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Discussion on: "UDU Factored Discrete-Time Lyapunov Recursions and Optimal Reduced-Order LQG Problems"

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A method has been proposed for designing both finite-horizon and infinite-horizon discrete-time reduced order LQG controls. The method solves the two recursive Lyapunov equations by UDU factorization, which is assumed to be more efficient than homotopy and other methods previously proposed. This method can be computationally attractive; however, its presentation also can make it inaccessible to many readers. The purpose of this note is to present a summary of these results in a reader friendly form so that the material can be accessed by many readers.

1. Introduction

A method for solving discrete time optimal reduced order linear quadratic Gaussian (LQG) control problems by recursion of UDU factorized Lyapunov equations is proposed in [7]. This method implements the results of [6] that modified the optimal projection method of [1] for the optimal reduced order LQG solution to guarantee optimality. Developments in [7] are based on an assumption that the plant has a varying dimension, hence the paper took additional effort to ensure that the variability of the plant dimension is included in all equations, which complicated the presentation. Although it may have theoretical merits, the variability of the plant dimension does not have to be considered for most applications. The purpose of this note is to present the major results of [7] for the finite horizon problem in a simplified format that does include dimensional variability of the plant. This reduces the number of variables and simplifies the method for practical applications. Additionally, a number of issues that were not explicitly or properly presented in [7] will be covered to ease the understanding of the method. There will be no comment on the results of [6] against those in [1]. Most of the symbols used in this note are inherited from [7].

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2. The Finite Horizon Reduced Order Optimal LQG Control Problem

The method of [7] is based on the linear time varying discrete time system

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i + v_i, \tag{1}$$

$$y_i = C_i x_i + w_i, \tag{2}$$

with $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R}^l$, and $i = 0, 1, 2, \ldots, N-1$; the signals v_i and w_i are system and observation white noises of zero mean and respective covariances $V_i \geq 0$ and $W_i > 0$ with cross covariance Y_i such that $\begin{bmatrix} V_i & Y_i \\ Y_i^T & W_i \end{bmatrix} \geq 0$. The primary problem of finding a reduced order controller

$$\widehat{x}_{i+1} = F_i \widehat{x}_i + K_i y_i, \tag{3}$$

$$u_i = -L_i \widehat{x}_i, \tag{4}$$

(with $\hat{x}_i \in \mathbb{R}^{n_{i_c}}$ and $n_{i_c} \leq n$) that minimizes the quadratic cost function

$$J = x_N^{\mathrm{T}} Q_N x_N + \sum_{i=0}^{N-1} \begin{bmatrix} x_i^{\mathrm{T}} & u_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q_i & M_i \\ M_i^T & R_i \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}, \quad (5)$$

where x_0 and x_N are known with $Q_i(=Q_i^{\mathrm{T}} \geq 0) \in \mathbb{R}^{n \times n}$, $R_i(=R_i^{\mathrm{T}} > 0) \in \mathbb{R}^{m \times m}$, and $M_i \in \mathbb{R}^{n \times m}$ chosen such that $\begin{bmatrix} Q_i & M_i \\ M_i^T & R_i \end{bmatrix} \geq 0$ has been solved in [6] based on the closed loop system

$$x'_{i+1} = \Phi'_i x'_i + v'_i, \quad (6)$$

with cost function

$$J' = x_N^{\mathrm{T}} Q_N x_N + \sum_{i=0}^{N-1} x_i'^{\mathrm{T}} Q'_i x'_i, \quad (7)$$

where

$$x'_i = \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix}, \quad \Phi'_i = \begin{bmatrix} \Phi_i & -\Gamma_i L_i \\ K_i C_i & F_i \end{bmatrix},$$
$$Q'_i = \begin{bmatrix} Q_i & -M_i L_i \\ -L_i^{\mathrm{T}} M_i^{\mathrm{T}} & L_i^{\mathrm{T}} R_i L_i \end{bmatrix}, \quad v'_i = \begin{bmatrix} v_i \\ K_i w_i \end{bmatrix}. \quad (8)$$

The desired controller matrices F_i, K_i and L_i are computed as

$$F_i = H_{i+1}[\Phi_i - K_i^0 C_i - \Gamma_i L_i^0] G_i^{\mathrm{T}}, \quad (9)$$

$$K_i = H_{i+1} K_i^0, \quad (10)$$

$$L_i = L_i^0 G_i^{\mathrm{T}}. \quad (11)$$

In this solution, the predetermined dimension n_{i_c} of the controller may be made to vary with time. The design problem addressed by [7] is that of computing H_{i+1}, K_i^0, G_i and L_i^0 that are used in the above controller equations. For $i = 1, 2, \ldots, N-1$, the covariance matrix P'_i of the closed loop state x'_i satisfies the Lyapunov recursion

$$P'_{i+1} = x'_{i+1} x_{i+1}'^{\mathrm{T}} = \Phi'_i P'_i \Phi_i'^{\mathrm{T}} + V'_i, \quad (12)$$

where

$$V'_i = \begin{bmatrix} V_i & Y_i K_i^{\mathrm{T}} \\ K_i Y_i^{\mathrm{T}} & K_i W_i K_i^{\mathrm{T}} \end{bmatrix}, \quad (13)$$

and the cost function can be written as

$$J' = \mathrm{trace}(P_N Q_N) + \sum_{i=0}^{N-1} \mathrm{trace}(P'_i Q'_i). \quad (14)$$

By using the Lagrange multiplier method, the minimization of (14) over P'_i subject to (12) is a two point boundary value problem that yields a backward Lyapunov recursion

$$S'_i = \Phi_i'^{\mathrm{T}} S'_{i+1} \Phi'_i + Q'_i, \quad (15)$$

where S'_i is an appropriate matrix of Lagrange multipliers. It is assumed that for $i = 0, 1, 2, \ldots, N$, matrices P'_i and S'_i can be compatibly partitioned as

$$P'_i = \begin{bmatrix} P_i^1 & P_i^{12} \\ P_i^{12^{\mathrm{T}}} & P_i^2 \end{bmatrix}, \quad S'_i = \begin{bmatrix} S_i^1 & S_i^{12} \\ S_i^{12^{\mathrm{T}}} & S_i^2 \end{bmatrix}, \quad (16)$$

so that H_i, and G_i are computed as

$$G_i = P_i^{2^+} P_i^{12^{\mathrm{T}}}, \quad (17)$$

$$H_i = -S_i^{2^+} S_i^{12^{\mathrm{T}}}, \quad (18)$$

where for any matrix A, the notation A^+ denotes its Moore–Penrose pseudo inverse. Additionally, by defining $P_i \triangleq P_i^1 - P_i^{12} P_i^{2^+} P_i^{12^{\mathrm{T}}}$, and $S_i \triangleq S_i^1 - S_i^{12} S_i^{2^+} S_i^{12^{\mathrm{T}}}$ for $i = 1, 2, \ldots, N-1$, the matrices K_i^0, and L_i^0 are determined as:

$$K_i^0 = (\Phi_i P_i C_i^{\mathrm{T}} + Y_i)(C_i P_i C_i^{\mathrm{T}} + W_i)^{-1}, \quad (19)$$

$$L_i^0 = (\Gamma_i^{\mathrm{T}} S_{i+1} \Gamma_i + R_i)^{-1} (\Gamma_i^{\mathrm{T}} S_{i+1} \Phi_i + M_i^{\mathrm{T}}). \quad (20)$$

It is clear that the solution to reduced order optimal LQG control problem depend largely on the solution of the Lyapunov recursions (12) and (15). Since these equations also depend on Φ'_i, which, as shown in (8), contains the sought controller parameters F_i, K_i and L_i, this interdependence is what makes (12) and (15) being referred to as nonlinearly coupled Lyapunov equations [7]. It makes the problem of finite horizon discrete time reduced order optimal LQG design problem hard to solve; note that Eq. (12) runs forward in time and (15) runs backward in time.

3. The Proposed UDU Factorization Approach

Although the proposed UDU factorization approach offers better numerical stability, its main advantage is on reducing the computational load. Note that the computations of each of the matrices G_i, H_i, K_i^0, and L_i^0 in (17), (18), (19) and (20), respectively require some form of matrix inversion, which can be computationally demanding depending on the structure of involved matrices. The proposed method reduces the computational load by inverting matrices whose structure require minimal inversion effort; this can be open for debate especially after considering the effort involved in performing UDU factorization.

UDU factorization methods for solving Lyapunov equations have been in existence for more than two decades (see, for instance [2,4,5]). Recall that any symmetric positive definite matrix X can be factorized into unit upper triangular and positive diagonal factors as

$$X = U_X D_X U_X^{\mathrm{T}}, \tag{21}$$

where U_X is unit upper triangular and D_X is positive definite diagonal. Therefore, since P'_i, and S'_i are symmetric positive definite, they can also be factorized as

$$P'_i = \begin{bmatrix} U^1_{P'_i} & U^{12}_{P'_i} \\ 0 & U^2_{P'_i} \end{bmatrix} \begin{bmatrix} D^1_{P'_i} & 0 \\ 0 & D^2_{P'_i} \end{bmatrix} \begin{bmatrix} U^1_{P'_i} & U^{12}_{P'_i} \\ 0 & U^2_{P'_i} \end{bmatrix}^{\mathrm{T}}, \tag{22}$$

$$S'_i = \begin{bmatrix} U^1_{S'_i} & U^{12}_{S'_i} \\ 0 & U^2_{S'_i} \end{bmatrix} \begin{bmatrix} D^1_{S'_i} & 0 \\ 0 & D^2_{S'_i} \end{bmatrix} \begin{bmatrix} U^1_{S'_i} & U^{12}_{S'_i} \\ 0 & U^2_{S'_i} \end{bmatrix}^{\mathrm{T}}. \tag{23}$$

First, Van Willigenburg and De Koning [7] use these $U-D$ factors to 'efficiently' solve the Lyapunov equations (12) and (15). This approach was first proposed by Lupash [5], that if $P'_i = U_{P'_i} D_{P'_i} U^{\mathrm{T}}_{P'_i}$ and $V'_i = U_{V'_i} D_{V'_i} U^{\mathrm{T}}_{V'_i}$ then (12) can be expressed as

$$\begin{aligned} P'_{i+1} &= U_{P'_{i+1}} D_{P'_{i+1}} U^{\mathrm{T}}_{P'_{i+1}} \\ &= U_{P'_i} D_{P'_i} U^{\mathrm{T}}_{P'_i} + U_{V'_i} D_{V'_i} U^{\mathrm{T}}_{V'_i}, \end{aligned} \tag{24}$$

from which a rank one factorization algorithm such as the Agee–Turner algorithm of [3] can be used to 'efficiently' solve the UDU factors $U_{P'_{i+1}}$ and $D_{P'_{i+1}}$ for the next covariance matrix P'_{i+1}. A similar approach is carried out for the matrix S'_i in (15).

Because of the presence of zeros and ones in the UDU factors, Van Willigenburg and De Koning [7] propose that matrices G_i, and H_i in (17) and (18) be computed with less effort using the $U-D$ factors as

$$G_i = U^{2^{-\mathrm{T}}}_{P'_i} U^{12^{\mathrm{T}}}_{P'_i}, \tag{25}$$

$$H_i = -U^{2^{-\mathrm{T}}}_{S'_i} U^{12^{\mathrm{T}}}_{S'_i}. \tag{26}$$

Matrices K^0_i, and L^0_i in (19), and (20) are computed using the same defining formulas but with P_i and S_{i+1} computed by

$$P_i = U^1_{P'_i} D^1_{P'_i} U^{1^{\mathrm{T}}}_{P'_i} \tag{27}$$

$$S_{i+1} = U^1_{S'_{i+1}} D^1_{S'_{i+1}} U^{1^{\mathrm{T}}}_{S'_{i+1}}. \tag{28}$$

The UDU factorization of the terms $(C_i P_i C_i^{\mathrm{T}} + W_i)$, $(\Gamma_i^{\mathrm{T}} S_i \Gamma_i^{\mathrm{T}} + R_i)$ and their inverses as proposed by [7] is an unnecessary process, since it does not provide any advantage.

4. Reader Observations

Despite the mix up of symbols and cluttering of the presentation by the need to accommodate the variability of the plant dimension, Van Willigenburg and De Koning [7] make a good attempt towards an efficient method for computing a solution to the reduced order optimal LQG control problem. However, it misses two main points that would have assisted in the application of the proposed approach. First, according to Lupash [5] the UDU factorization in (24) by rank one factorization algorithm works when Φ'_i is an upper triangular matrix. It is imperative that Van Willigenburg and De Koning [7] should have addressed how this triangularity requirement is handled, otherwise the accuracy of the method may be questioned. Second, the iteration in (15) runs backwards and the final value of S'_N is stated to be $\begin{bmatrix} Q_N & 0 \\ 0 & 0 \end{bmatrix}$ while the initial value of Φ'_0 is assumed known through controller initialization. It is unfortunate that Van Willigenburg and De Koning [7] did not clearly discuss how the S'_i needed in the subsequent iterations are computed especially S'_0 and S'_1. This is particularly important since P'_0 depends on both the initial state x_0 of the system and S'_0, which is used in estimating the initial controller state $\hat{x}_0$ through H_0 and the mean value of the plant state $\bar{x}_0$.

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Discussion on: "UDU Factored Discrete-time Lyapunov Recursions Solve Optimal Reduced-order LQG Problems"

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Practical modern control design and implementation requires the following four factors to be taken into consideration:

1. *Robustness*. The control design must be insensitive to both parametric errors and unmodeled dynamics in the design plant.
2. *Constraints on the compensator order*. Because of limited control processor throughput, there is an inherent implementation constraint on the order of the compensator. In addition, reduced-order compensators are desired because they are easier to analyze.
3. *Additional constraints on the compensator architecture*. All design constraints are not captured by modern cost functions (in particular H_2, H_∞, L_1 or ℓ_1 cost functions). Some of these additional constraints, e.g., the need for an integrator in the controller or the need for a decentralized control structure, place additional constraints on the control architecture.
4. *Digital implementation*. Almost all modern controllers will be implemented in a digital processor. In addition, it is very common to obtain the design plant through digital system identification, which naturally results in a discrete-time design plant. Hence, the need for digital implementation can be accommodated by designing a discrete-time compensator using a discrete-time representation of the plant.

The authors paper addresses two of these four fundamental factors (i.e. factors 2 and 4).

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The authors work is part of the evolution of algorithms for reduced-order control that were originally based on "optimal projection equations," which are coupled Riccati and Lyapunov equations that are derived from the first order necessary conditions that characterize the H_2-optimal reduced-order controller [1]. The initial focus of these algorithms was on the design of continuous-time controllers and used homotopy algorithms. However, it was shown in [11] that the discrete-time problem can be solved by iterating a "strengthened" set of optimal projections forward and backward in time. These algorithms have been shown to be much more efficient than the earlier homotopy algorithms. This paper develops an algorithm for discrete-time, H_2-optimal reduced-order control that is not based on optimal projection equations, but on a pair of non-linearly coupled recursive discrete-time Lyapunov equations that also characterize the optimal reduced-order controller. The algorithm is seen to outperform that of [11] when the dimension of the optimal reduced-order controller is small compared to that of the design plant.

The authors' algorithm is an important contribution when viewed from the perspective of algorithms based on the optimal projection equations. However, it is unclear whether this approach to reduced-order controller design has the ability to adequately take into account robustness aspects or additional controller constraints as eluded to in factors 1 and 3 above. Optimal projection equations have been developed for reduced-order controllers based on earlier forms of robustness theory [2,3,9]. However, these equations are very complex and are more difficult to solve. To my knowledge, optimal projection equations have not been developed for the much less conservative robustness theory based on mixed structured singular value (MSSV) and multiplier theories. These theories also require that the parameters of a multiplier be designed (see, e.g. [4]), which does not fit nicely into the optimal projection framework. In addition, it seems very difficult to incorporate additional controller constraints in this framework.

An obvious means of designing reduced-order controllers is to directly choose the controller parameters using some type of optimization algorithm. The authors elude to the perceived weaknesses of this approach in the second paragraph of the introduction when they say: "Unless the dimensions of the compensator are very small, this method becomes infeasible due to the large number of parameters, and the non-linear nature of the optimisation." Here it is assumed that the authors are referring to the computational intensity of the parameter optimization approaches and the tendency to find local minima instead of a global minimum. These problems were the original motivation of the search for algorithms for reduced-order controller synthesis based on the optimal projection equations. However, although I was one of the original developers of numerical algorithms for the optimal projection equations, I am no longer convinced that parameter optimization is not the correct approach.

One of the advantages of parameter optimization is that it has the ability to be used in conjunction with robustness theory, including the more complex robustness theory based on MSSV and multiplier theories. An example of such an algorithm is given in [4]. In addition, this approach can easily place additional constraints on the controller architecture. An example of the design of a robust, multivariable PI controller is given in [8]. (The same approach is used to design a fixed-architecture estimator in [7,8].) However, none of this discussion addresses the question as to whether these methods are computationally feasible, one of the major concerns expressed by the authors.

First, it should be said that the use of a parameterization of the controller can reduce the size of the problem, although minimal parameterizations can lead to ill-conditioned numerical algorithms and so nonminimal parameterizations may yield better results [5]. It should also be noted that for many practical control problems, for example, those involving multivariable PID control, the number of control parameters is not large. However, the main reason that I am now optimistic about parameter optimization approaches is that computer speed continues to increase and global search algorithms such as real-coded genetic algorithms (see [7,8,10]) have become more efficient, which alleviates the problem of local minima.

Finally, I would like to comment on the paradigm expressed in the first paragraph of the Introduction. The authors express a widely held view that there are two distinct approaches to reduced-order control design: (1) the indirect approach in which a full-order control design is followed by controller reduction, which offers no guarantees of stability or performance, and (2) the direct approach which is based either on solving coupled Riccati and/or Lyapunov equations as in the author's paper or alternatively on parameter optimization. However, this view is not complete. In reality the direct approaches are based on numerical algorithms that benefit from good initial conditions. In fact, the choice of initialization can determine whether or not the algorithm chosen to solve the direct approach actually converges to an acceptable solution [6]. As discussed in [6], the indirect approaches can be used to provide these initial conditions. What would be of interest is to demonstrate how indirect approaches can provide initial conditions for the authors' algorithm.

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Discussion on "UDU Factored Discrete-time Lyapunov Recursions Solve Optimal Reduced-order LQG Problems"

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The paper presents a new method for solving the reduced-order, discrete-time, LQG problem, in its most general setting. The system, the covariance matrices and the cost function may be time-varying, and the latter is in general finite horizon. Furthermore, the controller dimension is bounded but not fixed, and may vary from one time step to another. For this general case, the only existing solution method prior to the new algorithm, is recursive solution of the strengthened discrete-time optimal projection equations (SDOPE), which was suggested by the same authors.

This discussion is aimed at putting the current paper in perspective with the large body of works on numerical solution of order reduction problems, and to highlight some aspects of that problem. Since the new method is unique in its ability to handle the general cases, comparison requires that only time-invariant system and cost function, and infinite horizon, will be considered from now on. Some of the quoted results correspond to continuous-time, but their discrete-time counterparts exist and are similar.

Reduced order optimal control has some features that appear paradoxical. For example, the controllers that should be simpler to use, are much harder to design. The difficulty, similar to the one encountered in model order reduction, assumes different forms depending on the solution method, but it always exists. Also, despite the fact that the number of parameters needed to define the controller is much lower than the total number of entries in its (A_c, B_c, C_c) matrices, nonparametric methods that use that larger number are more efficient.

A classification that is not mentioned in the paper is between methods that are direct optimization algorithms (not to be confused with what the paper calls 'direct methods') and those that depend on the necessary conditions for optimality. That classification is fundamental, but is somewhat blurred by the fact that the equations obtained by the latter approach need to be solved numerically, often via iterations. As an example for an optimization algorithm, consider an n_cth order SISO controller, given by $2n_c$ parameters using standard parameterization. It is straightforward to write the cost in terms of those parameters and to set up a descending gradient algorithm. The difficulty then lies in ensuring stability via the nonlinear Routh conditions or equivalent conditions in the state space. Similar, yet more elaborated method is gradient flow [1], which solves the matrix differential equations

$$\dot{A}_c = -\partial J/\partial A_c, \quad \dot{B}_c = -\partial J/\partial B_c, \\ \dot{C}_c = -\partial J/\partial C_c, \tag{1}$$

where the partial derivative can be calculated analytically via Lyapunov equations. Upon convergence, the algorithm reaches an extremum. Here again the problem is stability, since in the basic scheme the trajectories may move into areas of destabilizing controllers. Stability can be guaranteed by introducing a

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barrier term to the cost function, but the calculation then becomes cumbersome.

In recent years linear matrix inequalities (LMI) were used to solve some L_2 as well as H_∞ problems. The main advantage of that approach is that after the problem is formulated as an LMI, reliable and powerful numerical tools exist for the actual solution. However when applying LMI to order reduction, the results contain, in addition to linear inequalities, a rank condition on a certain matrix, hence are not true LMIs [2]. This is not surprising since LMIs can solve problems defined by either Lyapunov equations or Riccati equations, linearized via Schur transformation. As the equations that define order reduction problems do not fall exactly into these categories, no straightforward method will yield LMIs. Recently, it was suggested to avoid that problem in estimator design and model reduction [3] by enforcing the order condition in a structural way. The system (estimator, model, controller) has full dimension, thus after some algebraic procedures suitable to LMI, but is modeled in the unobservable form

$$A_c = \begin{bmatrix} A_{c1} & 0 \\ A_{c2} & A_{c3} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}, \\ C_c = [A_{c1} \quad 0] \tag{2}$$

As a result, the minimum realization (A_{c1}, B_{c1}, C_{c1}) has the desired reduced order. However, as in all other approaches, the problem can only be shifted but not eliminated, since the optimization with the parameterization (2) was shown in [3] to be realization dependent. An algorithm for monotonically non-increasing iterations of state transformation was given, but in general it converges only to a local minimum.

We turn now to the class of methods that the one suggested in the paper belongs to, namely those that use the first order optimality conditions. Perhaps the main reason for the popularity of the full order LQG control is the well known separation principle, which acts on two levels. First the controller can be split into an observer and a state feedback. Secondly, the optimal state feedback and observer gains are given by two independent Riccati equations. In terms of the general optimal control problem, a two point boundary value problem (TPBVP) was replaced by two one sided problems. Each one of these problems can be solved relatively easily. One solution algorithm that is similar to the approach in the paper is integration (or iteration in the discrete time) of the corresponding Riccati differential equation. Due to separation, that is carried out only forward and only once. Reduced order control, on the other hand, is marked by lack of separation at all levels. This is demonstrated by the optimal projection result from [4] where the controller is given by

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} \Gamma(A - BR_2^{-1}B^{\mathrm{T}}P - QC^{\mathrm{T}}V_2^{-1}C)G^{\mathrm{T}} & \Gamma QC^{\mathrm{T}}V_2^{-1} \\ \hline R_2^{-1}B^{\mathrm{T}}PG^{\mathrm{T}} & 0 \end{array}\right] \tag{3}$$

and the equations that need to be solved are

$$AQ + QA^{\mathrm{T}} + V_1 - QC^{\mathrm{T}}V_2^{-1}CQ \\ + \tau_\perp QC^{\mathrm{T}}V_2^{-1}CQ\tau_\perp^{\mathrm{T}} = 0, \tag{4}$$

$$PA + A^{\mathrm{T}}P + R_1 - PBR_2^{-1}B^{\mathrm{T}}P \\ + \tau_\perp^{\mathrm{T}}PBR_2^{-1}B^{\mathrm{T}}P\tau_\perp = 0, \tag{5}$$

$$(A - BR_2^{-1}B^{\mathrm{T}}P)\hat{Q} + \hat{Q}(A - BR_2^{-1}B^{\mathrm{T}}P)^{\mathrm{T}} \\ + QC^{\mathrm{T}}V_2^{-1}CQ - \tau_\perp QC^{\mathrm{T}}V_2^{-1}CQ\tau_\perp^{\mathrm{T}} = 0, \tag{6}$$

$$(A - QC^{\mathrm{T}}V_2^{-1}C)^{\mathrm{T}}\hat{P} + \hat{P}(A - QC^{\mathrm{T}}V_2^{-1}C) \\ + PBR_2^{-1}B^{\mathrm{T}}P - \tau_\perp^{\mathrm{T}}PBR_2^{-1}B^{\mathrm{T}}P\tau_\perp = 0, \tag{7}$$

$$\mathrm{rank}(\hat{Q}) = \mathrm{rank}(\hat{P}) = \mathrm{rank}(\hat{Q}\hat{P}) = n_c, \tag{8}$$

$$\hat{Q}\hat{P} = G^{\mathrm{T}}M\Gamma, \quad \Gamma G^{\mathrm{T}} = I_{n_c}, \\ \tau = G^{\mathrm{T}}\Gamma, \quad \tau_\perp = I_n - \tau. \tag{9}$$

R_1, R_2, V_1 and V_2 are the state and control weighting matrices, and are the process and measurement noise intensities respectively. Denoting $L = QC^{\mathrm{T}}V_2^{-1}$, $F = R_2^{-1}B^{\mathrm{T}}P$, the controller appears to be the result of a projection order reduction applied to $(A\text{-}BF\text{-}KC, L, F)$ which is the standard form of an LQG controller. However L and F are different than those of the full order problem, and (6)–(9) are not exactly as in the order reduction problem for $(A\text{-}BF\text{-}KC, L, F)$. Hence, on the higher level, there is no separation between optimal control and order reduction.

Equations (4)–(9) have a clear and relatively compact structure, can provide insight on the problem and are independent of the controller realization. Unfortunately all this nice properties do not lead to any efficient method of solution as in the standard Riccati equation. The problems in solving this set of equations can be identified as the following:

(1) All the equations are coupled,
(2) Even separately, that is, fixing all other variables, neither one of the equations has a clear and

efficient solution algorithm. For example, the last term in (4), even with given τ, spoils the structure needed for Riccati equation solvers.

(3) The rank condition is hard to enforce.

The structural properties of (4)–(9) are destroyed when it comes to homotopy solutions [5,6] since enforcing the rank condition requires using Γ and G^{T} instead of τ. Equations (4)–(9) were derived by algebraic manipulations on the first order necessary optimality conditions. In previous works by Van Willington and De Koning, they applied an iterative scheme to solve SDOPE which is similar to (4)–(9). Since the compact structure does not seem to provide any advantage in the numerical solution phase, the authors chose in the current paper to base the iterative numerical scheme on an earlier stage in the development. At that stage the solution is given in terms of Lyapunov equations that are linear, easy to iterate, and suitable for the UDU factorisation. The paper maintains, and shows by means of an array of simulations, that this factorisation increases the numerical efficiency and accuracy. In a sense, an implicit conclusion of the paper is that a highly structured, compact, analytical form of a set of equations is not necessarily better from a numerical solution point of view.

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Discussion on: "UDU Factored Discrete-time Lyapunov Recursions Solve Optimal Reduced-order LQG Problems"

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The paper presents some interesting results on the direct design of optimal reduced-order linear quadratic Gauassian (LQG) controllers for time-invariant and time-varying discrete-time systems. In particular, in earlier work, the authors have developed the strengthened discrete time optimal projection equations (SDOPE) and then applied an original approach to solving the first order necessary conditions by using forward and backward in time recursion of the discrete-time equations. The present paper investigates a similar recursive forward/backward recursion approach, but this time applied to the necessary conditions in the form of two nonlinear, coupled Lyapunov equations, each of the dimension of the closed-loop system. The solution method set forth is particularly intriguing in that it appears to be simpler and more efficient than the homotopy methods proposed for this problem. In this discussion paper, I shall not attempt to offer corroboration of these results nor indulge in any criticism of the authors' excellent work. Instead, I should like to offer some perspectives, conjectures and observations on open problems from the point of view of one who helped originate the optimal projection equations (OPE). Many of the following comments pertain nearly equally well to both continuous-time and discrete-time settings and thus I shall phrase my specific remarks using the continuous time OPE since the notation is less complex.

The authors quickly fix attention on minimal compensators, present some interesting bounds on the dimension of a minimal compensator in Eq. (4.1) and, in Theorem 1, give the first-order necessary conditions for the minimal, optimal reduced-order compensator in terms of the solutions of the equations of the second moment of the closed-loop system and the dual closed-loop system. While they observe that "minimal compensators are the interesting ones", I would venture to say that this is not always the case. First, let me point out that if N, the time horizon, is large, the bounds on n_i^c are not particularly tight. In any case, our original and rather urgent motivation for developing the OPE was the need for low-order compensators for very high-order plants in which even the minimal optimal compensator would be too large. The immediate application was structural vibration control, that is, control of essentially infinite dimensional plants. Some previous work in this area showed that minimal compensators could be readily computed via well established model reduction algorithms applied to full order LQG solutions. Further, in several design problems of representative complexity, no minimal compensators of order less than full could be found to exist. Thus, from the outset, our work has sought very low-order, sub-minimal compensators. Of course, such compensators are suboptimal in comparison with minimal compensators. They are optimal *under the constraint of the fixed order of the compensation.* I am reminded of a famous dictum of David Hilbert: "On the imposition of constraints, the value of a minimum never decreases". In exploring this problem, it was our intention to characterize *the tradeoff between reduction of compensator complexity and suboptimality of performance.*

Within the above specialization, which we have observed may not be the only circumstance of interest, the authors set forward an efficient UDU factored numerical algorithm for solution of the second moment equations by repeated forward and backward recursion. These seem very well motivated and enlist our approbation and endorsement. Their discussion of the numerical damping approach in Eqs (18.1) and (18.2) should be similarly well received by those who have struggled with the reduced-order compensation problem. Perhaps as a further step, the authors may be encouraged to undertake a proof of the convergence of the algorithm.

At this point, let us cast our gaze over issues that lie beyond the specific results of the present paper and touch upon several broader issues. In particular, let me expostulate on the insights afforded by the OPE, the possibility of yet exploiting the structure of the OPE for efficient numerical algorithms, the multiplicity of solutions, the estimation of their number and, finally, the exigencies of very low-order, sub-minimal compensation of very high order plants. Again, because of the relative simplicity of notation, I shall illustrate my remarks in the context of the continuous-time, infinite horizon reduced-order compensation problem for linear, time-invariant plants.

I recall that the most tantalizing insight provided by the OPE was a clear relation between full-order (or minimal) compensation and reduced-order (and vastly sub-minimal) compensation. After considerable labor, my colleagues and I managed to express the first-order necessary conditions in several elegant (so we suppose) forms. The one that most closely links reduced- to full-order compensation is [1,2]:

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\overline{\Sigma}Q + \tau_\perp Q\overline{\Sigma}Q\tau_\perp^{\mathrm{T}}, \quad (1a)$$

$$0 = A^{\mathrm{T}}P + PA + R_1 - P\Sigma P + \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \quad (1b)$$

$$\begin{aligned} 0 = {} & (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} \\ & + Q\overline{\Sigma}Q - \tau_\perp Q\overline{\Sigma}Q\tau_\perp^{\mathrm{T}}, \end{aligned} \quad (1c)$$

$$\begin{aligned} 0 = {} & \left(A - Q\overline{\Sigma}\right)^{\mathrm{T}}\hat{P} + \hat{P}\left(A - Q\overline{\Sigma}\right) \\ & + P\Sigma P - \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \end{aligned} \quad (1d)$$

$$\tau = \sum_{i=1}^{n_c} \Pi_i[\hat{Q}\hat{P}], \quad (1e)$$

where $\tau_\perp \triangleq I_n - \tau$, n is the plant dimension and $\Pi_i(\ldots)$ denotes the ith eigenprojection of the indicated argument, that is, the dyadic formed from the left- and right-eigenvectors of $(\ldots)$. Clearly, these are generalizations of the full-order LQG equations. In the full order case, since the projection τ is I_n, $\tau_\perp = 0_{n\times n}$ and the terms involving $\tau_\perp$ vanish, leaving the two uncoupled Riccati equations and two uncoupled model-reduction equations which become, in this case, superfluous.

In this form, the OPE have inspired many relaxation-type solution algorithms. Most obvious is the procedure that begins with $\tau_\perp = 0_{n\times n}$, solves the full-order LQG Riccati equations and the model reduction equations [3], forms $\tau = \sum_{i=1}^{n_c} \Pi_i[\hat{Q}\hat{P}]$, where the eigen-projections associated with the n_c largest eigenvalues of $\hat{Q}\hat{P}$ are retained, and then iterates until τ converges. Such procedures, while often yielding very useful results, cannot be guaranteed to converge. Moreover, this entire class of approaches to the solution of the OPE has as its starting point the solution of the highest dimensional problem. In various practical problems ones faces the solution of full-order compensation for plants having several hundreds of states.

The above approach, while it seems a logical extension to modern control theory, utterly reverses the process followed by practical control designers. That is, regardless of the dimensionality of the plant, one typically begins by seeking a very low order compensator design via classical design techniques, inspiration and educated guesswork. Then one evaluates performance and if this is unacceptable, seeks to design a compensator of increased dimension. Eventually one finds a region of diminishing returns: The time and cost of designing a higher-order compensator are inadequately requited by the expected performance improvement. At this point, one stops, having expended the minimum of computation and labor. My personal hope was that OPE or *any* optimal reduced-order compensation theory could prove its worth by enabling us to deal reliably with multi-input, multi-output systems while guaranteeing the best possible performance and minimizing the necessary computational burden.

Existing computational techniques that exploit the structure of the OPE have resulted in many results of practical applicability, but the above hope has never been fully realized. The author of this discussion paper, although he has been diverted from the study of this problem by a lengthy series of distractions over the last decade, ventures to speculate that a truly satisfactory achievement of reliable, optimal design might spring from consideration of the following form of the OPE [1]:

$$0 = (A - \tau Q\bar{\Sigma})Q + Q(A - \tau Q\bar{\Sigma})^{\mathrm{T}} + V_1 + \tau Q\bar{\Sigma}Q\tau^{\mathrm{T}}, \tag{2a}$$

$$0 = (A - \Sigma P\tau)^{\mathrm{T}}P + P(A - \Sigma P\tau) + R_1 + \tau^{\mathrm{T}}P\Sigma P\tau, \tag{2b}$$

$$0 = \tau[(A - \Sigma P\tau)\hat{Q} + \hat{Q}(A - \Sigma P\tau)^{\mathrm{T}} + Q\bar{\Sigma}Q], \tag{2c}$$

$$0 = [(A - \tau Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - \tau Q\bar{\Sigma}) + P\Sigma P]\tau, \tag{2d}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c. \tag{2e}$$

Note that in this form, the dynamical matrices $A - \tau Q\bar{\Sigma}$ and $A - \Sigma P\tau$ represent the *reduced order* observer and regulator dynamics, respectively. Moreover, in the basis in which $\tau = \begin{pmatrix} I_{n_c} & 0 \\ 0 & 0 \end{pmatrix}$ and $\hat{Q} = \begin{pmatrix} \hat{Q}_1 & \hat{Q}_{12} \\ \hat{Q}_{12}^{\mathrm{T}} & \hat{Q}_2 \end{pmatrix}$, $\hat{P} = \begin{pmatrix} \hat{P}_1 & \hat{P}_{12} \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_2 \end{pmatrix}$, the third and fourth equations involve primarily $\hat{Q}_1$, $\hat{Q}_{12}$, $\hat{P}_1$, and $\hat{P}_{12}^T$. These are all low-dimensional quantities.

Equations (2a)–(2e) suggest an entirely different set of relaxation techniques. For example, suppose we start the solution algorithm with $\tau = \gamma\begin{pmatrix} I_{n_c} & 0 \\ 0 & 0 \end{pmatrix}$, where $\gamma \ll 1$. In other words, our point of departure is the open-loop system. Let us progressively increment γ. Suppose that at some given stage, we view the equations in the basis in which $\tau = \gamma\begin{pmatrix} I_{n_c} & 0 \\ 0 & 0 \end{pmatrix}$, and in this basis, let:

$$\hat{Q} = \begin{pmatrix} \hat{Q}_1 & \hat{Q}_{12} \\ \hat{Q}_{12}^{\mathrm{T}} & \hat{Q}_2 \end{pmatrix} \quad \text{and} \quad \hat{P} = \begin{pmatrix} \hat{P}_1 & \hat{P}_{12} \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_2 \end{pmatrix}, \tag{3a}$$

$$A - \tau Q\bar{\Sigma} \triangleq A_Q = \begin{pmatrix} A_{Q1} & A_{Q12} \\ A_{Q21} & A_{Q2} \end{pmatrix}, \tag{3b}$$

$$A - \Sigma P\tau \triangleq A_P = \begin{pmatrix} A_{P1} & A_{P12} \\ A_{P21} & A_{P2} \end{pmatrix}, \tag{3c}$$

etc. Then (2c) and (2d) reduce to:

$$0 = A_{P1}[\hat{Q}_1\ \hat{Q}_{12}] + [\hat{Q}_1\ \hat{Q}_{12}]\begin{pmatrix} A_{P1} & A_{P12} \\ A_{P21} & A_{P2} \end{pmatrix}^{\mathrm{T}} + [(Q\bar{\Sigma}Q)_1 (Q\bar{\Sigma}Q)_{12}], \tag{4a}$$

$$0 = A_{Q1}^{\mathrm{T}}[\hat{P}_1\ \hat{P}_{12}] + [\hat{P}_1 \hat{P}_{12}]\begin{pmatrix} A_{Q_1} & A_{Q_{12}} \\ A_{Q_{21}} & A_{Q_2} \end{pmatrix} + [(P\Sigma P)_1 (P\Sigma P)_{12}]. \tag{4b}$$

These equations are of dimension $n_c \times n$. Moreover, accepting previous iterate values for the coefficient matrices and the non-homogeneous terms and solving at each iterate for $[\hat{Q}_1\ \hat{Q}_{12}]$, and $[\hat{P}_1\ \hat{P}_{12}]$, we may expect the $\hat{Q}_{12}$, and $\hat{P}_{12}$ terms to be small in comparison with $\hat{Q}_1$, and $\hat{P}_1$. Then except for higher order quantities, the next approximation for $\hat{Q}\hat{P}$ is:

$$\hat{Q}\hat{P} \cong \begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_{12}^{\mathrm{T}} \end{bmatrix} [\hat{P}_1\ \hat{P}_{12}], \tag{5}$$

when we then perform an eigen-decomposition of this matrix, we see that all eigenvalues are zero except for n_c positive eigenvalues (or nonnegative in the case wherein the minimal compensator is of dimension $< n_c$). Also, it is clear that the resulting eigen-projections are simply rotations of the dyadics found in the previous iterate. With the new estimate of the projection, we then iterate through the four equations as before until τ converges.

The above sketch is merely a suggestion for future investigation. The general scheme outlined here has the virtues that it initiates solution of the P and Q equations at the convenient open-loop system configuration and involves solution of the lower order, $n_c \times n$ versions of the $\hat{Q}$ and $\hat{P}$ equations. The dyadics defining τ undergo smooth rotations and the rank of τ is fixed at n_c. The fundamental strategy of this solution process is to start with low gain design and lowest order compensators, compute solutions to well-behaved or low order design equations and, generally, explore the compensator order – performance trade-off starting from the lowest order compensators. Clearly this is more nearly in accord with prevailing (and largely successful) design practice.

A very important and still unresolved issue is the non-convexity of the fixed order compensation problem and the existence of multiple solutions to the first-order necessary conditions. That there can exist multiple solutions is easily seen from the following example. Suppose that the plant is open-loop stable and that $A, \Sigma, \bar{\Sigma}, V_1$, and R_1 are all diagonal, with the last four matrices positive definite. Then it follows that $Q, P, \hat{Q}$, and $\hat{P}$ are also diagonal and the eigen-projections of $\hat{Q}\hat{P}$ are merely the unit coordinate projections of the form:

$$\tau^{(k)} \in \mathbb{R}^{n\times n}, \quad (\tau^{(k)})_{l,m} \triangleq \begin{cases} 1, & l = m = k, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

Referring to (1e), the OPE are satisfied when τ is composed of any n_c distinct unit coordinate projections. Hence the number of admissible solutions is $\binom{n}{n_c}$. In this simple case, one can proceed to evaluate the cost function for each solution and identify the global minimum. But of course, in general, it is practically impossible to carry out such a program.

Much labor has been devoted to improving the above estimate of the number of solutions. However, the only hint of progress has arisen in connection with continuation and homotopy methods [4,5] applied to solution of OPE by Richter, Watson and DeCarlo. In particular, using topological degree theory, Richter derived an upper bound on the number of solutions to the OPE. Let n_u, l, and m denote the dimension of the unstable plant subspace, the number of measurements and the number of controls, respectively. Then the number of solutions for the case $n_c \geq n_u$ is not greater than:

$$\begin{matrix} \binom{\min(n,m,l) - n_u}{n_c - n_u}, & n_c \leq \min(n,m,l), \\ 1, & \text{otherwise.} \end{matrix} \tag{7}$$

This result is consistent with (6) and, in general, provides a much lower bound. However, for open-loop stable plants ($n_u = 0$) with a complex, multivariable control system (large m and l) the number of solutions is still quite large. Moreover, there remains the question of how to steer solution algorithms to converge on the solutions of interest.

Finally, let me offer some comments on the exigencies of very low-order, sub-minimal compensation of very high order plants and the appropriate choice of benchmark problems. Low order example problems are very useful for gaining insight into solution algorithms. On the other hand, reduced order compensation is most needed when the plant dimension (or rather the dimension of the subspace needing compensation) is large. Thus the authors are to be congratulated for their work on the European Journal of Control benchmark problem which displays a very respectable complexity. I would like to encourage investigators of reduced order compensation to continue to explore example problems of representative complexity. In particular, I think there is a need to develop algorithms capable of handling plants with hundreds, not tens of states. The difficulties of such problems are illustrated by the work reported in [6]. Here we experimentally demonstrated vibration suppression in a large, precision structure using two dozen inputs and outputs. There were over 150 vibration modes in the disturbance band. The 300 state plant model posed great difficulties to preliminary LQG design and model reduction, let alone OPE solution. Even at this time, it does not appear that the basic Riccati and Lyapunov solver software is well suited to problems of this dimension. This is unfortunate since this is precisely the sort of problem for which reliable low order compensator design is most needed. Returning to earlier remarks, many of the design algorithms are inspired by the structure of Eqs (1a)–(1e), in which the starting point is a full order LQG design. But this would appear to tackle the problem at the wrong end. It might be that a more suitable point of departure is given by Eqs (2)–(5), where at each iteration we succeed in decomposing the equations into $n_c \times n_c$ and $n_c \times (n - n_c)$ blocks and initiate the design process at small n_c. I believe there is substantial potential for development in this direction and encourage the authors and the community at large to continue their researches on this fascinating problem.

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Final Comments by the Authors L.G. Van Willigenburg and W.L. De Koning

First of all we would like to express our particular appreciation of the discussions organised by the European Journal of Control related to the journal's publications. We feel the journal is a notable exception to the rule to formalize publications, science and research. Like the journal we believe that discussions and debate are especially important and inspiring. Therefore secondly we would like to thank the authors that contributed to the discussion related to our paper. From these discussions it is obvious that the optimal reduced-order LQG problem, and its numerical solution, is by no means a full story, even though many avenues have been explored by now.

We would like therefore to add another perspective. Our algorithms have been motivated by the approach which successfully solved the full-order compensation problem for discrete-time systems with white parameters [1]. As a result the numerical algorithms proposed by us to solve these two problems are very similar. The major difference, we believe, is that in the algorithm to solve the reduced-order LQG problem we are always forced, at some stage, *to make a choice* out of several possibilities. If the order of the plant is n and the prescribed order of the compensator is $n^c < n$ then basically we have to decide on which $n^c < n$ eigenvalues or singular values to keep and which $n - n^c$ to skip. So the need to make a choice relates directly to the prescribed reduced-order $n^c < n$ of the compensator. We know of only one type of order-reduction problem, namely the Hankel norm model reduction problem for linear time-invariant systems, in which this choice is no longer a choice, but dictated by the criterion based on the Hankel norm. So we believe that, *unless we have a very specific criterion which dictates the selection*, there is no, and will be no best way to numerically solve a particular order-reduction problem such as the reduced-order LQG problem. There is a library of possible algorithms to numerically solve the reduced-order LQG problem, each one having its own merits and drawbacks. As an engineer or scientist you have to investigate which one is the best for the problem at hand.

We have added two efficient algorithms to the library. One in this paper and one in [2]. As opposed to the other algorithms, our algorithms are also able to solve the time-varying finite horizon problem. This problem has been very much ignored, but is crucial for the control of non-linear systems [3].

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