Temporal Linear System Structure

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***Abstract*—Piecewise constant rank systems and the differential Kalman decomposition are introduced in this note. Together these enable the detection of temporal uncontrollability/unreconstructability of linear continuous-time systems. These temporal properties are not detected by any of the four conventional Kalman decompositions. Moreover piecewise constant rank systems admit the state dimension to be time-variable. As demonstrated in this note linear continuous-time systems with variable state dimensions enable the well rounded realization theory suggested already by Kalman. The differential Kalman decomposition introduced in this note is associated with differential controllability and differential reconstructability. The system structure obtained from the differential Kalman decomposition may be interpreted as the temporal linear system structure. This note reveals that the difference between controllability and reachability as well as reconstructability and observability is entirely due to changes of the temporal linear system structure. Also, this note reveals how the differential Kalman decomposition relates to the conventional ones.**

***Index Terms*—Differential controllability, differential Kalman decomposition, piecewise constant rank systems, temporal system structure, time-varying system dimensions, time-varying system structure.**

I. Introduction

A general approach to control nonlinear systems is to compute an optimal control and state trajectory off-line using a nonlinear systems model. To accommodate for disturbances the *linearised* dynamic model about these trajectories is used to design, e.g., a linear quadratic perturbation feedback controller that operates on-line [1]. This approach depends critically on the controllability and reconstructability of the linearised dynamic model that is generally *time-varying*. If the systems model or the optimal control is not sufficiently smooth, e.g., if the control is *bang-bang, partly singular,* or *digital*, the time-varying linearised dynamic model may be *temporarily* uncontrollable and/or unreconstructable [2]. This implies that over the associated time intervals the feedback controller is partly ineffective and the system may become unstable. Therefore, the *detection* of *temporal* uncontrollability and unreconstructability is highly relevant to control engineers.

The properties controllability, reachability, reconstructability and observability are highly fundamental for linear control system design. These properties have a *global* character with respect to time and are associated with Kalman decompositions of the linear system [3]. As demonstrated in this note these decompositions do not detect the *temporal loss* of such properties which have a *local* character with respect to time. This note shows that the temporal loss is associated with changes in what we will call the *temporal linear system structure*. The temporal linear system structure is obtained from a suitable Kalman decomposition introduced in this note that has a *local* character with respect to time. The temporal linear system structure is obtained by *restricting* the time interval over which the system is considered to an infinitely small one. Then e.g., controllability turns into differential controllability. The idea of restricting the time interval over which to consider a system has been used before in [5] to obtain generalized balanced realizations.

Manuscript received April 13, 2007; revised October 18, 2007. Published August 27, 2008 (projected). Recommended by Associate Editor D. Henrion.

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Digital Object Identifier 10.1109/TAC.2008.921033

At each time instant, conventional Kalman decompositions provide a decomposition of the state vector into four parts consisting of respectively n_a, n_b, n_c, and n_d state variables. The integers n_a, n_b, n_c, and n_d determine the system structure and their sum equals the state dimension at any time. To obtain the results in this note piecewise constant rank systems having a *time-variable structure* need to be considered. Also, piecewise constant rank systems admit the system (state) dimension to be time-variable. As shown in this note they therefore enable the well rounded realization theory for time-varying linear systems suggested already by Kalman [4]. The main contributions of this note are represented by Fig. 1.

II. An Illustrative Example

Example 1: Consider the following time-varying linear system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t), t \in [0,1]\end{aligned} \tag{1}$$

where

$$\begin{aligned}&A(t) = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}, t \in [0, 0.25], A(t) = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}, \quad t \in (0.25, 0.5]\\ &A(t) = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}, t \in (0.5, 0.75], A(t) = \begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}, \quad t \in (0.75, 1]\\ &B(t) = \begin{bmatrix}1 & 0\end{bmatrix}^T, C(t) = \begin{bmatrix}1 & 0\end{bmatrix}, \quad t \in [0,1].\end{aligned} \tag{2}$$

If time in (1) would be restricted to $(0.25, 0.5)$ the system would be unreachable as well as uncontrollable. Similarly if time in (1) would be restricted to $(0.5, 0.75)$ the system would be unobservable and unreconstructable. If according to (1) $t \in [0,1]$ then we might call the system *temporarily* uncontrollable/unreachable over $(0.25, 0.5)$ because over this time interval the second state variable is not influenced by the input. Similarly we might call the system *temporarily* unreconstructable/unobservable over $(0.5, 0.75)$ because over this time interval the second state variable does not influence the output. Since moreover the second state variable grows exponentially over $(0.25, 0.5)$ and over $(0.5, 0.75)$ it cannot be stabilized by a controller over these time intervals. If we apply a similarity transformation at every time $t \in [0,1]$ to the system (1), (2) then these temporal properties are unchanged but no longer obvious from the system description. None of the four *conventional* Kalman decompositions recovers these temporal properties because the system (1), (2) is controllable and reconstructable from any time as well as reachable and observable at any time [3]. As demonstrated in this note the *differential* Kalman decomposition retrieves the temporal system properties and a system description similar to (1), (2). It is based on the properties differential controllability (d-controllability) and differential reconstructability (d-reconstructability) that have a *local* character with respect to time [3]. Observe that the system (1), (2) is not d-controllable over $(0.25, 0.5)$ and is not d-reconstuctable over $(0.5, 0.75)$. So temporal uncontrollability/unreconstructability is associated with d-uncontrollability/d-unreconstructability over *open* time intervals.

To circumvent technical difficulties caused by the discontinuities of $A(t), B(t)$ and $C(t)$ in (1), (2), but most important to obtain the well rounded realization theory suggested by Kalman for time varying linear systems [4], the next section introduces so called piecewise constant rank systems recently proposed in [6] where they appeared to be new.

III. Continuous-Time Systems With Variable Structure and Dimensions

Quoting Kalman from [4]: "The only possibility of getting a reasonably well-rounded realization theory is to generalize the notion of a dynamical system in such a way that the *dimension* of the state-space is

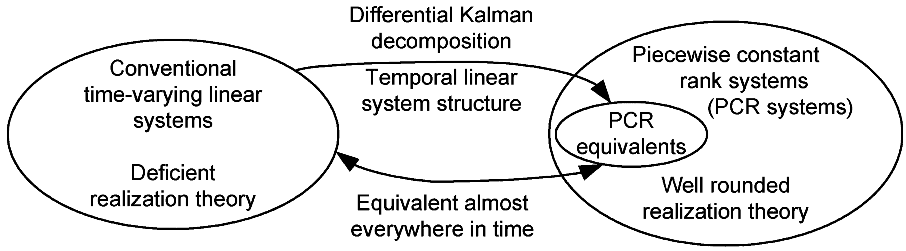

Fig. 1. The main contributions.

allowed to vary with time". Remarkably, except for [7], [6] continuous-time systems with variable dimensions seem to have been ignored. A reason for this might be that a general system with time-varying dimensions and system structure requires the following uncommon system description:

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t), y(t) = C(t)x(t) \\ &x(t) \in R^{n_i}, u(t) \in R^{m_i}, y(t) \in R^{l_i} \\ &t \in (t_i, t_{i+1}), i = 0, 1, \ldots, N-1 \\ &x(t_i^+) = A_i x(t_i^-), A_i \in R^{n_{i+1} \times n_i}, \quad i = 1, 2, \ldots, N-1. \end{aligned} \quad (3)$$

In (3) x denotes the state, u the input that is assumed to be bounded and y the output. The state, input and output dimensions n_i, m_i and l_i are bounded and depend only on the index i. Therefore the matrices $A(t), B(t)$ and $C(t)$ may only change dimensions *across* the *isolated* times $t_i, i = 1, 2, \ldots, N-1$ in (3). Over each single *open* interval (t_i, t_{i+1}) (3) represents a conventional linear system description the controllability and reconstructability matrices of which are assumed to have a constant rank. This implies that over each single open interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$ the system is a constant rank system [8]. Therefore the system (3) is called a *piecewise constant rank system (PCR system)*. The additional system matrices $A_i, i = 1, 2, \ldots, N-1$, describe the state transitions from t_i^- to t_i^+ where the superscripts $-, +$ denote respectively the left and right time limit. In (3) $x(t_0^+)$ should be identified as the *initial state*. Similarly $x(t_N^-)$ should be identified as the *terminal state*. The class of PCR systems is very broad and contains among others piecewise time-invariant and piecewise analytic systems [8]. A PCR system differs from the switched linear systems in [9], [10] because the time-instants $t_i, i = 1, 2, \ldots, N-1$ are *a priori fixed* and the linear system over each time interval (t_i, t_{i+1}) is *time-varying*. The time domain of a PCR system is denoted by T given by

$$\mathrm{T} = \cup(t_i, t_{i+1}), i = 0, 1, \ldots, N-1. \quad (5)$$

Three types of PCR systems will be considered with different time domains. We consider PCR systems with $t_0 = -\infty, t_N = +\infty$, with $t_0 = 0, t_N = +\infty$ and with, $t_0 = 0, t_N < +\infty$. These are denoted by $-+, 0+$ and 0N PCR systems respectively. In all the definitions in this note concerning $-+, 0+$ and 0N PCR systems time should always be considered *restricted* to the associated time domain of the PCR system. For $0+$ and 0N systems the deterministic initial state $x(t_0^+)$ is considered part of the system description.

In practice we often start from a continuous-time system description with constant dimensions defined over $[t_0, t_N]$, $[t_0, t_N)$ or (t_0, t_N) such as Example 1. Observe that the system in Example 1 is *not formally* a PCR system because its time domain $[t_0, t_N]$ does not comply with (5). To arrive at a PCR system description we have to find $t_i, i = 1, 2, \ldots, N-1$ that determine the open time intervals over which the system is constant rank. For the system in Example 1 these time instants are easily recognized as $t_1 = 0.25, t_2 = 0.5, t_3 = 0.75$. If the input does not affect the state transitions $x(t_i^-) \to x(t_i^+), i = 1, 2, \ldots, N-1$, then by selecting $A_1 = A_2 = A_3 = I$ in (3), where I denotes the identity matrix, *an associated* PCR *system* is obtained that is *equivalent almost everywhere* to the original, see Fig. 1.

Definition 1: A PCR system that is almost everywhere equivalent to another system is called a PCR equivalent of that system.

Observe from Example 1 that the determination of a PCR equivalent of a conventional linear system with constant dimensions and a bounded input comes down to determining $t_i, i = 1, 2, \ldots, N-1$. In Example 1 these are easily recognized. But this is not so in general e.g., for a linearized system obtained from finite differences. Then a means of *detecting* $t_i, i = 1, 2, \ldots, N-1$ becomes important. Observe that for Example 1 we may also select $t_1 = 0.1, t_2 = 0.25, t_3 = 0.5, t_4 = 0.75$ to obtain *another* PCR equivalent. Clearly the selection of $t_1 = 0.1$ is superfluous. In finding a PCR equivalent we prefer to *minimize* N in (3). Detecting $t_i, i = 1, 2, \ldots, N-1$ such that N is minimal is a key issue in this note and is achieved by application of the differential Kalman decomposition to be introduced in the next section, see Fig. 1. Most results known from linear systems theory also apply to PCR systems because the system description (3) combines an ordinary continuous-time system description with a discrete-time one without an input. The latter describes the transitions $x(t_i^-) \to x(t_i^+)$. During these transitions the dimension of the state may change. But most of linear systems theory also applies to discrete-time systems with variable state dimensions [11]. In obtaining and defining properties for PCR systems the key is to combine continuous and discrete-time results properly. Most of these can be found in [6]. In this note we only present definitions and results that are fundamental to the developments in this note like the next definition.

Definition 2: The *reachability/controllability grammian* $W_{s,t}$, $t > s$, of a PCR system is given by

$$\frac{dW_{s,t}}{dt} = A(t)W_{s,t} + W_{s,t}A^T(t) + B(t)B^T(t) \qquad t, s \in \mathrm{T}, t > s \quad (6)$$

$$W_{s,t_i^+} = A_i W_{s,t_i^-} A_i^T, i = 1, 2, \ldots, N-1 \qquad s \in \mathrm{T}, s < t_i^-, W_{s,s} = 0. \quad (7)$$

The transition rule (7) of $W_{s,t}$ from $t = t_i^-$ to $t = t_i^+$ equals the discrete-time rule without an input. This follows from the last line of (3). The *observability/reconstructability grammian* $M_{s,t}, t > s$, of the PCR system is given by

$$-\frac{dM_{s,t}}{ds} = A^T(s)M_{s,t} + M_{s,t}A(s) + C^T(s)C(s), t, s \in \mathrm{T}, t > s \quad (8)$$

$$M_{t_i^-,t} = A_i^T M_{t_i^+,t} A_i, i = 1, 2, \ldots, N-1 \qquad t \in \mathrm{T}, t > t_i^+, M_{t,t} = 0. \quad (9)$$

IV. The Differential Kalman Decomposition

The differential Kalman decomposition is introduced and applied in this section to obtain PCR equivalents, see Fig. 1. It computes the *local* system structure that is associated with differential controllability (d-controllability) and differential reconstructability (d-reconstructability). These properties reveal immediately temporal uncontrollability and temporal unreconstructability. We start this section by showing how the four conventional Kalman decompositions compute *global* system structures that are generally different but may all be interpreted as PCR sub systems.

As for linear systems with constant dimensions [3], [12], [13] *four conventional* Kalman decompositions may be computed for PCR systems (3) at any time $t \in \mathrm{T}$ based on the following four pairs of grammians that satisfy (6) and (7) and (8) and (9):

$$1)\ W_{t_0,t}, M_{t,t_N}\ 2)\ W_{t_0,t}, M_{t_0,t}\ 3)\ W_{t,t_N}, M_{t,t_N},\ 4)\ W_{t,t_N}, M_{t_0,t}. \quad (10)$$

In each case the system structure is of the following form:

$$\begin{aligned}
&x'(t) = [\,x_a'^T(t) \quad x_b'^T(t) \quad x_c'^T(t) \quad x_d'^T(t)\,]^T\\
&x_a'(t) \in R^{n_a}, x_b'(t) \in R^{n_b}, x_c'(t) \in R^{n_c}, x_d'(t) \in R^{n_d}\\
&A'(t) = \begin{bmatrix} A'_{\mathrm{aa}}(t) & A'_{\mathrm{ab}}(t) & A'_{\mathrm{ac}}(t) & A'_{\mathrm{ad}}(t)\\ 0 & A'_{\mathrm{bb}}(t) & 0 & A'_{\mathrm{bd}}(t)\\ 0 & 0 & A'_{\mathrm{cc}}(t) & A'_{\mathrm{cd}}(t)\\ 0 & 0 & 0 & A'_{\mathrm{dd}}(t)\end{bmatrix}\\
&B'(t) = [\,B'^T_a(t) \quad B'^T_b(t) \quad 0 \quad 0\,]^T,\\
&C'(t) = [\,0 \quad C'_b(t) \quad 0 \quad C'_d(t)\,], t \in (t_i, t_{i+1})\\
&\quad i = 0, 1, \ldots, N-1.
\end{aligned} \quad (11)$$

1) Decomposes the state-space at time $t \in \mathrm{T}$ into states x_a' being reachable at time t and unobservable at time t, into states x_b' being reachable at time t and observable at time t, into states x_c' being unreachable at time t and unobservable at time t and into states x_d' being unreachable at time t and observable at time t. 2) Does the same as 1) with "observable at" replaced by "reconstructable from". 3) Does the same as 1) with "reachable at" replaced by "controllable from". 4) Does the same as 1) with "reachable at" replaced by "controllable from" and "observable at" by "reconstructable from".

Theorem 1: 1) Using our definition of PCR systems (3) each conventional Kalman decomposition (10), (11) may be interpreted as a decomposition into four PCR *sub-systems* defined over T having *time-varying* dimensions in general. Over each separate time interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$, the number of states n_i, n_a, n_b, n_c, n_d of the PCR system and its sub systems are constant. But n_a, n_b, n_c, n_d may be different for each pair of grammians (10). 2) The input-output map of a PCR system is solemnly determined by PCR subsystem b) generated by Kalman decomposition 1) from (10). The PCR subsystem b) obtained from this Kalman decomposition is a *minimal realization* of the PCR system if $x(t_0^+) = 0$. If $x(t_0^+) \neq 0$ the same result is obtained if the reachability grammian $W_{t_0^+,t}$ is replaced by the so-called *weak* reachability grammian $W'_{t_0^+,t}$ that is also described by (6), (7) with W replaced by W' except for the initial condition $W'_{t_0^+,t_0^+} = x(t_0^+)x(t_0^+)^T$.

Proof: 1) Constant rank systems defined over the open time intervals (t_0, t_N) have constant dimensions, grammians with zero boundary conditions at t_0, t_N and the property that all four Kalman decompositions at every time $t \in (t_0, t_N)$ produce equal and constant dimensions n_a, n_b, n_c, n_d [8]. According to (6) and (7) and (8) and (9) for PCR systems the boundary conditions of the grammians for each separate time interval $(t_i, t_{i+1}), i = 1, 2, \ldots, N-2$ are generally nonzero and different for each Kalman decomposition. These different nonzero boundary conditions cause the dimensions n_a, n_b, n_c, n_d obtained from the four Kalman decompositions to be generally different. However from [8] it follows that the nonzero boundary conditions do *not* change the fact that the associated dimensions n_a, n_b, n_c, n_d are constant over each separate interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$. 2) This follows from application of the results in [3] and [8] over each separate interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$. In addition for $x(t_0^+) \neq 0$ the result follows from [14]. □

Definition 3: The *structure* of a PCR system at time $t \in \mathrm{T}$ is determined by the dimensions $n_a(t), n_b(t), n_c(t)$ and $n_d(t)$ obtained from a Kalman decomposition at time $t \in \mathrm{T}$.

As opposed to "reachability at" and "controllability from" a certain time $t \in \mathrm{T}$ "differential reachability at" (d-reachability at) and "differential controllability from" (d-controllability from) a certain time $t \in \mathrm{T}$ are *local* system properties with respect to time. Moreover for PCR systems these properties are *identical* due to (5), [3], [6]. They are determined by the d-reachability/d-controllability grammian W_t defined next [8], [6].

Definition 4: The so called *d-reachability/d-controllability grammian* W_t of a PCR system at every time $t \in \mathrm{T}$ is given by

$$\begin{aligned}
&W_t = C_j(t)C_j^T(t)\\
&C_j(t) = [\,P_0(t) \quad P_1(t) \quad \ldots \quad P_j(t)\,]\\
&P_0(t) = B(t), P_{k+1}(t) = -A(t)P_k(t) + \dot{P}_k(t)\\
&\quad k = 0, 1, \ldots, j-1
\end{aligned} \quad (12)$$

with j the smallest value such that $\mathrm{rank}(C_{j+1}(t)) = \mathrm{rank}(C_j(t))$. The so called *d-observability/d-reconstr. grammian* M_t of the PCR system at time t is given by

$$\begin{aligned}
&M_t = O_k^T(t)O_k(t)\\
&O_k = [\,S_0^T(t) \quad S_1^T(t) \quad \ldots \quad S_k^T(t)\,]^T\\
&O_0(t) = C(t), O_{l+1}(t) = O_l(t)A(t) + \dot{O}_l(t)\\
&\quad l = 0, 1, \ldots, k-1
\end{aligned} \quad (13)$$

with k the smallest value such that $\mathrm{rank}(O_{k+1}) = \mathrm{rank}(O_k)$. The values j, k exist and are constant over each time interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$ because the system is assumed to be piecewise constant rank [8]. Then according to [8] over each time interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$ the PCR system is a (j, k) constant rank system.

Definition 5: The so-called *differential Kalman decomposition* (d-Kalman decomposition) at every time t is computed from the *d-reachability/d-controllability grammian* W_t and the *d-observability/d-reconstructability grammian* M_t in exactly the same manner as the four conventional Kalman decompositions are computed from the grammian pairs (10).

Definition 6: The *temporal linear structure* of a PCR system at any time $t \in \mathrm{T}$ is determined by $n_a(t), n_b(t), n_c(t), n_d(t)$ obtained from the d-Kalman decomposition at time $t \in \mathrm{T}$.

Theorem 2: 1) The d-Kalman decomposition at every time $t \in \mathrm{T}$ decomposes a PCR system according to (11) into states that are a) d-reachable at/d-controllable from and d-unobservable at/d-unreconstructable from time t b) d-reachable at/d-controllable from and d-observable at/d-reconstructable from time t c) d-unreachable at/d-uncontrollable from and d-unobservable at/d-unreconstructable from time t d) d-unreachable at/d-uncontroll. from and d-observable at/d-reconstructable from time t. 2) Over each time interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$ the number of states n_a, n_b, n_c, n_d generated by the d-Kalman decomposition is constant. So a), b), c), and d) may be regarded as PCR *subsystems* of the PCR system. Like the original PCR system they have constant dimensions over $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$.

Remark 1: Suppose we start from a conventional system description with constant dimensions over $[t_0, t_N]$, such as Example 1. If we try to apply the d-Kalman decomposition at every time $t \in [t_0, t_N]$ what do we find? The d-Kalman decomposition at any time $t \in [t_0, t_N]$ requires the d-reachability/d-controllability and d-observability/d-reconstructability matrices $C_j(t), O_k(t)$ in (12), (13). They in turn require knowledge of a sufficient number of derivatives of $A(t), B(t)$ and $C(t)$. Now observe that $t_i, i = 0, 1, 2, \ldots, N$, are precisely those isolated times where the d-Kalman decomposition does not formally exist because some of these derivatives do not exist. This represents one of the technical difficulties mentioned at the end of Section I. On the other hand evaluation of the d-Kalman decomposition at the remaining times $t \in \mathrm{T}$, where T represents precisely the time domain of a PCR equivalent, *recovers* the *temporal* linear system structure $n_a(t), n_b(t), n_c(t), n_d(t)$ such as that of Example 1. Moreover it *detects* $t_i, i = 1, 2, \ldots, N-1$ as precisely those isolated times where the temporal linear structure *changes*. In this way it provides a PCR equivalent for which N is minimal, see Fig. 1. In practice the d-Kalman decomposition can be evaluated at a limited number of times only. This detects t_i only approximately.

Remark 2: Without attaching the special name to it the d-Kalman decomposition had already been presented in [8]. The paper [8], however, is restricted to constant rank systems. For this type of system all five Kalman decompositions considered in this note provide the same system structure that is constant over the entire time domain. The interesting difficult issues addressed in this note occur only if the temporal linear system structure *changes* over time. *Only then* the differences between controllability, reachability and differential controllability emerge and dually the differences between reconstructability, observability and differential reconstructability. Therefore to describe, detect and analyze changes of the temporal linear system structure the PCR systems introduced in this note are vital. Moreover, according to Theorem 1, they provide the well rounded realization theory suggested by Kalman that allows systems to have time variable state dimensions. On the other hand in addition to [3] the paper [8] deserves much credit because it provided very important insight as well as technical results related to the structure of linear systems on which this note heavily relies.

Theorem 3: In computing the d-Kalman decomposition *of a* PCR *system* the d-reachability/d-controllability grammian W_t of a PCR system at any time $t \in (t_i, t_{i+1})$ may be *interchanged* with 1) any reachability grammian $W_{t_s,t_f}, t_f > t_s, t_s, t_f \in (t_i, t_{i+1})$ with "initial condition" $W_{t_s,t_s} = 0$ or 2) any controllability grammian W_{t_s,t_f} with "terminal condition" $W_{t_f,t_f} = 0$. A dual result applies to the d-observability/d-reconstructability grammian M_t at every time $t \in (t_i, t_{i+1})$.

Proof: Over every separate time interval $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$ a PCR system is constant rank. Then the result follows from [8] □

Remark 3: Theorem 3 reveals that *given* a PCR equivalent, i.e., knowing $t_i, i = 1, 2, \ldots, N-1$, the d-Kalman decomposition can actually be computed without computing the d-reachability/d-controllability and d-observability/d-reconstructability grammians and the associated matrices $C_j(t), O_k(t)$ in (12), (13). This represents a way to circumvent the computation of derivatives of $A(t), B(t)$ and $C(t)$ needed to compute $C_j(t), O_k(t)$.

Remark 4: The following subtle picture emerges. As explained in Remark 1 and shown in Fig. 1, in practice we usually start from a system description that is *not* piecewise constant rank, like Example 1. Then a key issue is to find a PCR equivalent. This comes down to detecting $t_i, i = 1, 2, \ldots, N-1$. From Remarks 1 and 3, $t_i, i = 1, 2, \ldots, N-1$ can be detected *only* by the d-Kalman decomposition that uses the d-reachability/d-controllability grammian W_t and d-observability/d-reconstructability grammian M_t as an input. *Having detected* $t_i, i = 1, 2, \ldots, N-1$ the d-Kalman decomposition for the PCR equivalent can be computed alternatively and more easily according to Theorem 3 and Remark 3. For Example 1, $t_i, i = 1, 2, \ldots, N-1$ could even be established without computation. According to Remark 1 and (12), (13) *candidates* for $t_i, i = 1, 2, \ldots, N-1$ are isolated times where $A(t), B(t)$ and $C(t)$ are non smooth. One may in fact take all of these time instants accepting that N is not necessarily minimal in that case. For practical purposes, this makes no difference since $t_i, i = 1, 2, \ldots, N-1$ are isolated times. However, to determine $t_i, i = 1, 2, \ldots, N-1$ in this manner requires knowledge of isolated times where $A(t), B(t)$ and $C(t)$ may be non smooth. Unless analytical or piecewise analytical descriptions of $A(t), B(t)$ and $C(t)$ are available, as for Example 1, this knowledge may well be lacking.

Remark 5: The computation of derivatives of $A(t), B(t)$ and $C(t)$ needed to compute W_t, M_t which determine the d-Kalman decomposition may be circumvented in yet another way. Instead of W_t, M_t given by (12), (13) the grammians $W_{t,t+\Delta t}, M_{t,t+\Delta t}$ given by (6) and (8), with Δt sufficiently small, may be computed. Whereas W_t, M_t determine controllability/reachability and reconstructability/observability over an infinitely small time interval $W_{t,t+\Delta t}, M_{t,t+\Delta t}$ approximate these by determining these properties over a very small time-interval. According to (6) and (8) computation of $W_{t,t+\Delta t}$, can be done by means of numerical integration that only requires evaluations of $A(t), B(t)$ and $C(t)$ over $[t, t+\Delta t]$. Since it is $\mathrm{rank}(W_{t,t+\Delta t}), \mathrm{rank}(M_{t,t+\Delta t})$ that determine the outcome of the Kalman decomposition, in performing numerical integration from t to $t + \Delta t$ we should perform sufficient time steps since these contribute to changes of these ranks.

The next theorem states how knowing the *temporal* structure of a PCR system answers straightaway for any time $t \in \mathrm{T}$ whether the *global* system properties controllability and reconstructability apply.

Theorem 4: 1) A PCR system is controllable from time $t \in \mathrm{T}$ if and only if $\exists t' > t$ such that the PCR system is "d-reachable at" or equivalently "d-controllable from" time $t' \in \mathrm{T}$. Dually, 2) a PCR system is reconstructable from time $t \in \mathrm{T}$ if and only if $\exists t' < t$ such that the PCR system is "d-observable at" or equivalently "d-reconstructable from" time $t' \in \mathrm{T}$.

Proof: Over every separate interval (t_i, t_{i+1}) a PCR system is either controllable from and d-controllable from any time or no time [8]. □

As an immediate consequence of Theorem 4, we have the following corollary.

Corollary 1: 1) If a PCR system is "d-reachable at" or equivalently "d-controllable from" any time $t \in (t_{N-1}, t_N)$ then the PCR system is controllable from any time $t \in \mathrm{T}$. 2) If a PCR system is "d-observable at" or equivalently "d-reconstructable from" any time $t \in (t_0, t_1)$ then the system is reconstructable from any time $t \in \mathrm{T}$.

One may have expected results similar to Theorem 4 and Corollary 1 for reachability and observability. The following example demonstrates why these are not obtained.

Example 2: Consider a PCR system with $t_0 = 0, t_1 = 0.5, t_2 = t_N = 1$. The system matrices are

$$A(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C(t) = [1 \quad 0]$$
$$t \in (t_0, t_1), A(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C(t) = [1 \quad 0]$$
$$t \in (t_1, t_2), A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

One easily computes $\mathrm{rank}(W_t) = 2, t \in (t_0, t_1), \mathrm{rank}(W_t) = 1, t \in (t_1, t_2)$ so the PCR system is "d-reachable at" and equivalently "d-controllable from" any time $t \in (t_0, t_1)$ and is not for any time $t \in (t_1, t_2)$. Obviously, the second state variable is temporarily uncontrollable/unreachable for any $t \in (t_1, t_2)$. Due to $A_1 = [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$ this second state variable at time t_1^+ is zero and unaffected by the state $x(t_1^-)$. Therefore it remains zero for $t \in (t_1, t_2)$ and the system is not reachable at any time $t \in (t_1, t_2)$. This complies with $\mathrm{rank}(W_{t_0^+,t}) = 1, t \in (t_1, t_2)$. So although for any time $t \in (t_1, t_2), \exists t' < t$ such that the PCR system is "d-reachable at" and equivalently "d-controllable from" time t' we may *not* conclude that the system is reachable for any $t \in (t_1, t_2)$. Now notice how $\mathrm{rank}(W_{t_0^+,t}) = 1, t \in (t_1, t_2)$ is caused by $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ through (7). If $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathrm{rank}(W_{t_0^+,t}) = 2, t \in (t_1, t_2)$ and the second temporarily uncontrollable/unreachable state at time t_1^+ would be affected by $x(t_1^-)$. Then through manipulation of $x(t_1^-)$ we can manipulate the second state variable for $t \in (t_1, t_2)$. Then the system is reachable at any time $t \in (t_1, t_2)$.

V. Examples

Example 3: Minimal realizations. Consider the PCR equivalent of Example 1, i.e., with $t_1 = 0.25, t_2 = 0.5, t_3 = 0.75, A_1 = A_2 = A_3 = I$. One easily sees and computes $\mathrm{rank}(W_{t_0^+,t}) = 2, \mathrm{rank}(M_{t,t_N^-}) = 2, t \in (0,1)$. Therefore, the system is minimal despite the temporal uncontrollability/unreachability and unreconstructability/unobservability. The associated states may *not* be dropped because e.g., over $(0.25, 0.5)$ the second d-uncontrollable/d-unreachable state is nonzero in general because at $t = 0.25^+$ it is nonzero in general. This information is *transferred* by $W_{0+,0.25+}$ that satisfies $\mathrm{rank}(W_{0+,0.25+}) = 2$.

Consider Example 1 with the system descriptions on the intervals $[0, 0.25], (0.25, 0.5]$ swapped as well as those on the intervals $(0.5, 0.75], (0.75, 1]$. For the associated PCR equivalent, i.e., with $t_1 = 0.25, t_2 = 0.5, t_3 = 0.75, A_1 = A_2 = A_3 = I$, one easily sees and computes $\mathrm{rank}(W_{t_0^+,t}) = 1, t \in (0, 0.25), \mathrm{rank}(W_{t_0^+,t}) = 2, t \in (0.25, 1), \mathrm{rank}(M_{t,t_N^-}) = 2, t \in (0, 0.75), \mathrm{rank}(M_{t,t_N^-}) = 1, t \in (0.75, 1)$. Therefore, application of the conventional Kalman decomposition 1) from (10), according to theorem 1, provides the following minimal realization of this PCR equivalent

$$\begin{aligned}
&A(t) = 1, B(t) = 1, C(t) = 1, t \in (0, 0.25)\\
&A(t) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T\\
&\qquad\qquad\qquad t \in (0.25, 0.5)\\
&A(t) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T\\
&\qquad\qquad\qquad t \in (0.5, 0.75)\\
&A(t) = 1, B(t) = 1, C(t) = 1, t \in (0.75, 1)\\
&\quad A_1 = [1 \quad 0]^T\\
&\quad A_2 = I, A_3 = [1 \quad 0].
\end{aligned} \tag{14}$$

Example 4: Constant temporal linear system structure. A most interesting example that originates from statistical filtering is taken from [12] (symbolic computation revealed a few misprints in the original example). The system is described by

$$\begin{aligned}
A(t) &= \begin{bmatrix} -t^4/4f & 1 & 0 \\ -t^3/2f & 0 & 1 \\ -t^2/2f & 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} t^4/4f \\ t^3/2f \\ t^2/2f \end{bmatrix}\\
C(t) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T, \quad t \in (0,1)
\end{aligned} \tag{15}$$

where f is a positive scalar time function. Application of a similarity transformation $T(t)$ given by

$$\begin{aligned}
T(t) &= \begin{bmatrix} 0 & 0 & 1 \\ 2 & -t & 0 \\ 0 & 1 & -t \end{bmatrix}\\
T^{-1}(t) &= \begin{bmatrix} t^2/2 & 1/2 & t/2 \\ t & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{16}$$

gives

$$\begin{aligned}
A'(t) &= T(t)A(t)T^{-1}(t) + \dot{T}(t)T^{-1}(t)\\
&= \begin{bmatrix} -t^4/4f & -t^2/4f & -t^3/4f \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\\
B'(t) &= T(t)B(t) = \begin{bmatrix} t^2/2f \\ 0 \\ 0 \end{bmatrix}\\
C'(t) &= C(t)T^{-1}(t) = \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}^T
\end{aligned} \tag{17}$$

Although the shape of (17) suggests that it is a Kalman decomposition of the system (15) with $n_a = 0, n_b = 1, n_c = 0, n_d = 2$ one has to take care in drawing this conclusion as was clearly pointed out in [15]. Computation of any of the five Kalman decompositions presented in this note confirms that the similarity transformation (16) indeed produces a Kalman decomposition (17) as stated in [12].

Remark 6: The interesting feature of Example 4 is that although f can be any nonsmooth, discontinuous positive time function the system has a constant structure. This reveals that 1) constant rank systems [8] are not the only systems with a constant structure over their entire time-domain and, therefore, 2) PCR systems are not the only systems that have a piecewise constant structure over their entire time domain. On the other hand for practical purposes the class of PCR systems seems broad enough. If for example f is piecewise smooth the system (15) has a PCR equivalent. Interestingly in practice this PCR equivalent will *not* be detected by the d-Kalman decomposition because there are no *changes* of the system structure despite the fact that f is non smooth. On the other hand *at* the isolated times where f is nonsmooth the d-Kalman decomposition may not formally exist. For practical purposes all this is irrelevant since the temporal linear system structure is still described and detected correctly *almost everywhere* over the systems time domain. This indicates that from the sole point of temporal linear system structure, considering PCR systems and the associated d-Kalman decomposition computed from W_t, M_t is somewhat restrictive theoretically. For practical purposes, this seems irrelevant.

VI. Conclusion

One important practical and theoretical contribution of this note concerns the recognition, description and detection of the *temporal* linear system structure. This structure is associated with and also reveals immediately temporal unreachability/uncontrollability as well as temporal unobservability/unreconstructability. This is highly relevant information to systems and control engineers. Intuitively one easily imagines the temporal loss of reachability, controllability, observability and reconstructability. Interestingly to analyze them properly we have to extend the description of linear systems beyond the conventional ones. This led to the introduction of piecewise constant rank systems (PCR systems) that appear to be new. The other important contribution of this note is to show that these PCR systems admit time-varying state dimensions that provide the well rounded realization theory of continuous time-varying linear systems suggested already by Kalman [4].

PCR systems have a structure and dimensions that may change *instantaneously* at *isolated* time-instants. In between those isolated time instants both remain constant. In defining continuous-time systems, PCR systems appear to be the limit after which only highly pathological systems are obtained. In defining PCR systems the authors have struggled with the decision whether or not to include the isolated times $t_i, i = 0, 1, \ldots, N$ in the systems time domain. The advantage of not doing so is that on every part of the time domain the system behaves almost like a constant rank system and inherits its constant structure and well-known properties. On the other hand, ordinary linear systems are not *formally* a subset of PCR systems in this case, as is clearly illustrated by Fig. 1. One may include the isolated times $t_i, i = 0, 1, \ldots, N$, e.g., by defining the system over semi open intervals that connect. In that case, the theory and results related to these systems contain many exceptions and additions associated to the nonsmoothness at $t_i, i = 1, 2, \ldots, N-1$ and boundary conditions at t_0, t_N. Maybe this issue partly explains why continuous-time systems with variable structure and dimensions have been very much ignored. Since the issue concerns isolated time instants only it seems irrelevant for practical purposes. Clearly, PCR systems are open to further investigation.

A practical computational issue that remains concerns the computation of derivatives of the system matrices needed to compute the differential Kalman decomposition. If analytic expressions of the system matrices are unavailable samples of the system matrices may be taken in a sufficiently small time interval and numerical integration may then be employed to approximate the differential Kalman decomposition. Numerical issues related to this computation are currently under study.

Developments in discrete-time that parallel the ones in this note will be published elsewhere. Because discrete-time is not dense some interesting and remarkable differences are obtained.

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Nonlinear Stochastic H_2/H_∞ Control with (x, u, v)-Dependent Noise: Infinite Horizon Case

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***Abstract*—For general nonlinear stochastic Ito systems with state, control and external disturbance-dependent noise, the infinite horizon mixed H_2/H_∞ control is studied. It turns out that in order to solve the H_2/H_∞ control, one only needs to solve four cross-coupled Hamilton–Jacobi equations. A necessary condition and a sufficient condition for the existence of the solution of H_2/H_∞ control is respectively presented, which extends the previous results to more general models.**

***Index Terms*—Nonlinear systems, stochastic systems, H_2/H_∞ control, Hamilton–Jacobi equation, Nash equilibrium point.**

I. INTRODUCTION

It is well known that since the fundamental work [1] appeared, robust H_∞ control has become a popular research field and attracted many researchers' attentions in the last two decades. Among much of literature, interested reader can be referred to the monographs [2], [3]. H_∞ control is to seek a suitable control u^*_∞ to attenuate efficiently the exogenous disturbance below a given attenuation level $\gamma > 0$. If, in addition, the aforementioned u^*_∞ is also to minimize a penalty output when the worst case disturbance v^*_∞ is considered, then it is an objective of mixed H_2/H_∞ control. Mixed H_2/H_∞ control is more appealing in engineering practice, because not only H_∞, but also H_2 performance is considered simultaneously, and hence has been studied extensively; see [4], [8]–[10], [14], [19], [20], and [21] for the study of H_2/H_∞ control of deterministic systems or stochastic systems with additive white noise. However, it can be observed that many problems associated with H_2/H_∞ control of stochastic Ito systems are still open and deserve further study.

A very elaborate introduction for stochastic H_∞ control of linear Ito systems has been presented by [5] and [12] in recent years. [7] first generalized the results of deterministic H_2/H_∞ control [4] to linear stochastic Ito systems with only state-dependent noise, where a necessary and sufficient condition for the existence of finite and infinite horizon stochastic H_2/H_∞ control was respectively given in terms of two coupled generalized differential Riccati equations (GDREs) and generalized algebraic Riccati equations (GAREs). [15] generalized partial results of [7] to general linear stochastic systems. As known in practice, for most natural phenomena described by stochastic Ito systems, not only is it nonlinear, but also are the state, control input and external disturbance corrupted by noise. Hence, it is desirable to consider general nonlinear stochastic H_2/H_∞ control with all system state, control input and external disturbance-dependent noise ((x, u, v)-dependent noise for short). [11] studied finite and infinite horizon nonlinear stochastic H_∞ control with state and

Manuscript received November 1, 2005; revised January 24, 2007 and October 17, 2007. Published August 27, 2008 (projected). Recommended by Associate Editor I.-J. Wang. This work was supported by the National Natural Science Foundation of China under Project 60474013 and by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China, under Project CityU 112806.

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Digital Object Identifier 10.1109/TAC.2008.921024