Temporal Linear System Structure: The Discrete-Time Case

L.G. Van Willigenburg, W.L. De Koning

Abstract—**A Kalman decomposition for linear time-varying discrete-time systems is introduced that detects *temporal* uncontrollability/unreconstructability that is not detected by any of the four conventional Kalman decompositions. This new discrete-time Kalman decomposition is associated with j-step reachability and k-step observability. The system structure obtained from the Kalman decomposition may be interpreted as the *temporal linear system structure*. This paper reveals that the difference between controllability and reachability as well as reconstructability and observability is entirely due to *changes* of the temporal linear system structure. Finally this paper reveals how our Kalman decomposition relates to the conventional ones and how temporal discrete-time linear system properties relate to their well established non temporal counterparts.**

I. Introduction

A general approach to control non-linear continuous-time systems is to compute an optimal control and state trajectory off-line using a non-linear systems model. To accommodate for disturbances the *linearised* dynamic model about these trajectories is used to design e.g. a linear quadratic perturbation feedback controller that operates on-line [1]. If the optimal controller design explicitly takes into account the *digital nature* of the control system, the design presented in [1] may be adapted and extended in a manner described in [2]. The success of the perturbation feedback controller depends critically on the *controllability* and *reconstructability* of the *equivalent discrete-time* linearised model that is generally *time-varying*. Since digital controls are non-smooth the equivalent discrete-time linearised model may be *temporarily* uncontrollable and/or unreconstructable. Over the associated time intervals the feedback controller will be partly ineffective and the system may become unstable. Therefore the *detection* of *temporal* uncontrollability and unreconstructability of (equivalent) discrete-time linear systems is highly relevant to control engineers. If the controller design is performed entirely in continuous-time the temporal linear system structure, recently introduced in [3], [4] is equally important. Moreover in continuous-time the temporal linear system structure reveals precisely why and when the system properties reachability and controllability are equivalent or not [3], [4].

Interestingly, in continuous-time the introduction and recognition of temporal linear system structure and the associated Kalman decomposition required the introduction of unconventional so called piecewise constant rank systems (PCR systems) [3], [4]. The structure and dimensions of these may change instantaneously at isolated times. This requirement is related to continuous-time being *dense*. Discrete-time is not dense and therefore the development and recognition of temporal linear system structure differs significantly from the one in continuous-time. In discrete-time it will not be necessary to develop an unconventional type of system description although we do need to consider variable dimensions. In discrete-time variable dimensions appear to be introduced in [5]. On the other hand in discrete-time the demarcation of intervals on which the system is temporal controllable or temporal uncontrollable is not as clear as in continuous-time.

An illustrative example of temporal linear system structure has been presented in [3], [4]. A similar example applies in discrete-time. In section II discrete-time systems with variable structure and dimensions are introduced since these are needed to develop and describe the temporal linear system structure in section III. Section IV provides an example and section V conclusions dealing partly with significant differences between temporal system structure in discrete and continuous time.

II. Discrete-time systems with variable structure and dimensions

Although early theoretical results suggested linear systems having variable dimensions [5], [6], [7], [8] in continuous-time these have hardly been considered in the literature until very recently [3], [4], [9]. Discrete-time linear systems with variable dimensions have since about 1992. At that time they started to be used to describe, analyze, and design computational networks [5]. Later they appeared naturally as part of the solution of the discrete-time optimal reduced-order finite-horizon LQG problem [10]. Finally they were also used for discrete-time reduced-order modeling [9]. These discrete-time linear systems are described by,

$$x_{i+1} = \Phi_i x_i + \Gamma_i u_i, \\ y_i = C_i x_i,\ i = i_0, i_0+1, \dots, i_N,\ \ i_0, i_N \in Z, \tag{1}$$

where $x_i \in R^{n_i}$ is the state, $u_i \in R^{m_i}$ is the input and $y_i \in R^{l_i}$

L.G. Van Willigenburg is with the Systems & Control Group of Wageningen University, P.O. Box 17, 6700 AA Wageningen, The Netherlands, (corresponding author phone: +31317482941; e-mail: gerard.vanwilligenburg@wur.nl).

W.L. De Koning retired from the Department of Mathematics of Delft University of Technology and lives at Kroeskarper 6, Leiden, The Netherlands (e-mail: wilros@planet.nl).

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is the output. The dimensions n_i, m_i, l_i of the state, input and output may all vary over time. Therefore from now on we will call the system (1) a variable dimension discrete-time linear system (a VDD system). The matrices $\Phi_i \in R^{n_{i+1}\times n_i}$, $\Gamma_i \in R^{n_{i+1}\times m_i}$, $C_i \in R^{l_i\times n_i}$ are real valued and have compatible dimensions. Note Φ_i need not be square. Similar to the continuous-time case we will consider -+, 0+ and 0N VDD systems having a time-domain specified by respectively 1) $i_0=-\infty$, $i_N=+\infty$, 2) $i_0=0$, $i_N=+\infty$ and 3) $i_0=0$, $i_N=N<+\infty$. Observe from equation (1) that the time-domain I of -+, 0+ as well as 0N VDD systems satisfies,

$$\mathrm{I}=\{i_0, i_0+1, ..., i_N\}. \tag{2}$$

Recently in [11] different types of well known system properties like reachability, observability and minimality for different types of linear systems have been reviewed and their equivalences and differences investigated. This paper uses the definitions and results presented in [11] and only states the most important ones needed to obtain the main results of this paper.

Definition 1

The *reachability/controllability grammian* $W_{i,j}$, $j \geq i$, of a VDD system (1) is given by,

$$\begin{aligned} &W_{i,j}=\Phi_j W_{i,j-1}\Phi_j^T+\Gamma_j\Gamma_j^T \in R^{n_j\times n_j}, j>i, \\ &W_{i,i}=0\in R^{n_i\times n_i}. \end{aligned} \tag{3}$$

The *observability/ reconstructability grammian* $M_{i,j}$, $j>i$, of the VDD system (1) is given by,

$$\begin{aligned} &M_{i,j}=\Phi_i^T M_{i-1,j}\Phi_i+C_i^T C_i \in R^{n_i\times n_i}, i<j, \\ &M_{j,j}=0\in R^{n_j\times n_j}. \end{aligned} \tag{4}$$

Based on Definition 1 and similar to [3], [4] one *may* expect the four conventional Kalman decompositions at each time $i \in \mathrm{I}$ for VDD systems to be computed from the following grammians,

1) $W_{i_0,i}$, M_{i,i_N}, 2) $W_{i_0,i}$, $M_{i_0,i}$, 3) W_{i,i_N}, M_{i,i_N}, 4) W_{i,i_N}, $M_{i_0,i}$.

Now from equation (3) notice that only the two grammians mentioned under 1) both have dimensions $n_i \times n_i$ which is necessary to obtain a decomposition of the state with dimension n_i at time i. Therefore in general only 1) provides a Kalman decomposition for VDD systems unless the VDD system has a constant state dimension over its entire time domain I. In that case 2), 3) and 4) apply as well. 1) decomposes the state-space at time i into states $x_i'^a$ being reachable at time i and unobservable at time i, into states $x_i'^b$ being reachable at time i and observable at time i, into states $x_i'^c$ being unreachable at time i and unobservable at time i and into states $x_i'^d$ being unreachable at time i and observable at time i. Then the following structure is obtained,

$$\begin{aligned} &x_i'=\begin{bmatrix} x_i'^{aT} & x_i'^{bT} & x_i'^{cT} & x_i'^{dT}\end{bmatrix}^T, \\ &x_i'^a \in R^{n_i^a},\ x_i'^b \in R^{n_i^b},\ x_i'^c \in R^{n_i^c},\ x_i'^d \in R^{n_i^d},\ i\in \mathrm{I}, \\ &\Phi_i'=\begin{bmatrix} \Phi_i'^{aa} & \Phi_i'^{ab} & \Phi_i'^{ac} & \Phi_i'^{ad} \\ 0 & \Phi_i'^{bb} & 0 & \Phi_i'^{bd} \\ 0 & 0 & \Phi_i'^{cc} & \Phi_i'^{cd} \\ 0 & 0 & 0 & \Phi_i'^{dd}\end{bmatrix}, \\ &\Gamma_i'=\begin{bmatrix}\Gamma_i'^{aT} & \Gamma_i'^{bT} & 0 & 0\end{bmatrix}^T, \\ &C_i'=\begin{bmatrix}0 & C_i'^b & 0 & C_i'^d\end{bmatrix},\ i\in \mathrm{I}\backslash\{i_N\}. \end{aligned} \tag{5}$$

As opposed to continuous-time PCR systems n_i, n_i^a, n_i^b, n_i^c and n_i^d may be different for *every* value of the discrete-time i.

Theorem 1

1) Using our definition of VDD systems (1) the system decompositions (5) may be interpreted as decompositions into four *VDD sub-systems* defined over I having time-varying dimensions in general.

2) The input-output map of a VDD system is only determined by VDD sub-system b) generated by the Kalman decomposition. The VDD sub-system b) is a *minimal realization* of the VDD system if $x_{i_0}=0$. If $x_{i_0}\neq 0$ the reachability grammian $W_{i_0,i}$ should be replaced by the so-called *weak* reachability grammian $W'_{i_0,i}$ that is also described by equations (3) with W replaced by W' except for the initial condition $W'_{i_0,i_0}=x_{i_0}x_{i_0}^T$.

Proof

1) The proof follows from [3], [4], [5] and the part concerning the case $x_{i_0}\neq 0$ from [11] □

III. Temporal Structure of Discrete-time Linear Systems

Because continuous time is dense the time it takes to control a continuous-time system from one state to another can sometimes be made arbitrarily small. The property differential controllability is based on this possibility [6]. In discrete-time it always takes one or several time steps to control a system from one state to another. Therefore differential controllability does not have an obvious counterpart in discrete time. The counterpart that we will consider to detect the temporal structure of linear systems in discrete time considers controllability over finite time intervals, each interval being a *subset* of the time domain of the system. This is probably best represented by the following discrete-time counterpart of equation (12) in [4],

$$W_{i-j,i} = C_j(i)C_j^T(i), C_j(i) = \begin{bmatrix} P_0(i) & P_1(i) & \ldots & P_j(i) \end{bmatrix}, \\ P_k = \Phi_{i,i-k}\Gamma_{i-k-1},\ k = 0,1,..,j. \quad (6)$$

In equation (6) $W_{i-j,i}$ is the reachability/controllability grammian associated to state transitions from time $i-j$ to time i of a VDD system (1). Furthermore $C_j(i)$ is the associated reachability / controllability matrix and,

$$\Phi_{l,k} = \Phi_{l-1}\Phi_{l-2}...\Phi_k,\ l > k,\ \Phi_{l,k} = I,\ l = k. \quad (7)$$

The value of j in continuous time is determined by properties of the system matrices and their derivatives at time $t \in \mathrm{T}$. The value of j is set equal to the lowest value after which the rank of the matrix C_j no longer increases [7]. The value of j in discrete time determines the *length* of the time interval $[i-j,i]$ over which equation (6) considers reachability/controllability properties. Since we want to consider *temporal linear system structure* also now j has to be limited.

Definition 2

A VDD system is called *j-step reachable at time i* as well as *j-step controllable from time $i-j$* if $j \geq 0$, $i, i-j \in \mathrm{I}$ and any state $x_{i-j} \in R^{n_{i-j}}$ can be transferred to any state $x_i \in R^{n_i}$ through an appropriate choice of the input sequence $U_{i-j,i} = \{u_{i-j}, u_{i-j+1}, ..., u_{i-1}\}$.

Dual to equation (6) we have,

$$\begin{aligned} M_{i,i+k} &= O_k^T(t)O_k(t), \\ O_k &= \begin{bmatrix} S_0^T(t) & S_1^T(t) & .. & S_k^T(t) \end{bmatrix}^T, \\ S_l^T &= C_l\Phi_{i+l,i},\ l = 0,1,..,k. \end{aligned} \quad (8)$$

where $M_{i,i+k}$ is precisely the observability/ reconstructability grammian associated to the time interval $[i,i+k]$ and O_k the associated observability/ reconstructability matrix. Dual to definition 2 we have;

Definition 3

A VDD system is called *k-step observable at time i* as well as *k-step reconstructable from time $i+k$* if $k \geq 0$, $i, i+k \in \mathrm{I}$ and if from the output sequence $Y_{i,i+k} = \{y_i, y_{i+1}, .., y_{i+k-1}\}$ the state $x_i \in R^{n_i}$ can be determined.

From definitions 2, 3 and lemmas 2.2, 2.3 in [11] the following lemma is immediate.

Lemma 1

A VDD system is j-step reachable at time i and therefore j-step controllable from time $i-j$ if and only if $i, i-j \in \mathrm{I}$ and $W_{i-j,i} > 0$. Dually a VDD system is k-step observable at time i and therefore k-step reconstructable from time $i+k$ if and only if $i, i+k \in \mathrm{I}$ and $M_{i,i+k} > 0$.

Using the grammians $W_{i-j,i} \in R^{n_i \times n_i}$, $j \geq 0$, and $M_{i,i+k} \in R^{n_i \times n_i}$, $k \geq 0$ as inputs, a Kalman decomposition may be computed at every time i for -+, 0+ and 0N VDD systems similar to (5). The two grammians $W_{i-j,i} \in R^{n_i \times n_i}$, $j \geq 0$ and $M_{i,i+k} \in R^{n_i \times n_i}$, $k \geq 0$ are associated with respectively the properties j-step reachability at time i / j-step controllability from time $i-j$ and k-step observability at time i / k-step reconstructability from time $i+k$. Also note that both $j \geq 0$ and $k \geq 0$ may be chosen to depend on i i.e. $j(i) \geq 0$, $k(i) \geq 0$.

Definition 4

For -+, 0+ and 0N VDD systems the Kalman decomposition that uses the grammians $W_{i-j,i} \in R^{n_i \times n_i}$, $j \geq 0$ and $M_{i,i+k} \in R^{n_i \times n_i}$, $k \geq 0$ as inputs to compute a Kalman decomposition at every time i is called the *j-step, k-step Kalman decomposition*.

Theorem 2

1) For -+, 0+ and 0N VDD systems after application of the j-step, k-step Kalman decomposition at every time i the states x'_i decompose like the states in equation (5) i.e.: $x_i'^T = \begin{bmatrix} x_i'^{aT} & x_i'^{bT} & x_i'^{cT} & x_i'^{dT} \end{bmatrix}^T$, $x_i'^a \in R^{n_i^a}$, $x_i'^b \in R^{n_i^b}$, $x_i'^c \in R^{n_i^c}$, $x_i'^d \in R^{n_i^d}$. The states $x_i'^a$ are j-step reachable at time i / j-step controllable from time $i-j$ and k-step unobservable at time i / k-step unreconstructable from time $i+k$. The states $x_i'^b$ are j-step reachable at time i / j-step controllable from time $i-j$ and k-step observable at time i / k-step reconstructable from time $i+k$. The states $x_i'^c$ are j-step unreachable at time i / j-step uncontrollable from time $i-j$ and k-step unobservable at time i / k-step unreconstructable from time $i+k$. Finally the states $x_i'^d$ are j-step unreachable at time i / j-step uncontrollable from time $i-j$ and k-step observable at time i / k-step reconstructable from time $i+k$. Similarly $\Gamma_i'^T = \begin{bmatrix} \Gamma_i'^{aT} & \Gamma_i'^{bT} & 0 & 0 \end{bmatrix}^T$, $\Gamma_i'^a \in R^{n_{i+1}^a \times m_i}$, $\Gamma_i'^b \in R^{n_{i+1}^b \times m_i}$ and $C_i' = \begin{bmatrix} 0 & C_i'^b & 0 & C_i'^d \end{bmatrix}$, $C_i'^a \in R^{l_i \times n_i^a}$,

$C_i'^b \in R^{l_i \times n_i^b}$. The decomposition of the system matrices Φ_i' however can be different from those in equation (5).

Proof

At time i the grammians $W_{i-j,i} \in R^{n_i \times n_i}$, $j \geq 0$ are associated with j-step reachability at time i / j-step controllability from time $i-j$. Also the grammians $M_{i,i+k} \in R^{n_i \times n_i}$, $k \geq 0$ are associated with k-step observability at time i / k-step reconstructability from time $i+k$. Therefore similar to [8] the Kalman decomposition ensures the decomposition of the state and the system matrices as stated in Theorem 2. The decomposition of the system matrices Φ_{i-1}' however can be different from equation (5) due to the following. The structure of Φ_i' described by equation (5) is obtained only if the j-step reachable states $x_i'^a, x_i'^b$ at time i do not affect the j-step unreachable states $x_{i+1}'^c, x_{i+1}'^d$ at time $i+1$ and dually if the k-step unobservable states $x_i'^a, x_i'^c$ do not affect the k-step observable states $x_{i+1}'^a, x_{i+1}'^d$. *Conventional* reachable and unobservable states have this property [5], [8]. However j-step reachability and k-step observability and the associated grammians at every time i are based on the original VDD system *restricted* to the interval $[i-j, i+k]$. In general this interval *changes* with i so effectively at every time i a *different VDD system* is considered to determine the decomposition of the state of the original VDD system at time i. This is precisely what is needed to detect temporal reachability/controllability and temporal observability/ reconstructability. On the other hand due to this Φ_i' need no longer satisfy the structure represented by equation (5), as is further explained by lemma 2 and its proof. □

Definition 5

The *j-step, k-step structure* of a VDD system at time $i \in \mathrm{I}$ is determined by the dimensions $n_i^a, n_i^b, n_i^c, n_i^d$ obtained from the j-step, k-step Kalman decomposition at time i. If $n_h^a = n_i^a$, $n_h^b = n_i^b$, $n_h^c = n_i^c$ and $n_h^d = n_i^d$ then a VDD system has the same j-step, k-step structure at times $h, i \in \mathrm{I}$. Otherwise the j-step, k-step structure is different.

Lemma 2

According to theorem 2 the j-step, k-step Kalman decomposition does not in general produce the structure of Φ_i' at time i as represented by equation (5). However this structure *is* produced if $rank(W_{i-j+1,i+1}) = rank(W_{i-j,i+1})$ and $rank(M_{i,i+k}) = rank(M_{i,i+k+1})$.

Proof

From the proof of theorem 2 the structure of Φ_i' represented by equation (5) is destroyed only if a) some of the j-step reachable states $x_i'^a, x_i'^b$ affects some of the j-step unreachable states $x_{i+1}'^c, x_{i+1}'^d$ or b) some of the k-step unobservable states $x_i'^a, x_i'^c$ affect some of the k-step observable states $x_{i+1}'^b, x_{i+1}'^d$. If a) holds then the j-step unreachable state at time $i+1$ must be j+1-step reachable at time $i+1$, because otherwise $x_{i+1}'^c, x_{i+1}'^d$ are unaffected. If $rank(W_{i-j+1,i+1}) = rank(W_{i-j,i+1})$ however every j+1-step reachable state at time $i+1$ is also j-step reachable at time $i+1$ so a) cannot hold. Dually if b) holds the k-step unobservable state at time i must be k+1-step observable at time i because otherwise it does not affect $x_{i+1}'^b, x_{i+1}'^d$. If $rank(M_{i,i+k}) = rank(M_{i,i+k+1})$ every k+1-step unobservable state at time i is also k-step unobservable at time i so b) cannot hold □

Remark 1

As to the j-step, k-step Kalman decomposition a practical issue is the selection of j and k that may be chosen to depend on i i.e. $j(i)$, $k(i)$. Note that the number of input variables available to control the state from x_{i-j} to x_i equals $\sum_{k=i-j}^{i-1} m_k$. Therefore $j(i)$ should be selected such that $\sum_{k=i-j(i)}^{i-1} m_k \geq n_i$ because otherwise, whatever the system, it cannot be j-step reachable at time i. Dually $k(i)$ should be selected such that $\sum_{j=i}^{i+k(i)-1} l_j \geq n_i$ because otherwise, whatever the system, it cannot be k-step observable at time i. Clearly the larger we take $j(i)$ the larger the possibility that the system is j-step reachable at time i. Dually the larger we take $k(i)$ the larger the possibility that the system is k-step observable at time i. In the limit $j(i) = i - i_0$, j-step reachability at time i turns into reachability at time i. Dually in the limit $k(i) = i_0 + i_N - i$, k-step observability at time i turns into observability at time i. Assuming $m_i > 0$, $l_i > 0$, $\forall i \in \mathrm{I} \backslash \{i_N\}$ a natural choice is to select $j(i) = n_i$, $k(i) = n_i$. Note that for 0+ and 0N VDD systems this choice is invalid for $i_0 \leq i < i_0 + n_i$. If $i_0 \leq i < i_0 + n_i$. Then $j(i) = i - i_0$ is a natural choice. Similarly for 0N systems $k(i) = i_N - i$ is a natural choice if $i_N - n_i < i \leq i_N$.

The next theorem that applies to VDD systems is comparable to theorem 4 in [4] that applies to PCR systems but is less strong. The theorem states how the *temporal* structure of a VDD system associated to j-step reachability/j-step controllability and k-step observability/k-step reconstructability provides information about the *global* system properties controllability and reconstructability of a VDD system.

Theorem 3

1) A VDD system is j-step controllable from time i_1 $\Rightarrow$ the VDD system is controllable from any time $i \le i_1$, $i \in \mathrm{I}$. 2) A VDD system is k-step reconstructable from time i_1 $\Rightarrow$ the VDD system is reconstructable from any time $i \ge i_1$, $i \in \mathrm{I}$.

Proof

1) and 2) follow immediately from definitions 2.7, 2.16 in [11]. □

Theorem 4

The j-step, k-step structure of a VDD system is constant over $[i_1, i_2] \subseteq \mathrm{I}$ if and only if $rank\left(W_{i-j(i),i} M_{i,i+k(i)}\right)$, $rank\left(W_{i-j(i),i}\right)$, $rank\left(M_{i,i+k(i)}\right)$ and n_i are all constant for $i \in [i_1, i_2] \subseteq \mathrm{I}$.

Proof

Follows from definition 5 and the following four equalities: $n_i^b = rank\left(W_{i-j(i),i} M_{i,i+k(i)}\right)$, $n_i^a + n_i^b = rank\left(W_{i-j(i),i}\right)$, $n_i^b + n_i^d = rank\left(M_{i,i+k(i)}\right)$ and $n_i = n_i^a + n_i^b + n_i^c + n_i^d$ □

IV. Discrete-time Example

Unlike PCR systems VDD system of the form (5) are not *generically* Kalman decomposition as will be explained in this section. Still the example in this section is constructed starting from (5) where the non-zero elements are selected randomly. Next similarity transformations are applied to them to obtain system representations that no longer reveal the structure (5). To determine and verify the temporal linear system structure j-step, k-step Kalman decompositions are computed using the algorithm presented in [12].

Example 1

Consider a 0N discrete-time VDD system (1) with $i_0 = 0$, $i_N = 29$, $n_i = 4$, $0 \le i \le 29$, $m_i = 1$, $l_i = 1$, $0 \le i < 29$. For $0 \le i \le 10$ and $22 \le i < 29$ the system matrices Φ_i, Γ_i, C_i satisfy equation (5) with $n_i^a = 0$, $n_i^b = 4$, $n_i^c = 0$, $n_i^d = 0$. For $11 \le i \le 21$ the system matrices satisfy equation (5) with $n_i^a = 1$, $n_i^b = 1$, $n_i^c = 1$, $n_i^d = 1$. The non-zero elements of these matrices are selected randomly except for Φ_5 that is a singular matrix with rank 1 of which the first row is selected randomly and all the other elements equal to zero. Next a random time-varying similarity transformation is applied to the system. For the resulting system, using the algorithm from [12], both the conventional structure, obtained from the Kalman decomposition with $W_{0,i}$ and $M_{i,N}$ as inputs, as well as the j-step, k-step structure with $j(i) = \min(4, i)$, $k(i) = \min(4, i_N - i)$,obtained from the Kalman decomposition with $W_{i-j,i}$ and $M_{i,i+k}$ as inputs, is computed. The outcome of i and n_i^a, n_i^b, n_i^c, n_i^d, $n_i^r = n_i^a + n_i^b$, $n_i^o = n_i^b + n_i^d$ of the conventional and j-step, k-step Kalman decomposition respectively is tabulated below.

	Conventional						j-step, k-step					
i	n_i^a	n_i^b	n_i^c	n_i^d	n_i^r	n_i^o	n_i^a	n_i^b	n_i^c	n_i^d	n_i^r	n_i^o
0	0	0	0	4	0	4	0	0	0	4	0	4
1	0	1	0	3	1	4	0	1	0	3	1	4
2	0	2	0	2	2	4	0	2	0	2	2	4
3	0	3	1	0	3	3	0	3	1	0	3	3
4	2	2	0	0	4	2	2	2	0	0	4	2
5	0	2	0	2	2	4	0	2	0	2	2	4
6	0	3	0	1	3	4	0	3	0	1	3	4
7	0	4	0	0	4	4	0	4	0	0	4	4
8	0	4	0	0	4	4	0	4	0	0	4	4
9	0	4	0	0	4	4	1	3	0	0	4	3
10	0	4	0	0	4	4	2	2	0	0	4	2
11	0	4	0	0	4	4	2	2	0	0	4	2
12	0	4	0	0	4	4	2	2	0	0	4	2
13	0	4	0	0	4	4	1	2	1	0	3	2
14	0	4	0	0	4	4	1	1	1	1	2	2
15	0	4	0	0	4	4	1	1	1	1	2	2
16	0	4	0	0	4	4	1	1	1	1	2	2
17	0	4	0	0	4	4	1	1	1	1	2	2
18	0	4	0	0	4	4	0	2	1	1	2	3
19	0	4	0	0	4	4	0	2	0	2	2	4
20	0	4	0	0	4	4	0	2	0	2	2	4
21	0	4	0	0	4	4	0	2	0	2	2	4
22	0	4	0	0	4	4	0	3	0	1	3	4
23	0	4	0	0	4	4	0	4	0	0	4	4
24	0	4	0	0	4	4	0	4	0	0	4	4
25	0	4	0	0	4	4	0	4	0	0	4	4
26	1	3	0	0	4	3	1	3	0	0	4	3
27	2	2	0	0	4	2	2	2	0	0	4	2
28	3	1	0	0	4	1	3	1	0	0	4	1
29	4	0	0	0	4	0	4	0	0	0	4	0

From this table observe that the temporal j-step, k-step structure is constant for $i \in [7,8]$, $i \in [10,12]$, $i \in [14,17]$, $i \in [19,21]$, $i \in [23,25]$. The number of reachable as well as k-step reachable states $n_i^r = n_i^a + n_i^b$ builds up maximally starting from $i = 0$ as well as $i = 5$ where the maximum increase per time step equals $m_i = 1$ [10], [11]. Because Φ_5 is singular having rank 1 the build up starts again at $i = 5$. The number of observable and k-step observable states $n_i^o = n_i^b + n_i^d$ builds up maximally backward in time starting from $i = 29$ and $i = 5$ where the maximum increase per time step equals $l_i = 1$ [10], [11]. The singularity of the matrix Φ_5 causes a change in *both* the conventional as well as the j-step, k-step structure. Both the conventional and j-step, k-step result clearly show that although we generated the example with system matrices that change structure over just one time-step, the associated changes of the system structure involve several time steps. As opposed to PCR systems this also shows that VDD system matrices that posses the structure (5) do *not* generically represent a Kalman decomposition. Finally observe that the temporal j-step, k-step structure for $i \in [9,22]$ is not at all visible from the conventional structure. Only the j-step, k-step temporal structure indicates that just one state is both j-step reachable as well as k-step observable for $i \in [14,17]$ and that the system is neither j-step reachable nor k-step observable for $i \in [9,22]$.

V. CONCLUSIONS

The major practical and theoretical contribution of this paper concerns the recognition, description and detection of *temporal* linear discrete-time system structure. This structure is associated with and also reveals immediately temporal reachability/controllability as well as temporal observability/ reconstructability. This is highly relevant information to systems and control engineers. Intuitively one easily imagines the temporal loss of reachability, controllability, observability and reconstructability. Interestingly to analyze them properly we have to extend the description of linear systems beyond the conventional ones. In continuous-time this led to the introduction of piecewise constant rank systems (PCR systems) in [3], [4]. These continuous-time systems have variable structure and dimensions that may change *instantaneously* at *isolated* time-instants. In between those isolated time instants both the dimensions and temporal linear system structure remain constant. In discrete-time this led to the consideration of systems that also have variable dimensions (VDD systems). Since discrete-time is not dense properties like reachability *build up* over several time steps instead of instantaneously. Therefore in discrete time the temporal linear system structure also builds up over several time steps and therefore may change at *every* time instant.

Much of the results in this paper concerning the temporal linear system structure are directly obtained from the standard results by *restricting* the time interval over which we consider the original system. In continuous-time we restricted the time interval to an infinitely small one. Then e.g. controllability turns into differential controllability and for PCR systems the four conventional Kalman decompositions become *one*. In discrete-time to reveal any dynamics we need to consider several time-steps. Therefore finite time intervals with length j,k associated with j-step reachability/controllability and k-step observability/ reconstructability were considered where j,k have to be selected sensibly, as explained in this paper. For VDD systems only *one* of the four conventional Kalman decompositions applies at each time i.

In continuous-time the temporal linear system structure of a -+, 0+ and 0N PCR system may be constant over the entire time-domain. In that case the system is an ordinary constant rank system for which reachability, controllability, differential reachability and differential controllability and dually observability, reconstructability, diffential observability and differential reconstructability are *equivalent* [7]. This shows that in continuous-time the differences between the *global* system properties reachability and controllability as well as observability and reconstructability are entirely due to *changes* of the *temporal local* linear system structure associated to the differential versions of these properties.

In discrete-time due to the build up of the temporal linear system structure over several time-steps only -+ VDD systems with a constant state dimension can have a constant structure over their entire time-domain. For 0+ and 0N VDD systems the temporal linear system structure may be constant only over time intervals that cover part of the systems time domain. Because discrete-time is not dense the temporal linear system structure obtained from the j-step, k-step Kalman decomposition at time i is not entirely local since it depends on system properties over the time interval $[i-j, i+k]$.

The results obtained for PCR systems are generally stronger than those obtained for VDD systems. An interesting question is therefore if a class of VDD systems can be found that has more properties in common with PCR systems.

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