applies to continuous-time systems in the limit $T \downarrow 0$, where it becomes $\max_i(\mathrm{Re}(\lambda_i^{\delta}))$, $i = 1, 2, \ldots, n$. This corresponds to the well-known stability condition $\max_i(\mathrm{Re}(\lambda_i^{\delta})) < 0$ in continuous time that is a special case of (110).

Lemma 1: *When ρ and ρ^δ are stability measures of* corresponding *linear systems in the discrete time and delta domain, their minima $\rho_{\min}$ and $\rho^\delta_{\min}$ as a function of certain system parameters, are obtained for corresponding parameter values and are related by $\rho^\delta_{\min} = (\rho_{\min}-1)/T + \frac{1}{2}(\rho_{\min}-1)^2/T$.*

Proof: Because they are both *measures* of stability, the minima of ρ and ρ^δ must be obtained for corresponding system parameter values. Because ρ is a spectral radius, $\rho_{\min}$ is nonnegative real. Therefore, $\rho_{\min}$ is also achieved by the scalar discrete-time system $x_{i+1} = \rho_{\min} x_i$. The associated scalar system in the delta domain is $\delta x_i = (\rho_{\min}-1)/T x_i$. Because $\rho_{\min}$ is nonnegative real the corresponding value $\rho^\delta_{\min}$ according to (111) is $(\rho_{\min}-1)/T + \frac{1}{2}(\rho_{\min}-1)^2/T$.

Theorem 7 (delta domain full-order compensatability measure)**:** *Let* $(Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i}) = D^i(\theta,\theta,I,I)$. *Choose $Q=\theta$, $V=\theta$, $Q'=\theta$, $V'=\theta$, $R=\theta$, $W=\theta$, $\tau=I$, $n^c=n$ and apply the associations* (13), (14). *Let* $\sigma_i = \mathrm{tr}(Y_{1,i+1}+Y_{3,i+1})/\mathrm{tr}(Y_{1,i}+Y_{3,i})$. *As $i\to\infty$, σ_i converges to $\rho_{\min}$ the minimum of stability measure ρ of the closed-loop system* (1), (2) *achievable with a compensator. Let* $\sigma^\delta_i = \mathrm{tr}(\delta Y_{1,i}+\delta Y_{3,i})/\mathrm{tr}(Y_{1,i}+Y_{3,i})$. *As $i\to\infty$, $\sigma^\delta_i + \frac{T}{2}\sigma^{\delta^2}_i$ converges to $\rho^\delta_{\min}$ the minimum of stability measure* (111) *of the corresponding closed-loop system* (8), (9) *achievable with a compensator.*

Proof: For each value $T>0$, there is a one-to-one correspondence, given by Equations (13)–(15), between the discrete-time compensation problem (1)–(6) and the associated one formulated by means of the delta-operator (8)–(12). Also, this one-to-one correspondence leaves Y unchanged. Then, from De Koning (1992), as $i\to\infty$, $\sigma_i = \mathrm{tr}(Y_{1,i+1}+Y_{3,i+1})/\mathrm{tr}(Y_{1,i}+Y_{3,i}) \to \rho_{\min}$, the minimal spectral radius of the closed-loop system (1), (2) achievable with a compensator. Observe that,

$$\sigma^\delta_i = \frac{\mathrm{tr}(\delta Y_{1,i}+\delta Y_{3,i})}{\mathrm{tr}(Y_{1,i}+Y_{3,i})} = \frac{\mathrm{tr}(Y_{1,i+1}+Y_{3,i+1}) - \mathrm{tr}(Y_{1,i}+Y_{3,i})}{T} \times \frac{1}{\mathrm{tr}(Y_{1,i}+Y_{3,i})} = \frac{1}{T}\left(\frac{\mathrm{tr}(Y_{1,i+1}+Y_{3,i+1})}{\mathrm{tr}(Y_{1,i}+Y_{3,i})} - 1\right). \quad (112)$$

Therefore, as $i\to\infty$, $\sigma^\delta_i \to (\rho_{\min}-1)/T$. Then from Lemma 1 and its proof $\sigma^\delta_i + \frac{T}{2}\sigma^{\delta 2}_i \to \rho^\delta_{\min}$.

Remark 16: Although De Koning (1992) uses $[\mathrm{tr}(Y_{1,i}+Y_{3,i})]^{1/i}$, in the compensatability tests, $\mathrm{tr}(Y_{1,i+1}+Y_{3,i+1})/\mathrm{tr}(Y_{1,i}+Y_{3,i})$ is mentioned as an alternative that usually converges faster. Moreover, the latter expression translates more easily to the delta domain. Therefore, we use $\sigma_i = \mathrm{tr}(Y_{1,i+1}+Y_{3,i+1})/\mathrm{tr}(Y_{1,i}+Y_{3,i})$ in Theorem 7. Observe that both σ_i and σ^δ_i are easily calculated during each recursion $(Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i}) = D^i(\theta,\theta,I,I)$ mentioned in Theorem 7. Their convergence is used as a stop criterion.

Theorem 8 (delta domain reduced-order compensatability measure)**:** *Let* $(Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i}) = D^i(\theta,\theta,I,I)$. *Choose $Q=\theta$, $V=\theta$, $Q'=\theta$, $V'=\theta$, $R=\theta$, $W=\theta$, $\tau=I$, $n^c=n$ and apply the associations* (13), (14). *If, as $i\to\infty$,* $\sigma_i = \mathrm{tr}(Y_{1,i+1}+Y_{3,i+1})/\mathrm{tr}(Y_{1,i}+Y_{3,i}) \to \sigma$ *and* $Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i} \geq 0$; *then $\rho_{\min} = \sigma$ is a local minimum of stability measure ρ of the closed-loop system* (1), (2) *achievable with a compensator having state dimension n^c. In that case, $\sigma^\delta_i + \frac{T}{2}\sigma^{\delta 2}_i \to \rho^\delta_{\min}$ a local minimum of stability measure* (111) *of the corresponding closed-loop system* (8), (9) *achievable with a compensator having state dimension n^c, with σ^δ_i and $\rho^\delta_{\min}$ as in Theorem* 7.

Proof: The proof is identical to the proof of Theorem 7 with reference De Koning (1992) replaced by reference Van Willigenburg and De Koning (2000a). From Van Willigenburg and De Koning (2000a), observe that $Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i} \geq 0$ does not hold by definition, although in practice it generally holds. Therefore, $Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i} \geq 0$ is mentioned explicitly in Theorem 8.

Remark 17: Although the choice $W=\theta$, $R=\theta$ in Theorems 7 and 8 may lead to singularities, according to the proofs of Theorems 7 and 8, they can be handled as long as $T>0$ is ensured. Taking the limit $T\downarrow 0$, $t=iT$ fixed, the compensatability measures in Theorems 7 and 8 apply to the *continuous-time case*. Taking this limit, instead of $T=0$, ensures $T>0$. Therefore, taking the limit $T\downarrow 0$, $t=iT$ fixed, is *also* a means of circumventing difficulties with possible singularities in continuous-time. Although in the limit $T\downarrow 0$, $t=iT$ fixed, the compensator matrices may tend to infinity, the limits mentioned in Theorems 7 and 8 exist. This is illustrated by Example 3 in the following section.

7. Numerical examples

To check our delta operator equations and algorithms, we first ran discrete-time numerical examples from Van Willigenburg and De Koning (1999, 2000a) and De Koning and Van Willigenburg (2000). The delta operator algorithms provide identical solutions that are also invariant under changes of T. This provides strong evidence for the correctness of the delta operator equations and algorithms in this article. Since the new results presented in this article relate to continuous time, three out of four examples presented in this section are continuous-time examples. The uncertainty of continuous-time system parameters is

Table 1. ms-Compensatability and costs (Example 1) against parameter uncertainty.

β	0	0.03	0.06	0.1	0.12	0.15	0.18	0.3
$J_N/N, N\to\infty,\ n^c=2$	6.1593	11.645	37.589	898.77	62,513	9,48,800	$+\infty$	$+\infty$
$J_N/N, N\to\infty,\ n^c=1$	8.8945	28.414	150.79	6299.2	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\rho^{\delta}_{\min},\ n^c=2$	$-\infty$	-10.607	-4.5596	-1.8433	-1.0615	-0.18022	0.50414	2.4308
$\rho^{\delta}_{\min},\ n^c=1$	$-\infty$	-4.3842	-1.4920	-0.059412	0.39532	0.94498	1.4052	2.8782

Table 2. Costs (Example 2) against parameter uncertainty.

β	0	0.03	0.06	0.09	0.12	0.15	0.18	0.3
$J_N,\ n^c=2$	3.5957	4.1245	4.7752	5.5815	6.5866	7.8448	9.4248	21.094
$J_N,\ n^c=1$	3.6177	4.1557	4.8158	5.6334	6.6515	7.9248	9.5222	21.290

represented by their variances. In three of the four examples these variances will be selected as follows:

$$V^{AA}(t)=\beta\bar{A}(t)\otimes\bar{A}(t),\quad V^{BB}(t)=\beta\bar{B}(t)\otimes\bar{B}(t),$$
$$V^{CC}(t)=\beta\bar{C}(t)\otimes\bar{C}(t)$$
$$V^{AB}(t)=\theta,\quad V^{AC}(t)=\theta. \tag{113}$$

In Equation (113), $\beta\geq 0$ represents a *parameter uncertainty measure*. If $\beta=0$, the continuous-time system has deterministic parameters. As β increases, the parameter uncertainty increases. The outcome of several examples will be recorded for different values of the parameter uncertainty measure β.

Example 1:

$$\bar{A}(t)=\begin{bmatrix}0.2500 & 3.0171\\ -3.0171 & 0.2500\end{bmatrix},\ \bar{B}(t)=\begin{bmatrix}-0.0799 & 1.2867\\ 0 & 0.4218\end{bmatrix}$$
$$\bar{C}(t)=\begin{bmatrix}0 & -1.0907\\ -0.86280 & 1.8330\end{bmatrix},$$
$$Q(t)=R(t)=V(t)=W(t)=\mathrm{diag}[1\quad 1],$$
$$t_f\to\infty,\ V^{AA},\ V^{AB},\ V^{AC},\ V^{BB},\ V^{CC}$$

given by (113).

Observe that the infinite horizon continuous-time compensation problem in Example 1 comprises the $n^c=1$ compensatability test described by Theorem 6, applied to an unstable oscillatory system. For different values of the parameter uncertainty measure β, Table 1 records the minimum costs of the optimal compensation problem obtained with Algorithm 4. If the system is not n^c-ms-compensatable, Algorithm 4 does not converge and the costs are ∞. Also the full and reduced-order *ms-compensatability measures* obtained from Theorems 7 and 8 are computed. $T=0.005$ is selected for the Euler numerical integration time step (Remark 11).

Example 2: Equal to Example 1 except for $t_f=1$. Furthermore, $\bar{x}_0=[1\quad 1]^T$, $X=\mathrm{diag}(0.1\quad 0.1)$, $Z=X$.

For different values of the parameter uncertainty measure β, Table 2 records the minimum costs of this finite-horizon optimal compensation problem obtained with Algorithm 2, where $T=0.005$ is selected for the Euler numerical integration time step (Remark 11).

Example 3:

$$\bar{A}(t)=\begin{bmatrix}0.4 & 0\\ 0 & -0.05\end{bmatrix},\quad \bar{B}(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\bar{C}(t)=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\quad V(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad W(t)=1,$$
$$Q(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad R(t)=1,$$
$$V^{AA}=V^{AB}=V^{AC}=V^{CC}=\theta,\quad V^{BB}=\beta\bar{B}\otimes\bar{B}.$$

Example 3 is special in two respects. The computation of the full and reduced-order compensatability measures in Theorems 7 and 8 use $Q=\theta$, $V=\theta$, $Q'=\theta$, $V'=\theta$, $R=\theta$, $W=\theta$. For Example 3, they result in *singular* compensation problems when $T=0$. This can be observed from the final term in Equation (65), which for $T=0$ is entirely zero. This is due to Equations (60) and (64) which imply $\overline{TC_i^{\delta}P_iC_i^{\delta T}}=T\bar{C}P\bar{C}^T+V^{cc}=T\bar{C}P\bar{C}^T=\theta$, $\overline{T\tilde{C}_i^{\delta}\hat{P}_i\tilde{C}_i^{\delta T}}=\theta$. Furthermore $T\overline{W_i^{\delta}}=W=\theta$. Therefore, as opposed to the non-singular cases considered so far, $T\downarrow 0$ must *not* be implemented by taking $T=0$, but by taking $T=\varepsilon$ where ε is a small positive constant. Doing so, observe that the last term in Equation (65) becomes $\varepsilon\overline{C_i^{\delta}P_iC_i^{\delta T}}=\varepsilon\bar{C}P\bar{C}^T+V^{cc}=\varepsilon\bar{C}P\bar{C}^T$. Although this term is non-zero now, it is not full-rank, since $\bar{C}$ in Example 3 is not full rank. Therefore the last term in

Table 3. ms-Compensatability and costs (Example 3) against parameter uncertainty.

β	0	0.5	0.8	1.1	1.2	2
$J_N/N, N \to \infty,\ n^c = 2$	22.435	27.003	43.187	1510.4	$+\infty$	$+\infty$
$J_N/N, N \to \infty,\ n^c = 1$	22.436	27.103	44.250	1588.4	$+\infty$	$+\infty$
$\rho^{\delta}_{\min},\ n^c = 2$	$-\infty$	−1.9836	−0.52576	−0.026446	0.10556	0.55000
$\rho^{\delta}_{\min},\ n^c = 1$	$-\infty$	−1.6218	−0.41751	−0.020544	0.10830	0.54930

Table 4. Number of algorithm iterations in discrete time and in the delta domain.

T_s	0.1	0.06	0.03	0.01	0.006	0.003	0.001	0.0006	0.0003	0.0001
$\beta = 0,\ n_c = 1$	636	1465	826	2054	3233	7377	19,689	31,003	57,162	1,48,596
$\beta = 6 \bullet 10^{-3},\ n_c = 1$	987	3544	810	2171	4069	7577	20,239	31,884	58,829	1,53,166
$\beta = 0,\ n_c = 1$	744	1787	1106	2728	2763	3033	3003	2995	2991	2988
$\beta = 6 \bullet 10^{-3},\ n_c = 1$	1175	4325	1094	2846	3147	3107	3075	3069	3064	3061

Equation (65), although non-zero, is still not invertible. This reveals the second aspect in which Example 3 is special: it requires the use of the Moore–Penrose pseudoinverse instead of the standard inverse in Equation (65).

It turns out that the computation of the compensatability measures in Theorems 7 and 8 suffer most from numerical errors. This seems to relate to the norms of Y_1, Y_2, Y_3, Y_4 that diverge heavily as convergence is reached. Also, the compensator matrices F^{δ}, K^{δ} tend to infinity. Still, we managed to compute the compensatability measures for Example 3 as follows. We implemented the limit $T \downarrow 0$ by computing the equivalent discrete-time optimal compensation problem (Remark 2; Van Willigenburg and De Koning 2000c) for different values of the sampling period $T_s = T$ approaching zero. As the smallest value we took 10^{-6}. Next, we transformed the equivalent discrete-time problem data to the delta domain taking $T = T_s$ and solved each problem by initialising it with the previous solution. Table 3 shows the results.

Example 4:

$$\bar{A}(t) = \begin{bmatrix} 0.5 & -2.6467 \\ 2.6467 & 0.5 \end{bmatrix},\quad \bar{B}(t) = \begin{bmatrix} 0.1389 \\ -1.2370 \end{bmatrix},$$
$$\bar{C}(t) = \begin{bmatrix} 1.0193 & -0.6632 \end{bmatrix},$$
$$Q(t) = \mathrm{diag}([0.9851\ \ 0.2956]),\quad R(t) = 0.3381,$$
$$V(t) = \mathrm{diag}([0.4141\ \ 0.4348]),$$
$$W(t) = 0.1535,\quad t_f \to \infty,\quad V^{AA},\ V^{AB},\ V^{AC},\ V^{BB},\ V^{CC}$$

given by (113).

For different values of the sampling period T_s the equivalent discrete-time LQG problem associated with the data in Example 4 is computed (Remark 2; Van Willigenburg and De Koning 2000c). Next, the equivalent discrete-time problem is solved using algorithms described in Van Willigenburg and De Koning (1999) and De Koning and Van Willigenburg (2000). The second and third rows in Table 4 record the number of iterations required by these discrete-time algorithms. After transformation of the equivalent discrete-time problems to the delta domain taking $T = T_s$, they were solved using Algorithm 4. The third and fourth rows in Table 4 record the number of iterations required to reach the same high level of accuracy. Two cases are considered in Table 4. The case $\beta = 0$ implies that the system parameters are deterministic. In the second case, $\beta = 6 \bullet 10^{-3}$ the system parameters are stochastic. Table 4 clearly reveals the superior efficiency of Algorithm 4 when the sampling period becomes very small. Essentially, this is due to the fact that at some stage the sampling period T_s becomes smaller than the time-step required for Euler numerical integration in continuous time (Remark 11). Then, Algorithm 4 takes this Euler integration step instead of T_s. The Euler numerical integration time-step used by Algorithm 4 for $T_s \le 0.01$ in Table 4 equals 0.012 (Remark 11).

Note that the system in Example 4 is highly oscillatory and unstable and the compensator order $n_c = 1$ is reduced. To illustrate the ill-conditioning in discrete time, we just mention that $F = 0.997$ is the compensator matrix obtained when $T = T_s = 10^{-4}$. This illustrates the general problem mentioned in Middleton and Goodwin (1990) that in discrete time the compensator matrix F tends to the identity matrix when the sampling period becomes very small.

Then, the discrete-time compensator computation and implementation become ill-conditioned.

8. Conclusions

The SDOPE and the associated algorithms and compensatability tests were formulated in the delta domain. This provides a unification of several fundamental results concerning controller design for linear systems with deterministic and white stochastic parameters. It also provides new results. These concern the properties compensatability and ms-compensatability in continuous time, and a *measure* of ms-compensatability. Also the solution of the finite-horizon time-varying optimal compensation problem for systems with white stochastic parameters in continuous time is new.

Hyland, Bernstein and Haddad contributed largely to the development of the optimal projection equations. Apart from Haddad and Tadmor (1993) that applies to systems with deterministic parameters, they did not consider the finite-horizon time-varying case. Also, they did not consider the *system properties* called compensatability and ms-compensatability. Initiated by De Koning (1982), these properties were developed by us in De Koning (1992), Van Willigenburg and De Koning (2000a) and De Koning and Van Willigenburg (2000), together with the associated tests and measures that use the optimal projection equations. Our consideration of the finite-horizon time-varying compensation problem in discrete time (Van Willigenburg and De Koning 1999, 2002) was motivated by the desire to design robust *digital* controllers for nonlinear systems following the methodology proposed in Athans (1971). From an automatic control point of view, we considered the continuous-time results of less importance. On the other hand, continuous time is often preferred by theoretically oriented researchers. Moreover, oversampling may necessitate digital controller design to be performed in the delta domain. With the formulation in the delta domain, realised in this article, the unification and extension of LQG problems and algorithms to systems with white stochastic parameters seems complete. Among other things, systems with white stochastic parameters allow for robust compensator design under structured parameter uncertainty. In the finite-horizon time-varying case, we are currently investigating this topic that seems largely unexplored. Another thing we are currently researching is the formulation of the equivalent discrete-time optimal compensation problem (Van Willigenburg and De Koning 2000c) directly in the delta domain. This is expected to enhance the numerical accuracy of digital compensator computations and their associated compensatability measures.

An interesting difference between continuous and discrete time concerns the singularity of the compensation problem. These differences are immediately seen from the last term between brackets in Equation (65) and the first between brackets in Equation (66). In continuous time, the problem is singular precisely when these terms are not invertible (not full-rank). Since continuous time corresponds to $T \downarrow 0$, the first term in between the brackets goes to zero (not the second and third as explained in this article). But, as long as $T > 0$, the correspondence with discrete time remains. In that case, in Equations (65) and (66), we may use the Moore–Penrose pseudoinverse, instead of the standard inverse. This only affects the possible uniqueness of the solution (De Koning 1982, 1992). As demonstrated by Example 3 in Section 7, using both of these features enabled us to calculate the ms-compensatability measure in continuous time, even if the associated compensation problem in continuous time is singular. As $T \downarrow 0$, some of the associated compensator matrices tend to infinity due to the singularity, as in Example 3.

In this article, the optimal projection equations for several continuous-time cases were proved to be equivalent to first-order necessary optimality conditions and (pseudo) minimality of the compensator. In continuous time, the optimal projection equations are differential equations. But these cannot be numerically integrated in a straightforward manner, due to the rank conditions that are imposed. When formulated in the delta domain, our discrete-time algorithms may be interpreted in continuous time as performing Euler numerical integration while at the same time realising the rank conditions.

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Appendix

We start by proving Theorem 4 that concerns the compensation of continuous-time systems with white stochastic parameters, when the horizon is infinite. Theorem 3 is a special case of Theorem 4 since it concerns the same problem for systems with deterministic parameters. The unproved part of Theorem 2 will finally be proved in this Appendix. The structure of the equations making up Theorem 4 coincides with those making up Theorem 2. Therefore, from the proof of Theorem 4 the unproved part of Theorem 2 is easily proved.

To prove Theorem 4, we will show that, in the limit $T \downarrow 0$, $t = iT$ fixed, Equations (62), (63), (103), (104) in Bernstein and Hyland (1988) are equivalent to our Equations (69)–(72). Observe that (103), (104) in Bernstein and Hyland (1988) coincide with (64), (65) in Bernstein and Hyland (1988).

First, consider Equation (62) in Bernstein and Hyland (1988). It has six terms on the right when we consider the fourth term that starts with a summation symbol, as one term. Our Equation (69) has 11 terms on the right. In Table A1, we indicate which terms in both equations are equivalent. The numbers indicate the different terms on the right of each equation when counting them from left to right. The fourth term of Equation (62) in Bernstein and Hyland (1988) has two parts under the summation sign. These parts, denoted by 4.1 and 4.2, are considered separately in this table.

Table A1. Correspondence of equations.

Equation (62) in Bernstein and Hyland (1988)	1	2	3	4.1	4.2	5	6
Equation (63)	8	9	7	1	3 + 4 + 5 + 6	2	10

Table A2. Correspondence of equations.

Equation (63) in Bernstein and Hyland (1988)	1	2	3	4.1	4.2	5	6
Equation (70)	8	9	7	1	3 + 4 + 5 + 6	2	10

In our Equation (63), the 11th term goes to zero in the limit $T \downarrow 0$, $t = iT$ fixed, as explained in the proof of Theorem 2. Next, consider Equation (63) in Bernstein and Hyland (1988). This equation corresponds to our Equation (70). Due to the duality of the compensation problem, the equivalences given by Table A2 are identical.

Next, consider Equation (103) in Bernstein and Hyland (1988). This equation is equivalent to the first part of our Equation (71) premultiplied with τ_{i+1}, i.e.

$$\tau_{i+1}\frac{\Psi^1_{i+1} - \hat{P}_i}{T} = \tau_{i+1}\Psi^{1\delta}_{i+1}. \tag{114}$$

To see this observe that the term inside the square brackets in Equation (103) of Bernstein and Hyland (1988) is equivalent to $\Psi^{1\delta}_{i+1}$. Therefore, the right-hand side of Equation (103) of Bernstein and Hyland (1988) is equivalent to the right-hand side of our Equation (114). Furthermore, from Van Willigenburg and De Koning (1999),

$$\hat{P}_i = \tau_i\hat{P}_i = \tau_i\Psi^1_i, \tag{115}$$

and because $\tau_{i+1} \to \tau_i$ in the limit $T \downarrow 0$, $t = iT$ fixed, for the left-hand side of Equation (114) we find,

$$\tau_{i+1}\frac{\Psi^1_{i+1} - \hat{P}_i}{T} \to \frac{\hat{P}_{i+1} - \hat{P}_i}{T}. \tag{116}$$

From (116) the left-hand side of our Equation (114) is seen to be equivalent to the left-hand side of Equation (103) of Bernstein and Hyland (1988).

Dually, Equation (104) of Bernstein and Hyland (1988) is equivalent to the first part of our Equation (72) post-multiplied by τ_i, i.e.

$$\frac{\Psi^2_i - \hat{S}_{i+1}}{T}\tau_i = \Psi^{2\delta}_i\tau_i. \tag{117}$$

Using Van Willigenburg and De Koning (1999),

$$\hat{S}_i = \hat{S}_i\tau_i = \Psi^2_i\tau_i \tag{118}$$

and $\tau_i \to \tau_{i+1}$ in the limit $T \downarrow 0$, $t = iT$ fixed. Then, for the left-hand side of Equation (117), we find,

$$\frac{\Psi^2_i - \hat{S}_{i+1}}{T}\tau_i \to \frac{\hat{S}_i - \hat{S}_{i+1}}{T}. \tag{119}$$

Therefore also the left-hand sides of Equations (117) and (104) of Bernstein and Hyland (1988) are equivalent. In a similar fashion, it is easily recognised that the other expressions involved in Theorem 4, in the limit $T \downarrow 0$, $t = iT$ fixed, also match the ones in this article. This proves Theorem 4.

To prove Theorem 2, observe that Equations (3.2a)–(3.2d) in Haddad and Tadmor (1993) have the same structure as Equations (62)–(65) in Bernstein and Hyland (1988) after removing the fourth term in Equations (62), (63) of Bernstein and Hyland (1988). Therefore, the equivalence between Equation (3.2b) in Haddad and Tadmor (1993) and our Equation (63) is represented by Table A1 with '(62) in Bernstein and Hyland (1988)' replaced with '(3.2b) in Haddad and Tadmor (1993)' if we leave out the correspondences of 4.1 and 4.2. Dually, Table A2 describes the equivalence between (3.2a) in Haddad and Tadmor (1993) and our Equation (70). Because the problem in Haddad and Tadmor (1993) has a finite horizon, the left-hand sides of Equations (3.2a), (3.2b) in Haddad and Tadmor (1993) are time derivatives. These are easily seen to be equivalent to the left-hand sides of our Equations (63), (70). Similarly Equations (114)–(119) prove that (3.2c) is equivalent to our Equation (71) while (3.2.d) is equivalent to our Equation (72).