

The Role and Use of the Stochastic Linear-Quadratic-Gaussian Problem in Control System Design

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Abstract—The role of the linear-quadratic stochastic control problem in engineering design is reviewed in tutorial fashion. The design approach is motivated by considering the control of a nonlinear uncertain plant about a desired input-output response. It is demonstrated how a design philosophy based on 1) deterministic perturbation control, 2) stochastic state estimation, and 3) linearized stochastic control leads to an overall closed-loop control system. The emphasis of the paper is on the philosophy of the design process, the modeling issue, and the formulation of the problem; the results are given for the sake of completeness, but no proofs are included. The systematic off-line nature of the design process is stressed throughout.

I. INTRODUCTION

THE REAL underlying problem in engineering control system design almost invariably involves the on-line feedback control of an uncertain, usually nonlinear, physical process. The engineer usually likes to work with, and benefits from, a systematic approach to the design problem; such systematic approaches are often the outcome of past design experience.

Clearly, a universal engineering design approach must take into account: 1) the desired specifications, 2) actuator and sensor constraints, 3) measurement errors, 4) actuator errors, 5) design sensitivity due to plant parameter variations, 6) effects of unpredictable disturbances, 7) on-line versus off-line computational requirements, and 8) design simplicity.

The purpose of this paper is to indicate how the available theory of optimal control and estimation for the so-called linear-quadratic-Gaussian (LQG) problem provides such a unified design procedure. In particular, we wish to stress the advantages of this design process from the viewpoint of ease of computation, since the theory provides us with equations that can be readily solved by modern digital computers. Thus, the success of the design process hinges on the capability of the engineer to understand the physics of the problem and his ability to translate physical requirements and constraints into mathematical language.

Manuscript received July 26, 1971. Paper recommended by R. W. Brockett, Associate Guest Editor. This research was supported in part by the National Science Foundation under Grant GK-25781, in part by NASA under Grant NGL 22-009-124, and in part by the Air Force Office of Scientific Research under Grant 70-1941.

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Once this crucial "modeling" has been done, the digital computer algorithms will readily generate the quantitative details of the design.

Towards this goal, this paper is structured in the following manner. In Section II we discuss the problem under consideration in the most general terms, and we outline the design philosophy that we shall adopt. In Section III we discuss the deterministic aspects of the design problem, introduce the notions of the ideal input-state nominal time functions, dynamic linearization, provide the motivation for using quadratic criteria, and state the solution of the deterministic linear-quadratic problem. In Section IV we analyze the deterministic design from the viewpoint of uncertainty and sensor constraints. This leads us to the problem of estimating the state variables of the uncertain physical process on the basis of past measurements via the Kalman-Bucy theory. In Section IV we "hook up" together the stochastic estimator of Section IV with the deterministic controller of Section III to obtain the desired compensator that translates actual sensor measurements to commanded control inputs. Section VI contains a discussion of the results.

Of course, all the results outlined in this paper are available in the control literature. Hence the hoped-for contribution of this paper is that of unification, so that an engineer can see how seemingly diverse topics in control theory provide a systematic computer-aided design tool. Thus the emphasis will be on philosophy, interpretation, and critical discussion of the interplay between physical processes and the mathematical models required to apply the powerful tools of modern control theory. We also hope to convey the fact that this approach to design involves both "art and science," so that engineering creativity and know-how are (as always) the key ingredients of success.

At this point, it should be stressed that the LQG approach to design is only one of many available avenues to the control engineer. Other design philosophies are available, especially for linear constant systems. These can be classified roughly as follows: 1) frequency-domain and root-locus techniques; 2) pole-placement techniques (see [1], for example); and 3) deterministic compensator design techniques (see [2], [3], for example).

There is no such thing as an optimal design process, since it depends very strongly on the specific application, designer experience, and time available to do the job. The

LQG approach has been praised and criticized (see [4] for a critique). Only time and additional experience, as well as extensions of the available results, will settle some (but not all!) of the current controversies.

II. THE PHYSICAL PROBLEM AND THE DESIGN PHILOSOPHY

We commence our specific discussion with a brief description of the problem of controlling a physical process and a definition of the control problem.

A. Physical Plant, Actuators, and Sensors

We shall deal with the interconnected entity of a physical plant driven by physical actuators; measurements can be made by physical sensors. The interconnection is shown in Fig. 1, and it leads to the notion of a physical process.

1) *Actuators*: The actuators are physical devices that translate commanded inputs (time functions that can be specified by the designer, e.g., commanded accelerations for an aerospace vehicle) into actual plant inputs (e.g., thrust magnitudes and directions). This translation is not exact; this is modeled by the actuator uncertainties in Fig. 1. It is assumed that the actual plant inputs cannot be measured.

2) *Plant*: The plant is a physical device that translates the actual plant inputs as well as other plant disturbances (e.g., wind forces) into a set of time functions which we shall call the physical state variables of the plant (e.g., positions, velocities, bending modes). For our purposes, the plant state variables are the key physical variables that govern and specify completely the current plant behavior (for example, in a nonlinear network, these are the inductor flux linkages and the capacitor charges).

3) *Sensors*: We assume that it is either impossible or not desirable, for engineering reasons, to measure all the plant state variables. The physical sensors are devices that indicate which physical variables (state variables and/or combinations thereof) can be indeed measured. However, the actual sensor measurement signals are different than the signals that are sensed; these errors are illustrated in Fig. 1 by the inclusion of the sensor error and uncertainty signals, which take into account the measurement accuracy of any given sensor.

The design process itself may involve the selection of the number, type, and fidelity of at least some of the actuators and/or sensors. We shall comment later on how the available theory can aid the designer in this selection process.

B. Control System Objectives

The desired behavior of the physical process as a whole is often judged on the basis of the actual time evolution of all or some of the physical plant state variables. Of course, due to the sensor constraints one may not be able to deduce exactly what the plant is doing at each and every instant of time. Nonetheless, the plant state variables

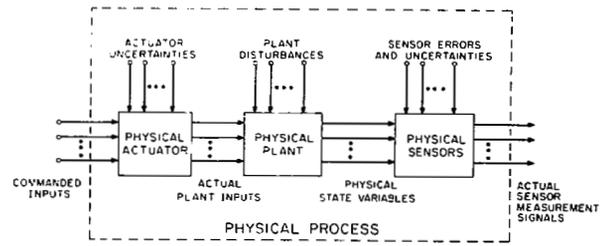


Fig. 1. Visualization of physical process, its subsystems, and important variables.

rather than the measurements are the key quantities that enter in the control problem formulation.

In many cases, the time evolution of the plant state variables may possess certain undesirable characteristics. These may be due to the effect of the plant disturbances and/or actuator errors, and due to inherent plant instability or sluggish response. In such cases, one must be able to control the time evolution of the plant state variables by the adjustment of the time evolution of the commanded inputs (which are the only variables that can be externally adjusted).

Hence, the control system objective can be loosely stated as follows.

Find the time-evolution of the commanded inputs such that the time evolution of the physical plant state variables is satisfactory for the application at hand.

C. Control System Structure

Since the control objective hinges on the time evolution of the physical state variables, and since we cannot sense them directly in view of the sensor constraints, it is intuitively obvious that the actual values of the commanded inputs at the present time must somehow (at least, partially) depend upon the current, and perhaps past, values of the sensor measurement signals.

Thus, we are admitting right at the start that some feedback is necessary. This leads us to visualizing that we must construct a physical device, which we shall call the *compensator*, whose task will be to translate the actual sensor measurement signals into the actual commanded inputs to the physical process. This feedback structure is illustrated in Fig. 2.

We can now reformulate the control objective of Section II-B as follows.

Find the compensator, driven by the sensor measurement signals and generating the commanded inputs to the physical process, such that the time evolution of the physical plant state variables is satisfactory for the application at hand.

D. Design Philosophy

It should be clear that the design of the compensator must hinge on: 1) natural dynamics of the physical process both in the absence of uncertainty (deterministic) and in the presence of uncertainty (stochastic); 2) the level of the uncertainty in the physical process (How big are the

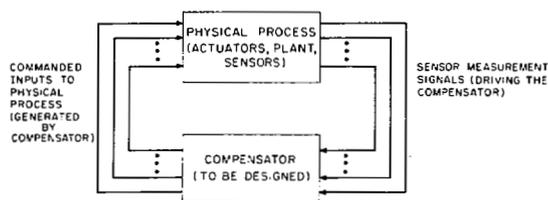


Fig. 2. Assumed structure of overall feedback control system.

probable actuator errors? How large are the plant disturbances? How accurate are the sensors?); and 3) the precise notion of what characterizes, for any given application, a satisfactory time evolution of the physical plant state variables.

In point of fact, our ability to construct such a compensator must depend upon our ability to predict (exactly or approximately) what the physical plant state variables will be doing for any given: 1) commanded input time functions; 2) actuator errors as a function of time; and 3) plant disturbances as a function of time.

Clearly, the design issue is clouded because it involves the interplay between the natural dynamics of the physical process, the stochastic nature of the uncertainties, and the effects of the deterministic commanded inputs. Nonetheless, one can adopt a design philosophy that involves the following three basic steps.

Step 1: Deterministic ideal response analysis and design.

Step 2: Stochastic estimation analysis and design.

Step 3: Stochastic feedback control system design.

In the rest of the paper, we shall elaborate on precisely what we mean by this three-part approach.

Deterministic Ideal Response Analysis and Design (Step 1): In this step, we pretend that there is absolutely no uncertainty. That is, we suppose that: a) actuator errors *do not* exist; b) there are *no* plant disturbances; c) we can measure *exactly* all the physical plant state variables and output variables; d) the actuator and plant dynamics are known *exactly*; e) all parameter values are known *exactly*. Under these assumptions, we can predict *exactly* what the plant state and output variables will do for any given commanded inputs.

If this is the case, then somehow (and this will be treated in detail in Section III) we should be able to *determine the ideal commanded inputs as functions of time which will give rise to an ideal set of plant state-variable time responses for the application at hand.*

In short, the basic end product of this first step of the design process is the specification of an ideal deterministic commanded-input state-variable pair that incorporates the specifications of the application and the natural constraints and dynamics of the physical process.

Stochastic Estimation Analysis and Design (Step 2): In this part of the design process, we reintroduce the uncertainty into our problem. In particular, we take into account that we cannot measure all of the plant state

variables and that any measurement is subject to sensor errors. The basic question that we answer at this step of the design process is the following.

Construct a device (state estimator, filter) that generates on the basis of the past sensor measurements a set of time functions which are as close as possible to the true values of the physical plant state and output variables at any instant of time.

The way that this state estimator is constructed is the subject of Section IV. The reason that this step is essential to the design process becomes apparent in the next step.

Stochastic Feedback Control System Design (Step 3): Let us recapitulate for a moment on what we have constructed up to now.

From Step 1 we have:

- a) an ideal deterministic set of commanded-input time functions;
- b) an ideal deterministic set of desired plant state-variable and output time functions.

From Step 2 we have:

- a) a set of estimated plant state and output variable time functions (which are hopefully close to the true plant state and output variables in the uncertain stochastic environment).

We now have the capability to compare the estimated state variables [from Step 2a] to the desired state variables [from Step 1b] at each and every instant of time. Their differences constitute a set of estimated deviations of the actual plant state variables from their ideal desired values at each instant of time.

Thus, we have an approximate idea on how close the response of the physical process is to its desired one.

In general, due to the presence of uncertainties and plant disturbances, one would expect to observe such an estimated deviation. One can now reformulate the control objective as follows.

Design the compensator such that all estimated deviations of the plant state variables from their ideal desired values are close to zero for all instants of time.

It should be clear that if we keep applying the ideal deterministic commanded-input time functions [from Step 1a] that the preceding objective will not be met, since the deterministic input was found under assumptions (no uncertainty!) that are violated. Hence, one would expect that the actual commanded inputs to the physical process must be somewhat different than the ideal deterministic inputs found in Step 1.

One can imagine that this is accomplished by constructing a set of control correction signals (generated on the basis of the estimated deviations of the state variables from their desired values) such that the actual commanded input to the physical process is the sum, at any instant of time, of the ideal inputs obtained in Step 1 of the control corrections.

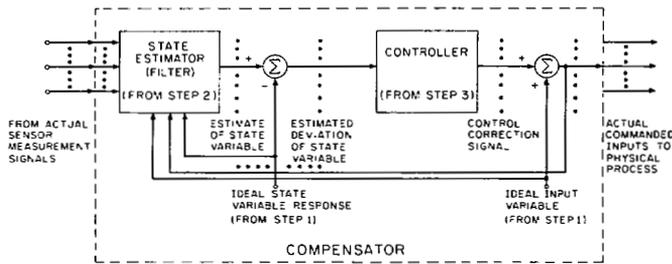


Fig. 3. Internal structure of compensator (see Fig. 2).

This structure of the compensator can now be visualized according to the block diagram of Fig. 3. Hence, our control objective in this step is as follows.

Design the controller which is driven by the estimated deviations of the state variables from their desired values and generates the control correction signals such that the deviation of the actual state variables from their ideal values is as small as possible for all time.

As we shall see, this design approach has many significant advantages from the conceptual, qualitative, and quantitative points of view.

III. DETERMINISTIC IDEAL RESPONSE ANALYSIS AND DESIGN (STEP 1)

A. Introduction

As indicated in Section II-D the first step in the proposed design process assumes that the physical process operates in the absence of uncertainty. In this section we elaborate on the steps and techniques which culminate in the deterministic ideal pair of inputs and associated state-variable responses.

Our objective here is to indicate that additional non-physical uncertainties can be introduced even if the physical process is assumed to operate in an otherwise deterministic physical environment. These nonphysical uncertainties are the results of modeling approximations. They lead to a structure similar to that of the overall stochastic problem (see Fig. 3) even if all physical plant state variables can be measured exactly.

B. Deterministic Modeling

It is essential for the overall design process that the physical process be modeled in a quantitative number.¹ This of course requires a blending of natural laws (e.g., Newton's law) as well as of experimentation so as to determine the nominal parameter values of the physical process.

Quite often assumptions that are made at this point are: 1) actuator dynamics are neglected; 2) sensor dy-

¹ We do not mean to minimize the difficulties associated with this step. Modeling is an extremely difficult process. The dynamics of aerospace systems are very simple and well understood as compared, say, to the dynamics of chemical process control systems or socioeconomic systems. This is one of the reasons that most of the successful applications of the LQG techniques have been to aerospace systems up to this time.

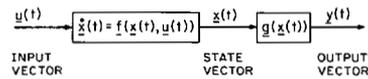


Fig. 4. Structure of deterministic mathematical model for physical process of Fig. 1. Underlined symbols indicate boldface.

namics are neglected; and 3) the plant is modeled as a lumped system.

1) *Actuator-Plant Model:* Under these assumptions the actuator and plant are modeled by a vector differential equation,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)); \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1)$$

where: $\mathbf{x}(t)$ is the plant state vector, an n -dimensional vector with components $x_1(t), x_2(t), \dots, x_n(t)$; $\mathbf{u}(t)$ is the plant control vector, an m -dimensional vector with components $u_1(t), u_2(t), \dots, u_m(t)$; $\mathbf{x}_0 = \mathbf{x}(t_0)$ is the initial state vector at the initial time t_0 ; $\dot{\mathbf{x}}(t) \triangleq d/dt \mathbf{x}(t)$; and $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ is the plant nonlinearity, a vector-valued nonlinear function with components $f_1(\mathbf{x}(t), \mathbf{u}(t)), f_2(\mathbf{x}(t), \mathbf{u}(t)), \dots, f_n(\mathbf{x}(t), \mathbf{u}(t))$.

Remarks: 1) The plant is assumed to be time invariant [i.e., $\mathbf{f}(\cdot, \cdot)$ does not explicitly depend on the time t]. This assumption can be removed [i.e., $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ replaces (1)]. 2) In general, actuator dynamics, if significant, can be absorbed together with the plant dynamics, thus increasing the dimensionality of the state vector $\mathbf{x}(t)$. 3) The function $\mathbf{f}(\cdot, \cdot)$ contains parameters whose values (nominal) are assumed known. 4) The function $\mathbf{f}(\cdot, \cdot)$ is assumed (for technical reasons) continuous and at least twice differentiable with respect to its arguments $\mathbf{x}(t)$ and $\mathbf{u}(t)$.

2) *Sensor Model:* We let the output vector $\mathbf{y}(t)$ denote the r -dimensional vector that represents the signals that can be measured. Thus the components $y_1(t), y_2(t), \dots, y_r(t)$ of $\mathbf{y}(t)$ denote the signals that can be measured by the sensors.

We assume that each output variable is at most a nonlinear time-invariant combination of the state variables. This is modeled by

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t)) \quad (2)$$

where $\mathbf{g}(\mathbf{x}(t))$ is called the output nonlinearity, a vector-valued function with components $g_1(\mathbf{x}(t)), g_2(\mathbf{x}(t)), \dots, g_r(\mathbf{x}(t))$.

Remarks: 1) Sensor dynamics, if significant, can be incorporated in the plant equation (1). 2) The vector $\mathbf{g}(\mathbf{x}(t))$ is assumed to be continuous and at least twice differentiable with respect to $\mathbf{x}(t)$ (once more for technical reasons).

The mathematical deterministic model of the physical process is illustrated in Fig. 4.

C. Ideal Input-State-Output Responses

Under our assumptions the following is true. Given: 1) the current value state vector $\mathbf{x}(t)$ and 2) the control input $\mathbf{u}(\tau), \tau \geq t$, to be applied in the future. Then 1) one can compute exactly the unique future state $\mathbf{x}(\tau), \tau \geq t$ and 2)

one can compute exactly the unique future output $\mathbf{y}(\tau)$, $\tau \geq t$.

This capability allows us to determine the ideal deterministic input-state pair for any given initial state \mathbf{x}_0 . In general, one is interested in the operation of the system over a time interval $[t_0, T]$. On the basis of the deterministic model one then obtains the following definitions.

- 1) Ideal deterministic input time function: $\mathbf{u}_0(t)$, $t \in [t_0, T]$.
- 2) Ideal deterministic state trajectory: $\mathbf{x}_0(t)$, $t \in [t_0, T]$.
- 3) Ideal deterministic output time function: $\mathbf{y}_0(t)$, $t \in [t_0, T]$.

These quantities are related by

$$\begin{aligned}\dot{\mathbf{x}}_0(t) &= \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t)); \mathbf{x}_0(t_0) = \mathbf{x}_0 \\ \mathbf{y}_0(t) &= \mathbf{g}(\mathbf{x}_0(t)).\end{aligned}\quad (3)$$

1) *Computation of Ideal Input-State Response:* The design procedure requires that to each initial state \mathbf{x}_0 we associate an input-state pair of time functions $\mathbf{u}_0(t)$ and $\mathbf{x}_0(t)$, $t \in [t_0, T]$. The interpretation of $\mathbf{x}_0(t)$ is that it represents the desired state evolution of the system, provided that the system initial state is \mathbf{x}_0 .

In principle, $\mathbf{u}_0(t)$ and $\mathbf{x}_0(t)$ can be obtained by experience, coupled with digital computer simulation. However, there is a systematic approach to the determination of $\mathbf{u}_0(t)$ and $\mathbf{x}_0(t)$ via the solution of a nonlinear deterministic optimal control problem. This involves the definition by the designer of a (nonquadratic in general!) scalar-valued cost functional,

$$I = \phi(\mathbf{x}(T)) + \int_{t_0}^T L(\mathbf{x}(t), \mathbf{u}(t)) dt \quad (4)$$

which incorporates any requirements on the terminal state $\mathbf{x}(T)$ by means of the penalty function $\phi(\mathbf{x}(T))$ and any state-variable constraints, control-variable constraints, and optimality criteria in the function $L(\mathbf{x}(t), \mathbf{u}(t))$. In this case then one can formulate a deterministic optimal control problem of the following form.

Given the system (1) and the initial state \mathbf{x}_0 . Find $\mathbf{u}_0(t)$ and the resultant $\mathbf{x}_0(t)$ for all $t \in [t_0, T]$, such that the cost functional (4) is minimized.

The detailed description of means of solution for this problem falls outside the scope of this paper. For our purposes it suffices to state the following.

a) The Pontryagin maximum principle or calculus of variations techniques (see, for example, [4, ch. 2], [5, ch. 5], [6, ch. 2, 3]) can be used to obtain a set of necessary conditions for optimality in the form of a nonlinear two-point boundary-value problem.

b) A host of off-line numerical iterative techniques exist (e.g., the gradient method and quasilinearization method [6, ch. 7], differential dynamic programming [7], to mention just a few) which generate for each initial state \mathbf{x}_0 the optimal control time function $\mathbf{u}_0(t)$ and the associated optimal state trajectory $\mathbf{x}_0(t)$ as time functions for all $t \in [t_0, T]$.

Once $\mathbf{x}_0(t)$ has been computed, the ideal output response $\mathbf{y}_0(t)$ —what the noiseless sensor measurements should be if the system followed $\mathbf{x}_0(t)$ —can be found from (3).

To proceed with our design process we shall view the triplet of time functions $\mathbf{u}_0(t)$, $\mathbf{x}_0(t)$, $\mathbf{y}_0(t)$, $t \in [t_0, T]$ as being precomputable so that they can be stored in the core memory of a digital computer or on tape.

D. Control Under the Deterministic Assumption

Let us now examine the interrelationship between our deterministic mathematical model and the physical process which operates in a deterministic environment.

Let $\mathbf{u}(t)$, $\mathbf{x}(t)$, $\mathbf{y}(t)$ denote the *true* input, state, and output of the physical process. By assumption, all can be measured exactly.

Let us imagine that we conduct the following experiment. We let

$$\mathbf{u}(t) = \mathbf{u}_0(t), \quad t \in [t_0, T] \quad (5)$$

that is, we excite the physical system with the ideal input found in Section III-C. Let us then measure the true state $\mathbf{x}(t)$ and output $\mathbf{y}(t)$ of the physical system.

The natural questions that arise are the following:

$$\text{Is } \mathbf{x}(t) = \mathbf{x}_0(t), \quad \text{for all } t \in [t_0, T]?$$

$$\text{Is } \mathbf{y}(t) = \mathbf{y}_0(t), \quad \text{for all } t \in [t_0, T]?$$

In general, the answer is *no*. The reason is that $\mathbf{x}_0(t)$ and $\mathbf{y}_0(t)$ were computed using a mathematical model of the physical process. However, the engineer has to make some approximations (often intentionally) to arrive at the mathematical model, often neglecting to include second-order effects.² Even if the equations were exact structurally, the values of the parameters used in the mathematical model are *nominal* ones and the true values may be slightly different. In addition, the actual initial state of the system $\mathbf{x}(t_0)$ may differ slightly from the ideally assumed one $\mathbf{x}_0(t_0)$.

It then follows that errors in the deterministic model may by themselves contribute to deviations of the true physical plant state $\mathbf{x}(t)$ from its ideal deterministic one $\mathbf{x}_0(t)$. In fact, small initial deviations, caused by the difference $\mathbf{x}(t_0) - \mathbf{x}_0(t_0)$, may get worse and worse as time goes on.

E. Deterministic Perturbation Control Problem

If we agree that our design objective is to keep the actual plant state $\mathbf{x}(t)$ near its ideal desired value $\mathbf{x}_0(t)$ for all $t \in [t_0, T]$, then it is clear that the actual plant input $\mathbf{u}(t)$ must be different from the precomputed ideal input $\mathbf{u}_0(t)$.

This leads to defining the following quantities (see Fig. 5).

- 1) State perturbation vector $\delta\mathbf{x}(t)$:

² For example, in aerospace problems one may neglect secondary effects in the equations of motion due to rotating earth, Coriolis forces, nonspherical earth, gravity harmonics, etc.

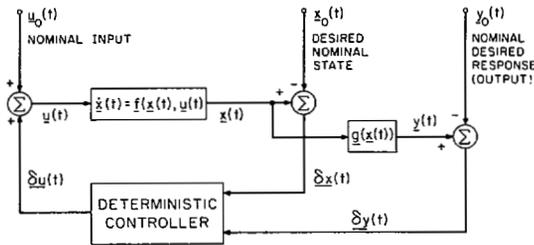


Fig. 5. Conceptual structure of deterministic control system. Underlined symbols indicate boldface.

$$\delta \mathbf{x}(t) \triangleq \mathbf{x}(t) - \mathbf{x}_0(t). \quad (6)$$

2) Output perturbation vector $\delta \mathbf{y}(t)$:

$$\delta \mathbf{y}(t) \triangleq \mathbf{y}(t) - \mathbf{y}_0(t). \quad (7)$$

3) Control correction vector $\delta \mathbf{u}(t)$:

$$\delta \mathbf{u}(t) \triangleq \mathbf{u}(t) - \mathbf{u}_0(t). \quad (8)$$

As illustrated in Fig. 5 we can imagine that the control correction vector $\delta \mathbf{u}(t)$ is generated by a deterministic controller³ which is possibly driven by: 1) the state perturbation vector $\delta \mathbf{x}(t)$ and 2) the output perturbation vector $\delta \mathbf{y}(t)$. Thus, even in this deterministic case, one must use feedback control to take care of errors that are primarily associated with errors in modeling.

The control objective can then be stated as follows.

Given $\delta \mathbf{x}(t)$ and $\delta \mathbf{y}(t)$, find $\delta \mathbf{u}(\tau)$, $\tau \geq t$, such that future state perturbation vectors, $\delta \mathbf{x}(\tau)$, are small for all $\tau \in [t, T]$. Find the deterministic controller in Fig. 5 that will accomplish this.

F. Linear-Quadratic Approach to the Deterministic Controller Design

Since the compensator to be designed involves a relationship between $\delta \mathbf{x}(t)$, $\delta \mathbf{u}(t)$, and $\delta \mathbf{y}(t)$, it is natural to ask at this point how these quantities are related. The sought-for relationship can be obtained by Taylor series expansions which lead to the use of dynamic linearization ideas.

1) *Linearized Perturbation Model*: The deterministic model for our system is still employed (since we have no other!). Thus, we assume that the true control $\mathbf{u}(t)$, true state $\mathbf{x}(t)$, and true output $\mathbf{y}(t)$ are related by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad (9)$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t)). \quad (10)$$

Similarly, the ideal nominal control $\mathbf{u}_0(t)$, state $\mathbf{x}_0(t)$, and output $\mathbf{y}_0(t)$ are related by

$$\dot{\mathbf{x}}_0(t) = \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t)) \quad (11)$$

$$\mathbf{y}_0(t) = \mathbf{g}(\mathbf{x}_0(t)). \quad (12)$$

Expanding $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, and $\mathbf{g}(\mathbf{x}(t))$ about $\mathbf{x}_0(t), \mathbf{u}_0(t)$ in a Taylor series expansion we obtain

$$\begin{aligned} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) &= \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t)) + \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_0 \delta \mathbf{x}(t) \\ &+ \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_0 \delta \mathbf{u}(t) + \alpha_0(\delta \mathbf{x}(t), \delta \mathbf{u}(t)) \end{aligned} \quad (13)$$

$$\mathbf{g}(\mathbf{x}(t)) = \mathbf{g}(\mathbf{x}_0(t)) + \left. \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right|_0 \delta \mathbf{x}(t) + \beta_0(\delta \mathbf{x}(t)) \quad (14)$$

where $\alpha_0(\delta \mathbf{x}(t), \delta \mathbf{u}(t))$ and $\beta_0(\delta \mathbf{x}(t))$ denote the higher order terms in the Taylor series expansions.

From the preceding we readily deduce that

$$\delta \dot{\mathbf{x}}(t) = \mathbf{A}_0(t) \delta \mathbf{x}(t) + \mathbf{B}_0(t) \delta \mathbf{u}(t) + \alpha_0(\delta \mathbf{x}(t), \delta \mathbf{u}(t)) \quad (15)$$

$$\delta \mathbf{y}(t) = \mathbf{C}_0(t) \delta \mathbf{x}(t) + \beta_0(\delta \mathbf{x}(t)). \quad (16)$$

In the preceding,

$$\mathbf{A}_0(t) \triangleq \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_0 \triangleq \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\substack{\mathbf{x}_0(t) \\ \mathbf{u}_0(t)}} \quad (17)$$

is an $n \times n$ time-varying matrix which is obtained by evaluating the elements of the Jacobian matrix $\partial \mathbf{f} / \partial \mathbf{x}$ along the known (precomputed) time functions $\mathbf{x}_0(t)$ and $\mathbf{u}_0(t)$.

$$\mathbf{B}_0(t) \triangleq \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_0 \triangleq \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{\substack{\mathbf{x}_0(t) \\ \mathbf{u}_0(t)}} \quad (18)$$

is an $n \times m$ time-varying matrix which is obtained by evaluating the elements of the Jacobian matrix $\partial \mathbf{f} / \partial \mathbf{u}$ along the known (precomputed) time functions $\mathbf{x}_0(t)$ and $\mathbf{u}_0(t)$.

$$\mathbf{C}_0(t) \triangleq \left. \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right|_0 \triangleq \left. \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right|_{\mathbf{x}_0(t)} \quad (19)$$

is an $r \times n$ time-varying matrix which is obtained by evaluating the elements of the Jacobian matrix $\partial \mathbf{g} / \partial \mathbf{x}$ along the known desired state $\mathbf{x}_0(t)$. Equations (15) and (16), including the higher order terms, represent the exact⁴ relationship between $\delta \mathbf{x}(t)$, $\delta \mathbf{u}(t)$, and $\delta \mathbf{y}(t)$.

The linearized perturbation model is obtained by setting the higher order terms equal to zero in (15) and (16) to obtain

$$\delta \dot{\mathbf{x}}(t) = \mathbf{A}_0(t) \delta \mathbf{x}(t) + \mathbf{B}_0(t) \delta \mathbf{u}(t) \quad (20)$$

$$\delta \mathbf{y}(t) = \mathbf{C}_0(t) \delta \mathbf{x}(t) \quad (21)$$

which is a standard state description of a linear time-varying system.

Remark: The linear perturbation model (20) and (21) represents only an approximate relationship between $\delta \mathbf{x}(t)$, $\delta \mathbf{u}(t)$, and $\delta \mathbf{y}(t)$, while (15) and (16) represent an exact model.

2) *Justification of the Quadratic Criterion*: As we have indicated before, the modeling aspects of a problem represent an extremely important part of the design process.

³ We make a distinction between a controller and a compensator for reasons that will become obvious in the sequel (the compensator will include a controller).

⁴ Assuming that $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ and $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t))$ are exact!

The type of model is up to the engineer; its relative accuracy is not of primary importance as long as one knows what are the effects of the approximations to be made. Up to this point, the fact that the mathematical model $\dot{x}(t) = f(x(t), u(t))$ was only an approximation to reality forced us to introduce feedback and to seek the feedback controller.

At this stage, we are also faced with a similar problem. The design engineer may wish to use the approximate linear perturbation model (20) and (21) rather than the more accurate nonlinear model (15) and (16). It is really up to him to do so, provided he can anticipate the effects of this choice upon the overall design.

The fact that must be kept in mind is that one cannot simply ignore the higher order terms and hope that they are indeed going to be small.

In order to trust the validity of the linear model, the engineer must guarantee that the higher order terms $\alpha_0(\delta x(t))$, $\beta_0(\delta x(t))$ and $\beta_0(\delta u(t))$ are indeed small for all $t \in [t_0, T]$.

To see how this philosophy leads to the use of quadratic criteria and the linear-quadratic optimal control problem, it becomes necessary to examine in more detail the higher order terms.

If we use Taylor's theorem which allows us to truncate a Taylor series at an arbitrary point we can represent exactly the higher order terms as follows:

$$\alpha_0(\delta x(t), \delta u(t)) = 1/2 \sum_{i=1}^n \phi_i \delta x'(t) \left[\frac{\partial^2 f_i(\cdot)}{\partial x^2(t)} \right]_{\sim} \delta x(t) + \delta u'(t) \frac{\partial^2 f_i(\cdot)}{\partial u^2(t)} \Big|_{\sim} \delta u(t) + 2\delta x'(t) \frac{\partial^2 f_i(\cdot)}{\partial x(t)\partial u(t)} \Big|_{\sim} \delta u(t) \quad (22)$$

$$\beta_0(\delta x(t)) = 1/2 \sum_{i=1}^n \phi_i \left[\delta x'(t) \frac{\partial^2 g_i(\cdot)}{\partial x^2(t)} \Big|_{\sim} \delta x(t) \right] \quad (23)$$

where ϕ_i are the natural basis vectors in R_n (i.e., $\phi_1' = [10 \dots 0]$) and the several second-derivative (Hessian) matrices are evaluated at functions $\tilde{x}(t)$, $\tilde{u}(t)$ which are in general different than $x_0(t)$ and $u_0(t)$; the values of $\tilde{x}(t)$ and $\tilde{u}(t)$ are, of course, not provided by Taylor's theorem.

The advantage of viewing the higher order terms in this context is that one can readily see that they are quadratic in $\delta x(t)$ and $\delta u(t)$. It is also clear that they involve certain unknown parameters since we do not know what $\tilde{x}(t)$ and $\tilde{u}(t)$ are!

This approach now leads to the following philosophy: to trust the validity of the linear model, one should select $\delta u(t)$ such that

$$\int_{t_0}^T \|\alpha_0(\delta x(t), \delta u(t))\| dt = \text{minimum} \quad (24)$$

$$\int_{t_0}^T \|\beta_0(\delta x(t))\| dt = \text{minimum.} \quad (25)$$

Since $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ are quadratic in $\delta x(t)$ and $\delta u(t)$, one

way of guaranteeing this is to select $\delta u(t)$ so that the standard quadratic cost functional⁵

$$J_0 = \delta x'(T) F_0 \delta x(T) + \int_{t_0}^T [\delta x'(t) Q_0(t) \delta x(t) + \delta u'(t) R_0(t) \delta u(t)] dt \quad (26)$$

is minimized, where F_0 and $Q_0(t)$ are symmetric, at least positive semidefinite, matrices and $R_0(t)$ is a symmetric positive definite matrix.

The weighting matrices $Q_0(t)$ and $R_0(t)$ are selected⁶ by the engineer as an upper bound to the effects of the second-derivative matrices in (22) and (23); the matrix F_0 and the terminal penalty cost $\delta x'(T) F_0 \delta x(T)$ are often included to insure that the $\delta x(t)$ stay near zero near the terminal time when the current actions of $\delta u(t)$ are not as strongly felt (since they take time to excite the system).

We can see that the state-dependent part $[\delta x'(T) F_0 \delta x(T) + \delta x'(t) Q_0(t) \delta x(t)]$ of the quadratic cost functional are consistent with the control objective of Section III-E which was to keep $\delta x(t)$ small. The difference is that the vague "smallness" requirement has been translated into something very specific, namely, to a quadratic penalty on the state deviations $\delta x(t)$ from their desired zero values.

The preceding arguments have hopefully communicated to the reader the notion that *quadratic criteria can be used to keep a linear model as honest as possible*. If the designer loved to work with nonlinear differential equations that were quadratic, then the Taylor series should have been terminated at the cubic terms and a cubic criterion should have been used to validate the quadratic model. Since the linear-quadratic problem has a nice solution, it may not be necessary to increase the complexity of the perturbation differential equation model further than the linear one.

G. Formal Statement and Solution of the Deterministic Linear-Quadratic Problem

Using the preceding philosophy (i.e., keeping our linearized model honest) we have arrived at the following precise mathematical optimization problem.

1) *Deterministic Linear-Quadratic Problem:* Given the linear deterministic time-varying system

$$\dot{\delta x}(t) = A_0(t) \delta x(t) + B_0(t) \delta u(t) \quad (27)$$

and given a fixed time interval of interest $t \in [t_0, T]$. Find the control perturbation vector $\delta u(t)$, $t \in [t_0, T]$, such that the following deterministic quadratic cost functional is minimized:

⁵ One could also include a cross term of the form $\delta x'(t) M_0(t) \delta u(t)$ in the integrand of (26). This causes no difficulty in its solution (see, for example, Bryson and Ho, [6, ch. 5, pp. 165-166]).

⁶ The selection of the weighting matrices will be discussed in Section III-I.

$$J_0 = \delta x'(T)F_0\delta x(T) + \int_{t_0}^T [\delta x'(t)Q_0(t)\delta x(t) + \delta u'(t)R_0(t)\delta u(t)] dt \quad (28)$$

where

$$F_0 = F_0' \geq 0 \quad (n \times n \text{ matrix}) \quad (29)$$

$$Q_0(t) = Q_0'(t) \geq 0, \quad \text{for all } t \in [t_0, T] \quad (n \times n \text{ matrix}) \quad (30)$$

$$R_0(t) = R_0'(t) > 0, \quad \text{for all } t \in [t_0, T] \quad (m \times m \text{ matrix}). \quad (31)$$

2) *Solution of the Linear-Quadratic Problem:* The optimal control perturbation vector $\delta u(t)$ is related to the state perturbation vector $\delta x(t)$ by means of the linear time-varying feedback relationship

$$\delta u(t) = -G_0(t)\delta x(t) \quad (32)$$

where $G_0(t)$ is an $m \times n$ time-varying control gain matrix. The value of $G_0(t)$ is given by

$$G_0(t) = R_0^{-1}(t)B_0'(t)K_0(t) \quad (33)$$

where the $n \times n$ matrix $K_0(t)$ is the solution of the Riccati matrix differential equation

$$\frac{d}{dt}K_0(t) = -K_0(t)A_0(t) - A_0'(t)K_0(t) - Q_0(t) + K_0(t)B_0(t)R_0^{-1}(t)B_0'(t)K_0(t) \quad (34)$$

subject to the boundary condition at the terminal time T

$$K_0(T) = F_0. \quad (35)$$

3) *Methods of Proof:* There are several ways of proving the preceding result. One way is using the maximum principle and by subsequent manipulation of the necessary conditions (see, for example, [5, pp. 756-763], [6, pp. 148-152], and the original work by Kalman [8]). Another way is through the use of the Hamilton-Jacobi-Bellman partial differential equation (see, for example, [5, pp. 764-766], [6, pp. 152-153], [9], and [10, pp. 23-28]). Another method is related to completing squares (see, for example, [1, pp. 130-141]). Another solution method [12] assumes that the optimal control is linear and of the form (32) and carries out a parameter optimization to determine the matrix $G_0(t)$. The result can also be obtained through the use of the projection theorem (see, for example, [13]).

H. Discussion

The solution of the deterministic linear-quadratic problem provides us with a deterministic feedback design that attempts to null out deviations of the true state $x(t)$ from its ideal response $x_0(t)$. Fig. 6 shows the structure of this overall system.

From a practical viewpoint this deterministic design is appealing because the control gain matrix $G_0(t)$ can be precomputed completely. To stress this point we outline the steps that must be followed.

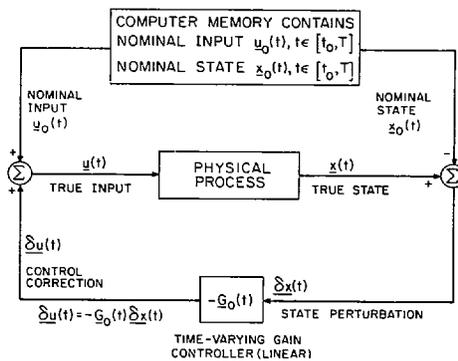


Fig. 6. Structure of deterministic feedback control system for general time-varying case. Note that $u_0(t)$, $x_0(t)$, and control gain matrix $G_0(t)$ are precomputable. Underlined symbols indicate bold-face.

1) Modeling:

Step 1: The engineer arrives at a deterministic model $\dot{x}(t) = f(x(t), u(t))$.

Step 2: The engineer determines the time functions $u_0(t)$ and $x_0(t)$ for all $t \in [t_0, T]$ (for example, by selecting the cost functional (4) and solving the resultant optimal control problem).

Step 3: The engineer selects the weighting matrices F_0 , $Q_0(t)$, $R_0(t)$ in the quadratic criterion (28).

2) Off-Line Computations:

Step 4: From the equations selected in Steps 1 and 2 one computes the matrices $A_0(t)$ and $B_0(t)$ according to (17) and (18); at this step the linear system (27) is determined.

Step 5: The matrices $A_0(t)$, $B_0(t)$, $Q_0(t)$, $R_0(t)$ are the coefficients of the Riccati equation (34), and F_0 is the boundary condition (35). Hence, one can numerically integrate the Riccati equation (34), backward in time, to obtain the matrix $K_0(t)$ for all $t \in [t_0, T]$ and $G_0(t)$ by (33).

3) On-Line Computations (see Fig. 6):

Step 6: Measure the true state $x(t)$; subtract $x_0(t)$ (precomputed in Step 2 from $x(t)$ to find $\delta x(t)$).

Step 7: Here we compute on-line $\delta u(t)$ by $\delta u(t) = -G_0(t)\delta x(t)$; only a matrix-vector multiplication is required in real time, since $G_0(t)$ has been precomputed in Step 5.

Step 8: Compute the true control input $u(t)$ by $u(t) = u_0(t) + \delta u(t)$ (recall that $u_0(t)$ was precomputed in Step 2).

Apart from the obvious difficulties that naturally occur in the modeling part of this design process (Steps 1, 2, and 3), the only practical deficiency of this scheme is associated with the fact that we cannot measure the true state vector $x(t)$ (see Step 6). This fact alone provides us with sufficient motivation to examine the stochastic aspects of the problem in Section IV.

I. Selection of the Weighting Matrices $Q_0(t)$, $R_0(t)$, F_0

The selection of the weighting matrices in the quadratic criterion (26) is not a simple matter. Usually they are selected by the designer on the basis of engineering experi-

ence coupled with alternate simulation runs for different trial values. There is no universal agreement on precisely how these are to be selected for any given application; in the design of classes of aerospace systems several workers have developed rules of thumb on the relative values of the elements of these weighting matrices.

In most practical applications F_0 , $Q_0(t)$, and $R_0(t)$ are selected to be diagonal. In this manner, specific components of the state perturbation vector $\delta x(t)$ and of the control perturbation vector $\delta u(t)$ can be penalized individually; it helps to have a physical set of state variables and control variables so that relative weightings can be rationally assigned.

From a pragmatic viewpoint one can develop certain qualitative properties which can help the designer in the choice of these important design parameters [these properties are deduced from the dependence of the Riccati equation upon F_0 , $Q_0(t)$, and $R_0(t)$].

1) The larger $\|F_0\|$, the larger the gain matrix $G_0(t)$ for values of time near the terminal time.

2) The larger $\|Q_0(t)\|$, the larger the gain matrix $G_0(t)$ and the faster the time during which state perturbations are reduced to small values. (One way of thinking about this is that an increase in $Q_0(t)$ results in an increase in the bandwidth of the closed-loop system).

3) The larger $\|R_0(t)\|$, the smaller the gain matrix $G_0(t)$ and the slower the system.

4) Often the state perturbation vector contains variables and their time derivatives (e.g., pitch error and pitch rate error). In general, if only output variables (e.g., pitch) are penalized by $Q_0(t)$, then the more oscillatory the response tends to be; to reduce overshoots one can penalize the derivatives of these variables by a different choice of $Q_0(t)$.

From the point of view of the justification of quadratic criteria and honesty of linearization the size of the state weighting matrix $Q_0(t)$ should somehow be proportional to estimates of the second-derivative matrices $\partial^2 f_i / \partial x^2(t)$ [see (22)], while the control weighting matrix $R_0(t)$ should be related in a proportional manner to the second-derivative matrices $\partial^2 f_i / \partial u^2(t)$. Estimates of these second-derivative matrices can often be obtained by evaluating them at the nominal values $u_0(t)$ and $x_0(t)$.

An alternate procedure has been suggested in the context of perturbation guidance or neighboring optimal control [6, pp. 177-197]. This approach is motivated by the fact that one can use the solution of an optimal control problem to determine the optimal control $u_0(t)$ and the optimal state $x_0(t)$ as outlined in Section III-C1 of this paper. The basic idea is to define the Hamiltonian function from (1) and (4)

$$H = H(x(t), p(t), u(t)) = L(x(t), u(t)) + p'(t)f(x(t), u(t))$$

where $p(t)$ is the costate vector.

Let us suppose then that we use the maximum principle to deduce the necessary conditions for optimality and then apply an iterative algorithm to solve the nonlinear two-point boundary-value problem. As we remarked in Section III-C, this procedure will yield the optimal control

$u_0(t)$ and state $x_0(t)$ for all $t \in [t_0, T]$. However, as a by-product, we also obtain the associated costate time function $p_0(t)$ for all $t \in [t_0, T]$.

The key idea behind the neighboring optimal control is to assume that the actual controls and states are somewhat different than the optimal ones. One then can substitute (6) and (7) into (4) and obtain the increase in the cost which is *approximately* measured by the second variation $\delta^2 J$ and given by

$$\delta^2 J = \delta x'(T) \frac{\partial^2 \phi}{\partial x(T)^2} \Big|_0 \delta x(T) + \int_{t_0}^T [\delta x'(t); \delta u'(t)] \begin{bmatrix} \frac{\partial^2 H}{\partial x(t)^2} \Big|_0 & \frac{\partial^2 H}{\partial x(t)\partial u(t)} \Big|_0 \\ \frac{\partial^2 H}{\partial u(t)\partial x(t)} \Big|_0 & \frac{\partial^2 H}{\partial u(t)^2} \Big|_0 \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt.$$

One then seeks the control $\delta u(t)$ which minimizes the second variation $\delta^2 J$ subject to the linear differential constraints (27). This leads to a linear-quadratic optimal control problem and can be viewed as another justification for quadratic criteria [6, p. 193].

In the preceding the second-derivative matrices of the Hamiltonian H

$$\frac{\partial^2 H}{\partial x(t)^2} \Big|_0, \quad \frac{\partial^2 H}{\partial x(t)\partial u(t)} \Big|_0, \quad \frac{\partial^2 H}{\partial u(t)\partial x(t)} \Big|_0, \quad \frac{\partial^2 H}{\partial u(t)^2} \Big|_0, \quad \frac{\partial^2 \phi}{\partial x(t)^2} \Big|_0$$

are all evaluated along $x_0(t)$, $p_0(t)$, $u_0(t)$, $t \in [t_0, T]$. Intuitively speaking, this approach attempts to minimize (to second order only!) increases in the cost functional. If we neglect the cross-coupling terms one could then make the association

$$F_0 \sim \frac{\partial^2 \phi}{\partial x(T)^2} \Big|_0$$

$$Q_0(t) \sim \frac{\partial^2 H}{\partial x(t)^2} \Big|_0$$

$$R_0(t) \sim \frac{\partial^2 H}{\partial u(t)^2} \Big|_0$$

However, there is *no* guarantee that these matrices enjoy any of the definiteness properties required for global existence and uniqueness of solutions to the linear-quadratic problem; if these definiteness assumptions are violated, then one may have to deal with singular problems (see [20]). Also, note that this philosophy neglects the contribution of the third, fourth, etc., variations in the cost; if these were going to be taken into account, then the partial derivatives of the Hamiltonian would have to be evaluated *not* at $x_0(t)$, $p_0(t)$, $u_0(t)$, but at some other time functions that are *not* known (as it was the case with the approach of keeping the linearizations honest). Nonetheless, this approach can often give the designer some clue as to the way these weighting matrices should be selected (see also [21]).

J. Time-Invariant Version

There is a subclass of control problems that can be attacked using the same philosophy and that lead to an even simpler realization, namely, involving the use of linear time-invariant gains in the feedback loops. We shall briefly outline this class of problems in the following paragraphs.

This class of problems is characterized by attempting to control a nonlinear time-invariant system about a constant equilibrium condition (or set point). A typical example is the design of an autopilot for an aircraft at steady flight (constant altitude, constant speed). For such problems, a repeat of the design philosophy leads to perturbation models with constant coefficients, and the use of an appropriate quadratic criterion leads to a constant gain linear feedback control system.

To be precise, let us suppose that the mathematical model of the physical process is, as before, described by the nonlinear differential equation

$$\dot{x}(t) = f(x(t), u(t)). \quad (36)$$

Let x_0 and u_0 be two constant vectors with the property that

$$f(x_0, u_0) = 0. \quad (37)$$

This implies that if the system (36) starts at the initial state $x(t_0) = x_0$ and if we apply the control

$$u(t) = u_0 = \text{constant for all } t \quad (38)$$

then (theoretically!)

$$x(t) = x_0 = \text{constant for all } t. \quad (39)$$

Under these conditions, one can interpret x_0 as the ideal equilibrium state of the process and u_0 as the ideal equilibrium input to the system. Thus, x_0 and u_0 play now the role of the time functions $x_0(t)$ and $u_0(t)$ of Section III-C.

The philosophy of control is the same as before. Application of u_0 itself will not be enough to have $x(t)$ stay at x_0 due to modeling errors. Hence, one needs to devise a feedback system that will indeed keep $x(t)$ near x_0 forever.

The philosophy of obtaining an approximate linear model and using a quadratic cost functional to keep the linearization honest can be repeated in this case. Thus, we define

$$\delta x(t) = x(t) - x_0 \quad (\text{state perturbation vector}) \quad (40)$$

$$\delta u(t) = u(t) - u_0 \quad (\text{control correction vector}). \quad (41)$$

The approximate linearized model is

$$\delta \dot{x}(t) = A_0 \delta x(t) + B_0 \delta u(t). \quad (42)$$

In this case, the matrices A_0 and B_0 are constant matrices; the reason is that they are defined as

$$A_0 \triangleq \left. \frac{\partial f}{\partial x} \right|_{x_0, u_0} \quad B_0 = \left. \frac{\partial f}{\partial u} \right|_{x_0, u_0}. \quad (43)$$

Clearly, these time-invariant Jacobian matrices are evaluated at the constant equilibrium pair x_0, u_0 .

If we use Taylor's theorem to truncate the Taylor

series at the quadratic term, then the second-derivative matrices [see (22)] are once more constant (but unknown). Thus, to minimize the effects of the approximations one is led to minimize the cost functional

$$\hat{J}_0 = \int_{t_0}^{\infty} [\delta x'(t) Q_0 \delta x(t) + \delta u'(t) R_0 \delta u(t)] dt \quad (44)$$

where Q_0 and R_0 are constant weighting matrices that must once more be selected by the designer. The terminal time is selected to be ∞ , because there is no natural final time associated with keeping the system near its ideal constant equilibrium condition.

In the cost functional the constant state weighting matrix Q_0 is selected to be symmetric and at least positive semidefinite; the constant control weighting matrix R_0 is selected to be symmetric and positive definite. Under these assumptions the value of \hat{J}_0 is nonnegative.

One can anticipate at this point that the optimal $\delta u(t)$ must lead to a stable system; heuristically, this is so because the minimum value of \hat{J}_0 must be finite. This implies that $\delta x(t) \rightarrow 0$ as $t \rightarrow \infty$. For this reason, a terminal weighting matrix F_0 [compare (44) with (28)] need not be included in this problem formulation.

1) *Formal Statement of the Optimization Problem:* Given the linear time-invariant system

$$\delta \dot{x}(t) = A_0 \delta x(t) + B_0 \delta u(t) \quad (45)$$

find $\delta u(t)$, $t \in [t_0, \infty]$ such that the quadratic cost functional

$$\hat{J}_0 = \int_{t_0}^{\infty} [\delta x'(t) Q_0 \delta x(t) + \delta u'(t) R_0 \delta u(t)] dt \quad (46)$$

is minimized. In (46) the constant matrices

$$Q_0 = Q_0' \geq 0; R_0 = R_0' > 0 \quad (47)$$

are selected by the engineer.

2) *Solution:* The optimal control correction vector $\delta u(t)$ is generated from the actual state perturbation vector $\delta x(t)$ by a linear constant gain feedback configuration

$$\delta u(t) = -\hat{G}_0 \delta x(t) \quad (48)$$

where \hat{G}_0 is a constant $m \times n$ feedback gain matrix

$$\hat{G}_0 = R_0^{-1} B_0' \hat{K}_0 \quad (49)$$

and \hat{K}_0 is a constant symmetric positive definite $n \times n$ matrix which is the solution of the algebraic matrix Riccati equation

$$0 = -K_0 A_0 - A_0' K_0 - Q_0 + K_0 B_0 R_0^{-1} B_0' K_0. \quad (50)$$

The structure of the time-invariant deterministic control system is shown in Fig. 7.

3) *Remarks:* The existence and uniqueness of solution for the preceding problem are guaranteed by the following assumptions: a) $[A_0, B_0]$ is a controllable pair; b) $[A_0, Q_0^{1/2}]$ is an observable pair. Under these assumptions the closed-loop system

$$\delta \dot{x}(t) = [A_0 - B_0 \hat{G}_0] \delta x(t)$$

is asymptotically stable in the large, i.e., all of the eigen-

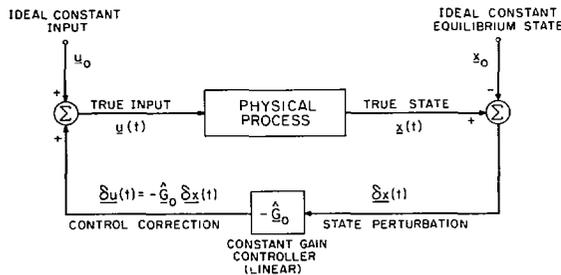


Fig. 7. Structure of deterministic feedback control system for time-invariant infinite-time case. Note that vectors $u_0(t)$, $x_0(t)$ are constant and precomputable; also gain matrix G_0 is constant and precomputable. Underlined symbols indicate boldface.

values of $[A_0 - B_0 G_0]$ are in the left-half complex plane.

The derivations of these results can be found in many places (see, for example, [5, pp. 771-776], [6, pp. 167-168], [14], [15], and [11]).

4) *Computation of \hat{K}_0* : There are many techniques available for computing \hat{K}_0 . One technique that brings into focus the relation between the Riccati matrix differential equation (34) and the algebraic Riccati equation (50) is the following.

The sought-for matrix \hat{K}_0 is given by

$$\hat{K}_0 = \lim_{t \rightarrow \infty} \bar{K}_0(\tau) \quad (51)$$

where $\bar{K}_0(\tau)$ is the solution of the matrix Riccati differential equation

$$\begin{aligned} \frac{d}{dt} \bar{K}_0(\tau) = & \bar{K}_0(\tau) A_0 + A_0' \bar{K}_0(\tau) + Q_0 \\ & - \bar{K}_0(\tau) B_0 R_0^{-1} B_0' \bar{K}_0(\tau) \end{aligned} \quad (52)$$

with the initial condition

$$\bar{K}_0(0) = 0. \quad (53)$$

Additional techniques are available that deal directly with the algebraic Riccati equation. References [16] to [19] represent typical approaches to this problem.

5) *Remarks*: If we examine Fig. 7 we note that there are no difficulties in storing x_0 , u_0 , and \hat{G}_0 since they are all constant and precomputable. From a practical viewpoint, the feedback controller can be an analog one, since the components of $\delta x(t)$ need only be multiplied by the constant gain elements of \hat{G}_0 .

The only practical disadvantage is, as before, that we require the exact measurement of the full state vector $x(t)$. This in general is not possible. For this reason we must proceed to the next step of the design process which deals with the development of techniques that yield a good estimate $\hat{x}(t)$ of the actual state $x(t)$.

IV. STOCHASTIC ESTIMATION ANALYSIS AND DESIGN (STEP 2)

A. Introduction

We have seen that even under the deterministic assumption we require a feedback controller to take care of errors in modeling. The main practical disadvantage of

the deterministic design step was that exact measurement of all state variables was necessary. This is seldom the case in practical applications.

Even if one could measure all of the state variables, one has to use physical sensors to carry out these measurements. Thus, this uncertainty in measurement must somehow be taken into account.

In addition, although the deterministic approach admitted errors in modeling (necessitating feedback) it did not explicitly take into account errors introduced by the actuators; furthermore, it did not take into account that in many practical applications there are disturbance inputs acting upon the physical process, which are not generated by the control system (e.g., wind gusts acting upon an airplane).

In this section we shall present the common means by which such "input" and "sensor" errors are introduced in the design process, and how they affect the generation of an estimate $\hat{x}(t)$ of the true state vector $x(t)$, through the use of the Kalman-Bucy filter. Towards this goal we present in Section IV-B some philosophical remarks pertaining to the use of white noise to model uncertainties in the design process. In Section IV-C we formulate the combined modeling problem of using linearized dynamics and white Gaussian noise. In Section IV-D we discuss the linear-Gaussian estimation problem and state its solution via the Kalman-Bucy filter. Finally, in Section IV-E we discuss the steady-state Kalman-Bucy filter which has great appeal from a practical implementation viewpoint.

B. Use of White Noise

It is common engineering practice to use a probabilistic approach to the modeling and implications of physical uncertainty. The reason is that a probabilistic approach is characterized by the existence of an extensive mathematical theory which has been already developed. Alternate approaches to uncertainty (e.g., via fuzzy sets, bounded but unknown uncertainty) have not as yet reached, from a mathematical viewpoint, the theoretical sophistication of the probabilistic approach.

In the design of dynamical systems the continuous existence in time of plant disturbances and sensor errors is modeled by representing these uncertain time functions by means of random processes. For example, suppose that $n(t)$ is a random process which represents the "noise" that is introduced by a sensor at any time t . Hence, we can model sensor uncertainty by

$$z(t) = s(t) + n(t) \quad (54)$$

where at time t , $z(t)$ is the actual sensor measurement, $s(t)$ is the actual signal to be measured, and $n(t)$ is additive measurement noise.

The statistical properties of $n(t)$ in essence define the accuracy of the sensor at time t . At any time t_1 , the scalar $n(t_1)$ is viewed as a random variable. Its probability density function $p(n(t_1))$ summarizes the statistical knowledge at time t_1 . However, since $n(t)$ is associated with a particular sensor, one must also specify any statistical properties of

the random variables $n(t_1)$ and $n(t_2)$ at any two distinct instants of time t_1 and t_2 . Such statistical information is specified by the joint probability density function $p(n(t_1), n(t_2))$ of the random variables $n(t_1)$ and $n(t_2)$.

If $n(t_1)$ and $n(t_2)$ are dependent, then from Bayes' rule we have

$$p(n(t_2)/n(t_1)) = \frac{p(n(t_1), n(t_2))}{p(n(t_1))} \quad (55)$$

which loosely implies that if we have observed $n(t_1)$ then we can say something about $n(t_2)$, e.g., estimate its average value, before we actually measure $n(t_2)$.

If on the other hand, $n(t_1)$ and $n(t_2)$ are independent (uncorrelated in the Gaussian case), then

$$p(n(t_1), n(t_2)) = p(n(t_1)) p(n(t_2)) \quad (56)$$

and Bayes' rule yields

$$p(n(t_2)/n(t_1)) = p(n(t_2)) \quad (57)$$

which means that the fact that we have already observed $n(t_1)$ does not help us at all to improve our knowledge about $n(t_2)$.

These properties have significant implications from the point of view of the hardware and software that we have to utilize in our control system. If a sensor noise is modeled as a correlated random process, then we must expect some sort of estimation algorithm (based on (55)) which attempts to guess properties of future values of sensor noise based upon past measurements. If this can be done (at the expense of, perhaps excessive, on-line computation) one can expect an improved "noise removing filter."

If on the other hand, we model the noise $n(t)$ as "uncorrelated," then past measurements do not help us at all in future guessing. In this case, the noise is *unpredictable* and no estimation algorithm that attempts to guess future values of the noise is required (and no on-line computations are required in this respect).

The preceding discussion dealt with the time structure of the noise $n(t)$. Let us return to the statistical properties of the noise at any instant of time t_1 . As we mentioned before, this statistical information is contained in the probability density function $p(n(t_1))$. It is well known that two important statistical parameters (from an engineering viewpoint) are the mean

$$E\{n(t_1)\} \triangleq \bar{n}(t_1) \quad (58)$$

and the variance

$$\text{var}[n(t_1)] = E\{(n(t_1) - \bar{n}(t_1))^2\}. \quad (59)$$

The mean $\bar{n}(t_1)$ is what we would expect to see on the average. The variance helps us understand how much this average $\bar{n}(t_1)$ is to be believed. A large variance means that the actual value $n(t_1)$ (in any given experiment) may be way off (with a large probability) from its mean value. A small variance means that the mean is a pretty good guess.

1) *Meaning of Continuous Time White Noise:* If we

model the random process $n(t)$ as continuous time white noise, then this corresponds in viewing the uncertainty as being the most unpredictable possible one. In other words, almost nothing can be said about white noise.

From a mathematical viewpoint a continuous time white noise process is a Gaussian one, and hence its density function is uniquely defined by its mean and variance. A Gaussian white noise at any time t_1 is a random variable with infinite variance (this means that its mean value is almost useless for guessing purposes). Furthermore, the random variables $n(t_1)$ and $n(t_2)$, $t_1 \neq t_2$, are independent for all t_1, t_2 (hence, past measurements do not help in future prediction).

Thus, if an engineer models a physical uncertainty by means of white noise, then (loosely speaking) he is communicating to the mathematics that he is relatively ignorant about the uncertainty. In particular, he is preventing the mathematics from requiring any second-guessing estimation algorithm that contributes to the complexity of the system.

Of course, white noise is a mathematical fiction. The engineer usually knows much more about the uncertainty than he may care to communicate to the mathematics. He may choose to do so for two main reasons: a) to carry out a worse case type design (because of the infinite variance); b) to minimize future guessing by complex estimation algorithms.

It is the opinion of the author that the use of white noise in control system design is primarily a modeling issue. The engineer has to make a judgment on how to model uncertainties via white noise. There are not available cook-book procedures for doing this; the success of the design depends on the ability of the engineer to know the physics of his problem and to subjectively translate this into mathematical probabilistic models. We shall comment on these problems in Section IV-G in some more detail.

2) *Mathematical Description of White Noise:* The mathematical specification of white noise is as follows.

Let $\mathbf{n}(t)$ be a vector-valued white noise process. Then $\mathbf{n}(t)$ is Gaussian with mean

$$E\{\mathbf{n}(t)\} = \mathbf{0}, \quad \text{for all } t \quad (60)$$

and covariance matrix

$$\text{cov}[\mathbf{n}(t); \mathbf{n}(\tau)] = E\{\mathbf{n}(t)\mathbf{n}'(\tau)\} = \mathbf{N}(t)\delta(t - \tau) \quad (61)$$

where

$$\mathbf{N}(t) = \mathbf{N}'(t) \geq \mathbf{0}. \quad (62)$$

We shall call $\mathbf{N}(t)$ the *intensity matrix* of the white noise; essentially it governs the magnitude of the delta function in (61). If $\mathbf{N}(t) = \mathbf{N} = \text{constant}$ for all t , then we deal with stationary white noise. In this case, one can give an alternate interpretation to white noise by means of its spectral density function. If $\mathbf{n}(t)$, and hence \mathbf{N} , are scalars, then its spectral density is equal to N for all frequencies (constant power for all frequencies) and hence a continuous white noise process is characterized by infinite average

power (see [22, pp. 109–115], [23, pp. 81–85], [24, pp. 29–32]). Thus, the value of N governs the strength of the white noise process in terms of the power level at each frequency; the larger the value of N , the stronger the noise and the more pronounced are its effects upon a system.

C. Stochastic Modeling for Control System Design

Let us now return to the modeling issues associated with the control system design problem.

1) *Actuator-Plant-Input Disturbance Models:* Recall that in the deterministic version the relation of the true command input to the actuator $\mathbf{u}(t)$ and that the true plant state $\mathbf{x}(t)$ were related by the deterministic model (see (1))

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)). \tag{63}$$

In the stochastic case we can model⁷ the actuator-plant input disturbance part of our physical process by the stochastic differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) + \boldsymbol{\xi}(t) \tag{64}$$

where $\boldsymbol{\xi}(t)$ is a continuous time white noise process. The addition of $\boldsymbol{\xi}(t)$ to the otherwise deterministic model (63) implies that the engineer is communicating to the mathematics one or more of the following “facts of life”: a) that there are additional stochastic disturbances that drive the system; b) that the deterministic equations may be in error due to oversimplification; c) that some of the parameters in $\mathbf{f}(\cdot, \cdot)$ may not be exact (true parameters may vary from their nominal values); d) that the actuators introduce errors.

If we examine (64) we can see that since $\boldsymbol{\xi}(t)$ is white, then $\dot{\mathbf{x}}(t)$ can change, in part, in an unpredictable way. The deterministic part of the equation, $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, represents the contribution of our completely predictable model; the stochastic part of the equation $\boldsymbol{\xi}(t)$ stresses the unpredictable element of the real world. Loosely speaking, the use of (64) is a way of saying to the mathematics “watch out! The deterministic equation is in error, but I will not tell you the structure of the error, so that you will not try to second-guess it in the future.”

2) *Sensor Modeling:* Recall that in the deterministic case the type of sensors that could be used led to the definition of the output vector $\mathbf{y}(t)$ whose components were the signals that could be measured by the available sensors. The deterministic model was [see (2)]

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t)) \tag{65}$$

The simplest way of modeling sensor errors is to assume that the sensor that measures the output variable $y_i(t)$ yields the measurement (data) signal $z_i(t)$ which equals $y_i(t)$ and additive white noise $\theta_i(t)$

$$z_i(t) = y_i(t) + \theta_i(t) \tag{66}$$

or, in vector notation,

$$\mathbf{z}(t) = \mathbf{y}(t) + \boldsymbol{\theta}(t) = \mathbf{g}(\mathbf{x}(t)) + \boldsymbol{\theta}(t) \tag{67}$$

where $\boldsymbol{\theta}(t)$ is vector-valued white noise. Once more, sensor errors are modeled by the worst possible type of noise.

3) *Initial Uncertainty:* In the deterministic context we assume that the initial state of the plant $\mathbf{x}(t_0) \triangleq \mathbf{x}_0$ was known. Since the state variables cannot be measured, we can no longer make this assumption. The simplest way of modeling this is to view the initial state \mathbf{x}_0 as a vector-valued Gaussian random variable. Its mean and covariance matrix represent our *a priori* information about the initial conditions of our plant.

4) *Statistical Description:* We can see that the uncertainty in the overall physical process has been modeled in three separate parts.

- a) Initial uncertainty: The initial state \mathbf{x}_0 is viewed as a random variable.
- b) Plant uncertainty: The system is driven by the white noise $\boldsymbol{\xi}(t)$ which implies that $\dot{\mathbf{x}}(t)$ has an unpredictable component.
- c) Measurement uncertainty: The output vector is corrupted by the additive white noise $\boldsymbol{\theta}(t)$, so that the measurement vector $\mathbf{z}(t)$ has an unpredictable component.

The quantitative description of this uncertainty is as follows.

The initial state vector is Gaussian with known mean and covariance matrix, i.e.,

$$E\{\mathbf{x}_0\} \triangleq \bar{\mathbf{x}}_0 \quad (\text{assumed known}) \tag{68}$$

$$\text{cov}[\mathbf{x}_0; \mathbf{x}_0] \triangleq E\{(\mathbf{x}_0 - \bar{\mathbf{x}}_0)(\mathbf{x}_0 - \bar{\mathbf{x}}_0)'\} = \boldsymbol{\Sigma}_0 \tag{69}$$

(assumed known) $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_0' \geq \mathbf{0}$.

The plant driving noise $\boldsymbol{\xi}(t)$ is white, Gaussian, with zero mean and known covariance matrix for all $t \geq t_0$, i.e.,

$$E\{\boldsymbol{\xi}(t)\} = \mathbf{0}, \quad \text{for all } t \geq t_0 \tag{70}$$

$$\text{cov}[\boldsymbol{\xi}(t); \boldsymbol{\xi}(\tau)] \triangleq E\{\boldsymbol{\xi}(t)\boldsymbol{\xi}'(\tau)\} \triangleq \boldsymbol{\Xi}(t)\delta(t - \tau) \tag{71}$$

$$\boldsymbol{\Xi}(t) = \boldsymbol{\Xi}'(t) \geq \mathbf{0}, \quad \text{for all } t \geq t_0 \quad (\text{assumed known}).$$

The measurement noise $\boldsymbol{\theta}(t)$ is white, Gaussian, with zero mean and known covariance matrix for all $t \geq t_0$, i.e.,

$$E\{\boldsymbol{\theta}(t)\} = \mathbf{0}, \quad \text{for all } t \geq t_0 \tag{72}$$

$$\text{cov}[\boldsymbol{\theta}(t); \boldsymbol{\theta}(\tau)] \triangleq E\{\boldsymbol{\theta}(t)\boldsymbol{\theta}'(\tau)\} = \boldsymbol{\Theta}(t)\delta(t - \tau) \tag{73}$$

$$\boldsymbol{\Theta}(t) = \boldsymbol{\Theta}'(t) > \mathbf{0}, \quad \text{for all } t \geq t_0 \text{ (known)}.$$

Furthermore, one usually assumes that \mathbf{x}_0 , $\boldsymbol{\xi}(t)$, and $\boldsymbol{\theta}(\tau)$ are mutually independent, i.e.,

$$\text{cov}[\mathbf{x}_0; \boldsymbol{\xi}(t)] = \mathbf{0}, \quad \text{for all } t \geq t_0 \tag{74}$$

$$\text{cov}[\mathbf{x}_0; \boldsymbol{\theta}(t)] = \mathbf{0}, \quad \text{for all } t \geq t_0 \tag{75}$$

$$\text{cov}[\boldsymbol{\xi}(t); \boldsymbol{\theta}(\tau)] = \mathbf{0}, \quad \text{for all } t, \tau \geq t_0. \tag{76}$$

⁷ This is the simplest possible model; more complex models have also been considered, e.g.,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{H}(\mathbf{x}(t), \mathbf{u}(t))\boldsymbol{\xi}(t)$$

where $\boldsymbol{\xi}(t)$ is white noise or alternate representations using Wiener processes (see [22, pp. 149–169], [23, pp. 93–140], [24, pp. 44–86]).

This assumption is reasonable in most physical applications.

We shall discuss later the selection of the intensity matrices $\Xi(t)$ and $\Theta(t)$, which govern the "strength" of the white noise processes $\xi(t)$ and $\theta(t)$, respectively, as well as of the initial covariance matrix Σ_0 .

5) *Linearized Dynamics Modeling*: Let us recall that one of the byproducts of the deterministic analysis was to specify an ideal deterministic time function $u_0(t)$, an ideal deterministic state response $x_0(t)$, and an ideal output response $y_0(t)$ for all $t \in [t_0, T]$. Our control system objective was to augment $u_0(t)$ by the control correction vector $\delta u(t)$ so that the commanded control $u(t) = u_0(t) + \delta u(t)$ had the property that the state deviation vector $\delta x(t) = x(t) - x_0(t)$ was small for all $t \in [t_0, T]$.

Our control objective has not changed except that $x(t)$, $u(t)$, $\delta x(t)$, and $\delta u(t)$ are now random processes rather than deterministic. We would still expect to generate $\delta u(t)$ by means of some feedback arrangement which is based on the actual sensor measurements $z(t)$.

Let us also recall that associated with the ideal state response $x_0(t)$ we had an ideal measurement vector $y_0(t)$ [see (4)] and an output perturbation vector $\delta y(t) = y(t) - y_0(t)$ [see (7)].

Since our measurement vector is given by $z(t) = y(t) + \theta(t)$, then

$$z(t) = y_0(t) + \delta y(t) + \theta(t). \quad (77)$$

Arbitrarily we define

$$z(t) \triangleq z_0(t) + \delta z(t) \quad (78)$$

where

$$z_0(t) \triangleq y_0(t) = g(x_0(t)) \quad (79)$$

and

$$\delta z(t) = \delta y(t) + \theta(t). \quad (80)$$

Note that $z_0(t)$ is a deterministic precomputable quantity (see Fig. 8). Hence $\delta z(t)$ can be evaluated.

A repeat of the development of the Taylor series expansions about $x_0(t)$, $u_0(t)$, $y_0(t)$, using the stochastic models leads to the following set of equations:

$$\delta \dot{x}(t) = A_0(t)\delta x(t) + B_0(t)\delta u(t) + \xi(t) + \alpha_0(\delta x(t), \delta u(t)) \quad (81)$$

$$\delta y(t) = C_0(t)\delta x(t) + \beta_0(\delta x(t)) \quad (82)$$

$$\delta z(t) = C_0(t)\delta x(t) + \theta(t) + \beta_0(\delta x(t)). \quad (83)$$

In (81)–(83) note the following. a) The matrices $A_0(t)$, $B_0(t)$, $C_0(t)$ are still given by (17), (18), and (19), respectively. They are deterministic and precomputable. b) The vectors $\alpha_0(\cdot, \cdot)$ and $\beta_0(\cdot)$ represent the effects of the quadratic and higher order terms; they are stochastic processes, since at least $\delta x(t)$ is a stochastic process (for partial justification see, for example, [25] and [23, pp. 336–340]). Once more we define the linearized stochastic (approximate) model by

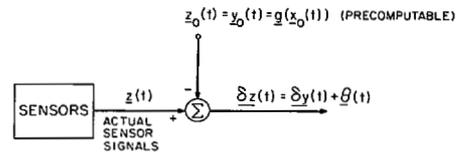


Fig. 8. On-line generation of signal $\delta z(t)$ from actual sensor signal $z(t)$ and precomputed ideal output $y_0(t)$. Underlined symbols indicate boldface.

$$\delta \dot{x}(t) = A_0(t)\delta x(t) + B_0(t)\delta u(t) + \xi(t) \quad (84)$$

$$\delta z(t) = C_0(t)\delta x(t) + \theta(t) \quad (85)$$

simply by ignoring $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ in (81) and (83), respectively.

To be sure, (84) and (85) represent only approximations to (81) and (83). However, both equations contain the white noise driving term which at least is a "flag" to the mathematics that the linearized equations are "in error."

We can now see even more clearly the role of the white noises in modeling. Up to this point, the noise $\xi(t)$ could be used to model input uncertainties and deterministic modeling errors. Now we see that it can also be used to model the fact that the higher order terms have been neglected in the use of (84) instead of (81). Thus, the choice of the covariance matrix for $\xi(t)$

$$\text{cov}[\xi(t); \xi(\tau)] = \Xi(t)\delta(t - \tau) \quad (86)$$

i.e., the value of $\Xi(t)$ selected by the designer, should incorporate his judgment on the importance of the higher order terms in the validity of the linearized model. Thus, the "more nonlinear" the system dynamics, the larger $\Xi(t)$ should be used.

The white noise $\theta(t)$ (assumed independent of $\xi(t)$) in the observation equation (85) plays a similar role. Not only should it reflect the inherent uncertainty of the measurements due to sensor inaccuracies, but it should also be used to model the implications of neglecting $\beta_0(\cdot)$ in (83) to obtain the linear equation (85). Since

$$\text{cov}[\theta(t); \theta(\tau)] = \Theta(t)\delta(t - \tau) \quad (87)$$

then the more nonlinear the output nonlinearity $g(x(t))$ is, the "larger" $\Theta(t)$ should be selected.

Admittedly, we are cheating in our quest for linear models. However, the use of white noises allows us to communicate to the mathematics our estimate of the "degree of cheating." This is extremely important because as we shall see in the next section we shall ask some very precise questions of the mathematics. If we ask precise but stupid questions, we shall get precise but stupid answers!

D. Estimation (Filtering) Problem

We have seen in Fig. 8 that we can construct the signal $\delta z(t)$ from the actual sensor measurement $z(t)$. The state perturbation $\delta x(t)$ is still the deviation of the actual state $x(t)$ from the desired ideal state response $x_0(t)$. However, $\delta x(t)$ cannot be measured directly; it is related, however, to the available signal $\delta z(t)$ by (85). The future evolution of $\delta x(t)$ can be influenced by the control correction vector $\delta u(t)$ according to (84).

We still want to keep $\delta x(t)$ small by selecting $\delta u(t)$. We have seen how this can be done in the deterministic case if $\delta x(t)$ were known. Since in this case $\delta x(t)$ is not directly available, then we can ask the following question.

Is it possible to generate a "good" estimate $\delta \hat{x}(t)$ of $\delta x(t)$, based on the measurements made up to time t , for any given time function $\delta u(t)$?

The Kalman-Bucy filter presents a precise way of obtaining such an estimate.

1) *Statement of the Filtering Problem:* Given the linear dynamic stochastic system

$$\frac{d}{dt} \delta x(t) = A_0(t) \delta x(t) + B_0(t) \delta u(t) + \xi(t) \quad (88)$$

and the linear stochastic measurement equation

$$\delta z(t) = C_0(t) \delta x(t) + \theta(t). \quad (89)$$

It is assumed that $A_0(t)$, $B_0(t)$, $C_0(t)$, $\delta u(t)$ are deterministic and known. It is assumed that the white noise $\xi(t)$ has the statistics specified by (70) and (71). It is also assumed that the white noise $\theta(t)$ has the statistics specified by (72) and (73). It can also be shown that $\delta x(t_0)$ is a Gaussian random vector with mean [see (68)]

$$E\{\delta x(t_0)\} \triangleq \bar{\delta x}_0 = \bar{x}_0 - x_0(t_0) \quad (\text{known}) \quad (90)$$

and covariance matrix Σ_0 [see (69)].

Given the measured signal $\delta z(\tau)$ for all $\tau \in [t_0, t]$.

Find a vector $\delta \hat{x}(t)$, an estimate of the true $\delta x(t)$ which is "optimal" in a well-defined statistical sense.

We remark that the linear-Gaussian nature of the hypotheses allows us to define a variety of optimization criteria (least squares, minimum variance, maximum likelihood, etc.). They all lead to the same answer. For example, one can show that the preceding assumptions imply that the *a posteriori* density function of $\delta x(t)$.

$$p(\delta x(t) | \delta z(\tau); \tau \in [t_0, t]) \quad (91)$$

is Gaussian and $\delta \hat{x}(t)$, as generated by the Kalman-Bucy filter, is its mean.

2) *The Kalman-Bucy Filter:* The optimal estimate $\delta \hat{x}(t)$ is generated by

$$\frac{d}{dt} \delta \hat{x}(t) = A_0(t) \delta \hat{x}(t) + B_0(t) \delta u(t) + H_0(t) [\delta z(t) - C_0(t) \delta \hat{x}(t)] \quad (92)$$

starting at the initial condition

$$\delta \hat{x}(t_0) = \bar{\delta x}_0 = \bar{x}_0 - x_0(t_0). \quad (93)$$

The filter gain matrix $H_0(t)$ is given by

$$H_0(t) = \Sigma_0(t) C_0'(t) \Theta^{-1}(t) \quad (94)$$

where $\Sigma_0(t)$ is the covariance matrix of the estimation error vector

$$\delta x(t) - \delta \hat{x}(t). \quad (95)$$

It turns out that

$$E\{\delta x(t) - \delta \hat{x}(t)\} = 0 \quad (96)$$

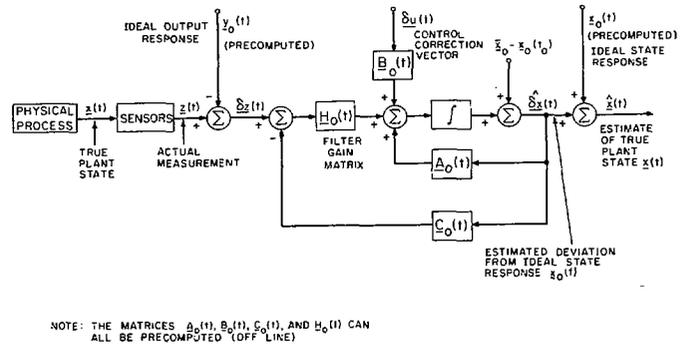


Fig. 9. Structure of linearized Kalman-Bucy filter for general time varying case. Underlined symbols indicate boldface.

so that

$$\Sigma_0(t) = E\{(\delta x(t) - \delta \hat{x}(t))(\delta x(t) - \delta \hat{x}(t))'\}. \quad (97)$$

Furthermore, the error covariance matrix $\Sigma_0(t)$ is the solution of the matrix Riccati differential equation

$$\frac{d}{dt} \Sigma_0(t) = A_0(t) \Sigma_0(t) + \Sigma_0(t) A_0'(t) + \Xi(t) - \Sigma_0(t) C_0'(t) \Theta^{-1}(t) C_0(t) \Sigma_0(t) \quad (98)$$

starting at the initial condition

$$\Sigma_0(t_0) = \Sigma_0 \quad (\text{initial covariance matrix of } x_0). \quad (99)$$

The structure of the linearized Kalman-Bucy filter is illustrated in Fig. 9. We remark that: a) $\Sigma_0(t)$ and hence $H_0(t)$ are precomputable so that all gains in the Kalman-Bucy filter can be predetermined; b) the actual estimate $\delta \hat{x}(t)$ must be generated on line as a function of the actual $\delta z(t)$; and c) as shown in Fig. 9, once $\delta \hat{x}(t)$ has been obtained, one can construct an estimate $\hat{x}(t)$ of the true plant state $x(t)$ by

$$\hat{x}(t) = x_0(t) + \delta \hat{x}(t) \quad (100)$$

where $x_0(t)$ is the precomputed ideal state response estimate.

E. Steady-State Kalman-Bucy Filter

Significant practical advantages occur if the following are true.

1) The linearized system defined by (88) and (89) is time invariant, i.e.,

$$\begin{aligned} A_0(t) &= A_0 = \text{constant matrix} \\ B_0(t) &= B_0 = \text{constant matrix} \\ C_0(t) &= C_0 = \text{constant matrix.} \end{aligned}$$

2) The noise statistics are stationary, i.e.,

$$\begin{aligned} \Xi(t) &= \Xi = \text{constant matrix} \\ \Theta(t) &= \Theta = \text{constant matrix.} \end{aligned}$$

3) The initial time t_0 , at which observations have started is in the distant past, i.e.,

$$t_0 \rightarrow -\infty.$$

We remark that the linearized system will be time invariant if (compare with Section III-J) one desires to control

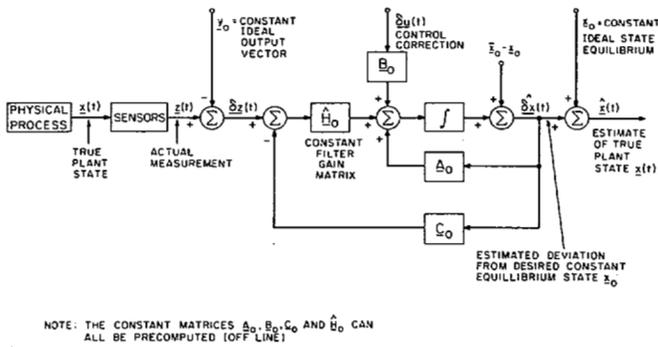


Fig. 10. Structure of steady-state Kalman-Bucy filter, which can be realized with time-invariant gain elements using an analog computer. Underlined symbols indicate boldface.

the nonlinear system about a constant equilibrium condition so that application of a constant u_0 yields a constant state x_0 and, hence, a constant output y_0 .

Under these assumptions the estimate $\delta\hat{x}(t)$ can be generated by the steady-state Kalman-Bucy filter

$$\frac{d}{dt} \delta\hat{x}(t) = A_0 \delta\hat{x}(t) + B_0 \delta u(t) + \hat{H}_0 [\delta z(t) - C_0 \delta\hat{x}(t)] \quad (101)$$

$$\delta\hat{x}(t_0) = \bar{x}_0 - x_0. \quad (102)$$

The filter gain matrix \hat{H}_0 is constant and is given by

$$\hat{H}_0 = \hat{\Sigma}_0 C_0' \Theta^{-1}. \quad (103)$$

The constant symmetric (at least) positive semidefinite matrix $\hat{\Sigma}_0$ is interpreted as the steady-state estimation error covariance matrix, and it satisfies the algebraic Riccati matrix equation

$$0 = A_0 \hat{\Sigma}_0 + \hat{\Sigma}_0 A_0' + \Xi - \hat{\Sigma}_0 C_0' \Theta^{-1} C_0 \hat{\Sigma}_0. \quad (104)$$

The constant matrix $\hat{\Sigma}_0$ can be computed by several methods.⁸ In particular,

$$\hat{\Sigma}_0 = \lim_{\tau \rightarrow \infty} \Sigma_0(\tau) \quad (105)$$

where

$$\begin{aligned} \frac{d}{d\tau} \Sigma_0(\tau) &= A_0 \Sigma_0(\tau) + \Sigma_0(\tau) A_0' + \Xi \\ &\quad - \Sigma_0(\tau) C_0' \Theta^{-1} C_0 \Sigma_0(\tau); \quad \Sigma_0(0) = 0. \end{aligned} \quad (106)$$

The structure of the steady-state Kalman-Bucy filter is shown in Fig. 10. Its major practical advantage is that it is characterized by constant gains and, hence, it can be readily realized via analog computer network elements.

F. Derivations of the Kalman-Bucy Filter

Since the original publications of Kalman [27] and Kalman and Bucy [28], there have been many different derivations of these results, each contributing to enhanced understanding to the advantages and shortcomings of these techniques, as well as extensions to the nonlinear

case. In this issue, [29]–[31] present alternate viewpoints and derivations. Often the derivations are obtained for the discrete time case and using limiting arguments the continuous time version can be obtained if proper care is exercised. Both probabilistic and deterministic derivations are available in the literature (see [29]).

G. Discussion

Most of the difficulties that are encountered with the Kalman-Bucy filter are primarily related to: 1) model mismatching (i.e., the model used in the implementation of the Kalman-Bucy filter is different than the physical process), and 2) correct selection of Σ_0 , and the white noise intensity matrices $\Xi(t)$ and $\Theta(t)$. In pure filtering situations these contribute to the so-called divergence of the Kalman-Bucy filter (see [32]). There are several analyses that have been carried out that considered the effects and implications of selecting the wrong covariance matrices (see, for example, [33, pp. 376–419], [34], and [35]). The existence of unknown biases in the noises $\xi(t)$ and $\theta(t)$ are not as troublesome since they can be estimated by an augmented Kalman-Bucy filter, [36], at the expense of introducing additional state variables. Some research efforts have been directed toward simultaneous estimation of the state variables and the intensity matrices (see [32] and [37]). Encouraging preliminary results have been obtained [38] for the elimination of bias errors due to model mismatching by integrating the residuals (innovations process). See also [44].

The sensitivity, and possible divergence, of the Kalman-Bucy filter is then intimately related to the modeling issues. If we view the (wrong) linearized model as a constraint, then the engineer can attempt to minimize the filter sensitivity by the judicious choice of the intensity matrices $\Xi(t)$ (or Ξ) and $\Theta(t)$ (or Θ). Considerable success has been obtained in certain classes of application problems (reentry vehicle tracking, orbit determination) by increasing the intensity matrix $\Xi(t)$ to compensate for modeling errors, which arise primarily in the dynamical equations. However, these techniques were developed only after extensive Monte-Carlo simulations and trial-and-error approaches. There is need for systematic approaches to this most important problem of selecting $\Xi(t)$ and $\Theta(t)$, and this represents an exciting research area.

Loosely speaking, the effect of increasing the magnitude of the intensity matrix $\Xi(t)$ (fake plant noise) results in larger values of the error covariance matrix $\Sigma_0(t)$ [see (98) and (104)] and this leads to an increase in the filter gain matrix $H_0(t)$ [see (94) and (103)]. Qualitatively speaking, the residuals are then weighted more (the filter is paying more attention to the actual measurements to compensate for errors in the *a priori* values of $A_0(t)$, $B_0(t)$, $\Xi(t)$, $\Theta(t)$ and Σ_0) and one obtains a high-gain filter. Thus, an increase in $\Xi(t)$ causes the filter to have a wider bandwidth. This bandwidth interpretation is useful since an increased $\Xi(t)$ means that the plant white noise $\xi(t)$ has more power and, hence, causes more "wiggles" in the actual state $x(t)$; the filter must estimate these wiggles in $x(t)$, and this requires higher bandwidth. Of course, a

⁸ The same techniques that can be used in the solution of the algebraic Riccati equation for the linear-quadratic problem [16]–[19] can also be used to solve (104). See also [26].

higher bandwidth passes more of the measurement noise $\theta(t)$ and this is the price that one must pay. Hence, the choice of distinct pairs of $\Xi(t)$ and $\Theta(t)$ by the designer can be interpreted as one way of controlling the filter bandwidth. In fact, it appears that the class of applications in which increased values of $\Xi(t)$ "cured" the sensitivity of the Kalman-Bucy filter were characterized by relatively accurate measurements [low values of $\Theta(t)$].

The preceding discussions point out the relative effects of using white versus colored noise in the modeling stage. If we model the plant uncertainties as colored noise (which may be more realistic since modeling errors are certainly not white), then we may get a better filter but at the expense of adding extra state variables in the dynamics. The issue of using colored measurement noise has been investigated (see, for example, [6, pp. 400-407], [33, pp. 333-346]); its accurate modeling will certainly yield better results than its replacement with white noise. However, in the majority of applications measurement noise is a wide-band process, while the bandwidth of the system is much smaller. Hence, in such applications, one would not expect too much improvement by the more accurate modeling of the measurement noise.

In short, there are no general techniques currently available that can be applied with confidence by the designer when he has to select the noise intensity matrices $\Xi(t)$ and $\Theta(t)$. Nonetheless, physical intuition, common sense, and off-line simulations represent effective tools that have been used to obtain excellent designs.

This brings us to a final word of caution. The *ad hoc* techniques that have been developed for decreasing the sensitivity of Kalman-Bucy filters do *not* necessarily carry over when the problem is one of stochastic control (in which the Kalman-Bucy filter is a subsystem in the compensator). Many of the sensitivity problems that arise in filtering can be traced to the lack of a valid trajectory for linearization purposes. In the control problem, one does have a much more valid trajectory, $u_0(t)$, $x_0(t)$, $y_0(t)$, on which to base the linearizations. The reason is that one would select the control to keep the system near its desired precomputed trajectory. Hence, even if a Kalman-Bucy filter is by itself relatively sensitive, this does not necessarily imply that, when it is used in the control problem (as part of the compensator), the closed-loop control system will be as sensitive. Intuitively speaking, in the latter problem there are many more feedback loops that help to reduce sensitivity. Thus, the selection of the matrices Σ_0 , $\Xi(t)$ and $\Theta(t)$ by the designer should depend on whether or not the problem is that of state estimation or stochastic control. Additional discussion on this point will be presented in the sequel.

V. STOCHASTIC CONTROL SYSTEM DESIGN (STEP 3)

A. Introduction

We have seen how the linearized Kalman-Bucy filter can be designed so as to generate in real time the estimated deviation $\delta\hat{x}(t)$ of the actual plant state $x(t)$ from its ideal deterministic response $x_0(t)$. Of course $\delta x(t)$ also depends

on the control correction vector $\delta u(t)$. Hence, one can now think of the final step of the design processes as the techniques necessary for generating on line the control correction vector $\delta u(t)$ as a function of the measurements so as to keep $\delta x(t)$ small.

The remarkable property of the LQG control problem is that the optimal control correction $\delta u(t)$ is generated from the estimated state deviation $\delta\hat{x}(t)$, generated by the Kalman-Bucy filter by means of the relationship

$$\delta u(t) = -G_0(t)\delta\hat{x}(t) \quad (107)$$

where the gain matrix $G_0(t)$ is precisely the one determined in the solution of the deterministic linear-quadratic problem (see Section III-G). Recall that the deterministic solution was

$$\delta u(t) = -G_0(t)\delta x(t) \quad (108)$$

under the assumption that the complete state perturbation vector $\delta x(t)$ is measured exactly. Furthermore, recall that in the statement and solution of the filtering problem (see Section IV-D) the control correction vector $\delta u(t)$ was assumed deterministic. Clearly, from (107), $\delta u(t)$ is *not* deterministic (since $\delta\hat{x}(t)$ is a random process).

Thus, it is neither apparent nor intuitively obvious why the generation of the control correction vector according to (107) should be optimal since in the true stochastic problem: 1) the deterministic assumptions on $\delta x(t)$ that led to the generation of $G_0(t)$ are violated, and 2) the deterministic assumptions on $\delta u(t)$ that led to the generation of $\delta\hat{x}(t)$ are also violated.

Thus, the purpose of this section is to precisely state how the overall LQG problem solution separates into the solution of a linear-quadratic deterministic problem and the solution of a linear-Gaussian estimation problem. The key theorem that shows this property is often called the *separation theorem* (see [6, pp. 408-432], [24, pp. 256-292], and [39]-[42]).

B. LQG Problem

We have seen in Section IV-C5 that the (approximate) linearized relation between the actual state deviation vector $\delta x(t)$ and the control correction vector $\delta u(t)$ is

$$\delta\dot{x}(t) = A_0(t)\delta x(t) + B_0(t)\delta u(t) + \xi(t) \quad (109)$$

while the true relation was that of (81) which includes the effects of the higher order terms in the function $\alpha_0(\delta x(t), \delta u(t))$.

Similarly, we have seen that the (approximate) linearized measurement relation between $\delta z(t)$ and $\delta x(t)$ is

$$\delta z(t) = C_0(t)\delta x(t) + \theta(t) \quad (110)$$

while the true relation was that of (83) which includes the effects of the higher order terms in the function $\beta_0(\delta x(t))$.

We can motivate the use of quadratic criteria by mimicking the development of Section III-F2 in the deterministic case; there we remarked that use of Taylor's theorem can be used to justify the fact that the control correction vector $\delta u(t)$ could be selected so as to "maxi-

mize the validity of the linearized model" by minimizing the quadratic cost to go [see (26)]

$$J_0 = \delta x'(T)F_0\delta x(T) + \int_t^T [\delta x'(\tau)Q_0(\tau)\delta x(\tau) + \delta u'(\tau)R_0(\tau)\delta u(\tau)] d\tau. \quad (111)$$

However, in our case J_0 is a scalar-valued random variable, because both $\delta x(t)$ and $\delta u(t)$ are random processes.

To formulate then a meaningful optimization problem one needs to minimize a nonrandom scalar. Since the cost functional J_0 is random, then a natural criterion is to minimize its expected value \bar{J}_0 conditioned on past measurements up to the present value of time t

$$\bar{J}_0 \triangleq E\{J_0|\delta z(\sigma); t_0 \leq \sigma \leq t\}. \quad (112)$$

The use of the cost functional \bar{J}_0 implies that we wish to "maximize on the average the validity of our linearized stochastic models." Since $\xi(t)$ and $\theta(t)$ are white, and hence most unpredictable, in individual experiments they may cause the system to deviate significantly from the region in which the linearization is more or less valid. Since we have no control over the specific outcome of the white noise processes, we cannot guarantee the validity of the linearization for any specific experiment. However, we can attempt to design the control system so as to optimize its average behavior.

1) *Formal Statement of the LQG Stochastic Control Problem:* Given the linearized dynamical system (109) and the linearized observation equation (110). Given the measurements $\delta z(\sigma)$, $t_0 \leq \sigma \leq t$. Find a system that generates the control correction vector $\delta u(\tau)$, $t \leq \tau \leq T$ such that the average cost to go

$$\bar{J} = E\{\delta x'(T)F_0\delta x(T) + \int_t^T [\delta x'(\tau)Q_0(\tau)\delta x(\tau) + \delta u'(\tau)R_0(\tau)\delta u(\tau)] d\tau|\delta z(\sigma); t_0 \leq \sigma \leq t\} \quad (113)$$

is minimum. The weighting matrices $Q_0(\tau)$, $R_0(\tau)$, and F_0 are those defined in Section III-G1, while the statistical properties of the noises are those given in Section IV-C4.

2) *Separation Theorem: Solution of the LQG Stochastic Control Problem:* The optimal control correction vector $\delta u(t)$ is generated by (see Fig. 11).

$$\delta u(t) = -G_0(t)\delta \hat{x}(t). \quad (114)$$

Specification of $G_0(t)$: The control gain matrix $G_0(t)$ is obtained by the solution of the deterministic linear-quadratic problem (see Section III-G), forgetting completely the stochastic aspects. Thus

$$G_0(t) = R_0^{-1}(t)B_0'(t)K_0(t) \quad (\text{control gain matrix}) \quad (115)$$

where $K_0(t)$ is defined by the solution of the control matrix Riccati equation

$$\frac{d}{dt}K_0(t) = -K_0(t)A_0(t) - A_0'(t)K_0(t) - Q_0(t) + K_0(t)B_0(t)R_0^{-1}(t)B_0'(t)K_0(t) \quad (116)$$

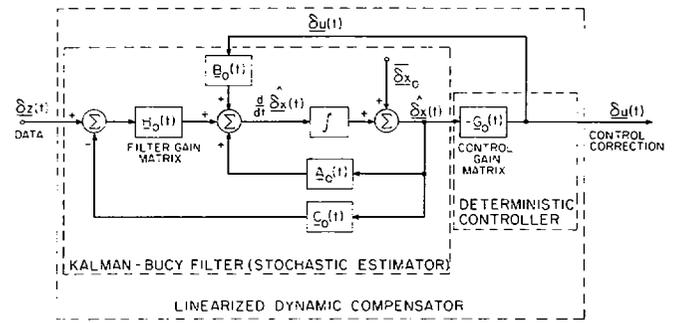


Fig. 11. Overall dynamic compensator that generates control correction vector $\delta u(t)$ from sensed signal $\delta z(t)$, as well as its internal decomposition. Underlined symbols indicate boldface.

$$K_0(T) = F_0. \quad (117)$$

(Note that $G_0(t)$ and $K_0(t)$ do not depend on the statistics, i.e., Σ_0 , $\Xi(t)$, $\Theta(t)$, and the data $\delta z(t)$).

Specification of $\delta \hat{x}(t)$: The vector $\delta \hat{x}(t)$ is generated by the Kalman-Bucy filter (see Section IV-D) under the assumption that $\delta u(t)$ is deterministic, forgetting completely the control problem. Thus

$$\frac{d}{dt}\delta \hat{x}(t) = A_0(t)\delta \hat{x}(t) + B_0(t)\delta u(t) + H_0(t)[\delta z(t) - C_0(t)\delta \hat{x}(t)] \quad (118)$$

$$\delta x(t_0) = \bar{x}_0 - x_0(t_0) \quad (119)$$

$$H_0(t) = \Sigma_0(t)C_0'(t)\Theta^{-1}(t) \quad (\text{filter gain matrix}) \quad (120)$$

and $\Sigma_0(t)$ is defined by the solution of the filter matrix Riccati equation

$$\frac{d}{dt}\Sigma_0(t) = A_0(t)\Sigma_0(t) + \Sigma_0(t)A_0'(t) + \Xi(t) - \Sigma_0(t)C_0'(t)\Theta^{-1}(t)C_0(t)\Sigma_0(t) \quad (121)$$

$$\Sigma_0(t_0) = \Sigma_0.$$

(Note that $H_0(t)$ and $\Sigma_0(t)$ are independent of the control problem weighting matrices $Q_0(t)$, $R_0(t)$, and F_0 .)

It is also possible to evaluate the minimum value of expected cost to go (113). The expression is ([24, p. 290])

$$\bar{J}^* = \delta \hat{x}(t)'K_0(t)\delta \hat{x}(t) + \text{tr}[K_0(t)\Sigma_0(t)] + \int_t^T \text{tr}[K_0(\tau)\Xi(\tau)] d\tau + \int_t^T \text{tr}[K_0(\tau)B_0(\tau)R_0^{-1}(\tau)B_0'(\tau)K_0(\tau)\Sigma_0(\tau)] d\tau. \quad (122)$$

The last three terms in (122) reflect the additional cost incurred due to the stochastic nature of the problem. The second term in (122) reflects the contribution due to the current uncertainty in the state estimate (as measured by the error covariance matrix $\Sigma_0(t)$); the third term in (122) reflects the contribution due to future plant uncertainty (as reflected by the intensity matrix $\Xi(\tau)$, $\tau \in [t, T]$); the last term reflects the effects of future uncertainty in the state estimation (as reflected by the error covariance matrix $\Sigma_0(\tau)$, $\tau \in [t, T]$).

3) *Discussion:* We shall now make some brief remarks regarding the interpretation that should be attached to the formal solution to the LQG problem.

First, we shall discuss how tradeoff studies regarding the accuracy as well as the type of sensors and actuators to be used affect the solution to the control problem as a whole. Let us suppose that the weighting matrices F_0 , $Q_0(t)$, and $R_0(t)$ have been somehow selected. In this context, the solution $K_0(t)$ of the control Riccati equation (112) is available.

Sensor selection: Let us suppose that we are faced with the problem of selecting between two types of sensors which, except for accuracy, otherwise perform the same tasks. Suppose that the more accurate sensor(s) is characterized by a measurement noise intensity matrix $\Theta_1(t)$ while the less accurate by $\Theta_2(t)$, such that $\Theta_1(t) \leq \Theta_2(t)$. On the other hand, the more accurate sensor(s) cost more money. For each sensor, we can solve the filter problem Riccati equation (121) and obtain the corresponding error covariance matrices, say, $\Sigma_1(t)$ and $\Sigma_2(t)$; it turns out that $\Sigma_1(t) \leq \Sigma_2(t)$, i.e., use of the more accurate sensor improves the estimation accuracy. In fact, from the filtering viewpoint the increase in state estimation accuracy may justify the increase in monetary cost. However, it does not necessarily follow that the expected improvement in the control system performance will necessarily justify the monetary cost. The reason is that only the last term in the minimum cost functional (122) will decrease [since $\Sigma_1(t) \leq \Sigma_2(t)$] from the use of the more accurate sensor. However, the relative percentage in increased performance is also governed by the first three terms in (122). It may turn out that for a doubling of invested money we may double estimation accuracy, but only buy a few percent in improving the control system performance as measured by (122). In particular, if we assume that at the initial time the state deviations are small, then we can carry out off-line (non-Monte-Carlo) studies by examining essentially percentage changes in the last three terms of (122).

Similar remarks can be made regarding the selection of the number of sensors. In our context, this would change the $C_0(t)$ matrix (both dimensionwise and numerically) in (120) and (121); this in turn will change the value of the $\Sigma_0(t)$ matrix that affects only the last term in (123).

Actuator and model accuracy tradeoffs: In a similar vein we can carry out tradeoff studies which involve the selection of the plant noise intensity matrix $\Xi(t)$. As we have remarked before, this models the total uncertainty in the dynamics (due to actuator errors as well as modeling errors). Let us suppose that we can buy two sets of actuators characterized by $\Xi_1(t) \leq \Xi_2(t)$ so that the first are more accurate (and more costly) than the second. Once more, from (121) we can deduce that $\Sigma_1(t) \leq \Sigma_2(t)$, i.e., more money buys increased state estimation accuracy (one can make a similar argument that more accurate modeling requires more engineering and experimentation time). As far as the effects of less plant uncertainty on the control system performance is concerned, different values of $\Xi(t)$ affect the last two terms in the cost (122), directly

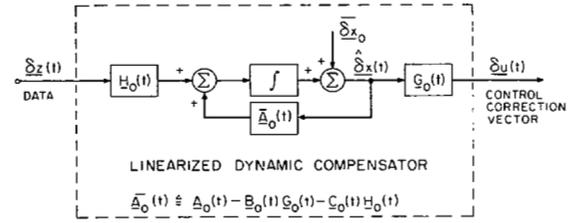


Fig. 12. Alternate representation of compensator of Fig. 11. Underlined symbols indicate boldface.

in the third term and indirectly via $\Sigma_0(t)$ in the fourth term.

Even more interesting (off line and non-Monte-Carlo!) tradeoffs can be carried out in the wisest allocation of funds partly to buy some better sensors, partly to buy some better actuators, and partly to invest in additional engineering time for better modeling.

4) *Compensator Interpretation:* The preceding results can be used to interpret the generation of the control correction vector $\delta u(t)$ in terms of the measurement vector $\delta z(t)$ as a multi-input $[\delta z(t)]$ multioutput $[\delta u(t)]$ compensator. The compensator is a linear n th order time-varying dynamical system with compensator state vector $\delta \hat{x}(t)$, compensator input vector $\delta z(t)$, and compensator output vector $\delta u(t)$. The compensator has the following description.

Compensator state equation:

$$\frac{d}{dt} \delta \hat{x}(t) = [A_0(t) - B_0(t)G_0(t) - H_0(t)C_0(t)] \delta \hat{x}(t) + H_0(t) \delta z(t). \quad (123)$$

Compensator output equation:

$$\delta u(t) = -G_0(t) \delta \hat{x}(t). \quad (124)$$

Fig. 12 shows an alternate realization of this linear dynamic compensator.

Note that the dynamics of this compensator are summarized in the compensator system matrix

$$A_0(t) - B_0(t)G_0(t) - H_0(t)C_0(t) \triangleq \bar{A}_0(t) \quad (125)$$

and that both the control objectives [via $G_0(t)$] and the stochastic aspects [via $H_0(t)$] are simultaneously important. The separation is only reflected in the fact that $G_0(t)$ and $H_0(t)$ are computed independently.

C. Steady-State LQG Problem

We have seen in the previous section that the dynamic compensator derived via the separation theorem contains both the control gains of the deterministic linear-quadratic controller and the Kalman-Bucy filter. As shown in Fig. 11, the determination of this dynamic compensator hinges on the evaluation of: 1) the filter gain matrix $H_0(t)$, and 2) the control gain matrix $G_0(t)$.

We have seen in Section III-I that if our linearized system is time invariant and the control problem is optimized over the semi-infinite control interval $(0, \infty)$, then the

control gain matrix turns out to be a constant, i.e., [see (48)] the control is generated by

$$\delta u(t) = -\hat{G}_0 \delta x(t). \quad (126)$$

On the other hand, we have seen in Section IV-E that if the linearized system is time invariant, the noise statistics stationary, and the initial time $t_0 \rightarrow -\infty$, then the estimate $\delta \hat{x}(t)$ of $\delta x(t)$ is generated by the steady-state Kalman-Bucy filter

$$\frac{d}{dt} \delta \hat{x}(t) = A_0 \delta \hat{x}(t) + B_0 \delta u(t) + \hat{H}_0 [\delta z(t) - C_0 \delta \hat{x}(t)] \quad (127)$$

and it is characterized by the constant filter-gain matrix \hat{H}_0 .

Intuitively then, one can suspect that: 1) under suitable assumptions of time invariance, 2) whenever measurements have been made over a sufficiently long time interval, and 3) if control is to be exercised for an infinite time into the future, then the separation theorem may still carry through in the sense that the control correction vector $\delta u(t)$ is generated by the steady-state Kalman-Bucy filter estimate $\delta \hat{x}(t)$ according to the relation

$$\delta u(t) = -\hat{G}_0 \delta \hat{x}(t). \quad (128)$$

Indeed this is the case. It remains to indicate precisely the cost functional which is minimized.

1) *Precise Statement of the Steady-State LQG Stochastic Control Problem:* Given the completely controllable and observable linear time-invariant system

$$\delta \dot{x}(t) = A_0 \delta x(t) + B_0 \delta u(t) + \xi(t) \quad (129)$$

and the time-invariant measurement relation

$$\delta z(t) = C_0 \delta x(t) + \theta(t). \quad (130)$$

Suppose that the noises $\xi(t)$ and $\theta(t)$ are both Gaussian, white, zero mean, mutually independent, and stationary, i.e.,

$$\text{cov}[\xi(t); \xi(\tau)] = \varepsilon \delta(t - \tau); \quad \varepsilon = \varepsilon' \geq 0 \quad (131)$$

$$\text{cov}[\theta(t); \theta(\tau)] = \Theta \delta(t - \tau); \quad \Theta = \Theta' > 0. \quad (132)$$

Find $\delta u(t)$ for all $t \in [-T, T]$, such that the cost functional

$$\bar{J}_0 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} [\delta x'(t) Q_0 \delta x(t) + \delta u'(t) R_0 \delta u(t)] dt \quad (133)$$

is minimized, where the weighting matrices Q_0 and R_0 are such that

$$Q_0 = Q_0' \geq 0 \quad (134)$$

$$R_0 = R_0' > 0. \quad (135)$$

2) *Solution of the Steady-State LQG Control Problem:* The optimal control correction vector $\delta u(t)$ is given by

$$\delta u(t) = -\hat{G}_0 \delta \hat{x}(t). \quad (136)$$

Generation of control gain matrix \hat{G}_0 : The matrix G_0

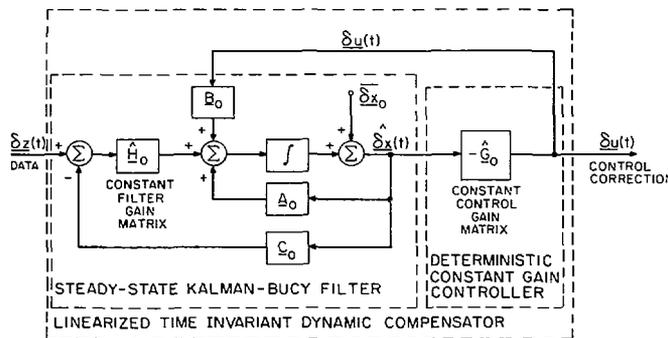


Fig. 13. Linear time-invariant dynamic compensator that generates $\delta u(t)$ from $\delta z(t)$. Undelined symbols indicate boldface.

is the solution to the associated linear-quadratic deterministic control problem (see Section III-I), i.e.,

$$\hat{G}_0 = R_0^{-1} B_0' \hat{K}_0 \quad (137)$$

where \hat{K}_0 satisfies the (control) algebraic Riccati equation [see (50)]

$$0 = -\hat{K}_0 A_0 - A_0' \hat{K}_0 - Q_0 + \hat{K}_0 B_0 R_0^{-1} B_0' \hat{K}_0. \quad (138)$$

Generation of $\delta \hat{x}(t)$: The vector $\delta \hat{x}(t)$ is generated on line by the steady-state Kalman-Bucy filter

$$\frac{d}{dt} \delta \hat{x}(t) = A_0 \delta \hat{x}(t) + B_0 \delta u(t) + \hat{H}_0 [\delta z(t) - C_0 \delta \hat{x}(t)]. \quad (139)$$

The constant filter gain matrix H_0 is given by

$$\hat{H}_0 = \Sigma_0 C_0' \Theta^{-1} \quad (140)$$

where the matrix Σ_0 satisfies the (filter) algebraic Riccati equation [see (104)]

$$0 = \Sigma_0 A_0 + A_0' \Sigma_0 + \varepsilon - \Sigma_0 C_0' \Theta^{-1} C_0 \Sigma_0. \quad (141)$$

3) *Structure of the Dynamic Compensator:* The preceding equations can be manipulated so as to clarify the nature of the dynamic compensator which generates $\delta u(t)$ on the basis of the measurement vector $\delta z(t)$, as shown in Fig. 13.

This compensator is characterized by: a) its state vector $\delta \hat{x}(t)$; b) its input vector $\delta z(t)$; c) its output vector $\delta u(t)$. By substituting (136) into (139) we can obtain the state-variable characterization of the compensator

$$\begin{aligned} \frac{d}{dt} \delta \hat{x}(t) &= [A_0 - B_0 \hat{G}_0 - \hat{H}_0 C_0] \delta \hat{x}(t) + \hat{H}_0 \delta z(t) \\ \delta u(t) &= -\hat{G}_0 \delta \hat{x}(t). \end{aligned} \quad (142)$$

Since all matrices in (142) are constant, we conclude that the required compensator is linear and time invariant. This, of course, is a major advantage from an implementation viewpoint; this compensator can be synthesized using analog components.

The compensator, being linear and time invariant, can be characterized by its transfer matrix $M(s)$ [from $\delta z(t)$ to $\delta u(t)$]

$$\mathcal{L}\{\delta u(t)\} = M(s) \mathcal{L}\{\delta z(t)\}. \quad (143)$$

The compensator transfer matrix $\mathbf{M}(s)$ is given by

$$\mathbf{M}(s) = -\hat{\mathbf{G}}_0[s\mathbf{I} - \mathbf{A}_0 + \mathbf{B}_0\hat{\mathbf{G}}_0 + \hat{\mathbf{H}}_0\mathbf{C}_0]^{-1}\hat{\mathbf{H}}_0. \quad (144)$$

Under mild controllability and observability assumptions it can be shown that both the compensator as well as the closed-loop system (about \mathbf{x}_0 and \mathbf{u}_0) are strictly stable.

4) *Dynamics of the Closed-Loop System:* By combining (142) with (129) and (130) it is easy to deduce that the closed-loop system satisfies the differential equation

$$\frac{d}{dt} \begin{bmatrix} \delta\mathbf{x}(t) \\ \delta\hat{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0 & -\mathbf{B}_0\hat{\mathbf{G}}_0 \\ \hat{\mathbf{H}}_0\mathbf{C}_0 & \mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0 - \hat{\mathbf{H}}_0\mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}(t) \\ \delta\hat{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{H}}_0 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix}. \quad (145)$$

An alternate state representation of the closed-loop system is obtained through the use of the state estimation error vector

$$\delta\mathbf{e}(t) = \delta\mathbf{x}(t) - \delta\hat{\mathbf{x}}(t). \quad (146)$$

From (145) and (146) we obtain

$$\frac{d}{dt} \begin{bmatrix} \delta\mathbf{x}(t) \\ \delta\mathbf{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0 & \mathbf{B}_0\hat{\mathbf{G}}_0 \\ \mathbf{0} & \mathbf{A}_0 - \hat{\mathbf{H}}_0\mathbf{B}_0 \end{bmatrix}}_{\mathbf{P}_0} \begin{bmatrix} \delta\mathbf{x}(t) \\ \delta\mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\hat{\mathbf{H}}_0 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix}. \quad (147)$$

The dynamic response of the closed-loop system will be governed by the eigenvalues of the $2n \times 2n$ matrix, the roots of the polynomial $\det(\mathbf{P}_0 - \lambda\mathbf{I})$. But

$$\begin{aligned} \det(\mathbf{P}_0 - \lambda\mathbf{I}) &= \det \begin{bmatrix} \mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0 - \lambda\mathbf{I} & \mathbf{B}_0\hat{\mathbf{G}}_0 \\ \mathbf{0} & \mathbf{A}_0 - \hat{\mathbf{H}}_0\mathbf{C}_0 - \lambda\mathbf{I} \end{bmatrix} \\ &= \det(\mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0 - \lambda\mathbf{I}) \det(\mathbf{A}_0 - \hat{\mathbf{H}}_0\mathbf{C}_0 - \lambda\mathbf{I}). \end{aligned} \quad (148)$$

The preceding shows an extremely interesting property of the $2n$ eigenvalues (poles) of the closed-loop system. Half of the eigenvalues can be independently adjusted by the value of $\hat{\mathbf{G}}_0$ (which only depends on the weighting matrices \mathbf{Q}_0 and \mathbf{R}_0), while the other half are adjusted independently by the value of $\hat{\mathbf{H}}_0$ (which only depends on the noise intensity matrices Ξ and Θ). This shows still another separation. Since both $(\mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0)$ and $(\mathbf{A}_0 - \hat{\mathbf{H}}_0\mathbf{C}_0)$ are strictly stable matrices, it follows that the overall closed loop system is stable.

D. Discussion

We conclude this section with some remarks pertaining to the selection of the control weighting matrices and the noise intensity matrices. Our discussion will be primarily directed to the time-invariant design, although it is equally applicable to the time-varying one.

The fact that from a mathematical point of view the separation theorem allows us to solve the control and

filter problems separately *does not imply that these two problems should be solved separately by two distinct design groups and hooked together by the supervisor.*

Unfortunately, this is how the theorems have been used in many engineering designs leading to unsuccessful results. For this reason, we shall briefly elaborate on the proper usage of this theorem.

In general, if we could solve the overall nonlinear non-quadratic stochastic control problem, the optimal design would *not* obey this nice separation property. Since we cannot solve the true problem, we employ the LQG approach to arrive at a set of problems that we can solve. The key question is then: What is available to the design engineer to control the goodness of the overall design so that he can obtain a satisfactory system? The answer to this question, for the time-invariant design, is the selection of the four matrices \mathbf{Q}_0 , \mathbf{R}_0 , Ξ , and Θ . For any arbitrary selection, the mathematical problem separates. However, *this does not mean that \mathbf{Q}_0 , \mathbf{R}_0 , and Ξ , Θ should be selected independently of each other.* The discussion of Section V-B3 points this out to a certain degree since the changes in uncertainty by changes in Ξ and Θ are modulated by the values of \mathbf{R}_0 and \mathbf{Q}_0 (via \mathbf{K}_0) in the cost (122).

Unfortunately, there seems to be no published literature on the preceding point, with the notable exception of the note by Mendel [43]. We have already commented that \mathbf{Q}_0 and \mathbf{R}_0 can be used in the quadratic criterion to maximize linearization validity (see Section III-F). We have also commented that Ξ and Θ can be used to communicate to the mathematics the existence of modeling errors due to linearization (see Section IV-C5). Clearly both sets of matrices are partly used for the same purpose; hence, their selection should not be done independently.

Unfortunately, as we have stressed throughout this paper, there are no systematic procedures available for the specification of \mathbf{Q}_0 , \mathbf{R}_0 , Ξ , and Θ . Additional theoretical research and applications are necessary. However, some qualitative properties of the overall design can be made, by reexamining the implications of Section V-C4.

Let us suppose that we fix \mathbf{R}_0 and Θ and examine the effects of Ξ and \mathbf{Q}_0 . Let us suppose that a large \mathbf{Q}_0 is selected; in this case we get high control gains and the deterministic system has high bandwidth, because half of the poles of the closed-loop system (the eigenvalues of $\mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0$) are pushed away from the $j\omega$ axis in the left half-plane. If a small value of Ξ is selected, then the bandwidth of the Kalman filter is small (to prevent the measurement noise from passing through); thus the other half of the poles of the closed-loop system (the eigenvalues of $\mathbf{A}_0 - \hat{\mathbf{H}}_0\mathbf{C}_0$) will be dominant, and *the transient response of the closed-loop system will be drastically different from that of the deterministic one.* On the other hand, if there is large uncertainty in the system dynamics (large Ξ) and small measurement noise (low Θ) the bandwidth of the Kalman filter may be so large that only the eigenvalues of $\mathbf{A}_0 - \mathbf{B}_0\hat{\mathbf{G}}_0$ are dominant; in this case, the transient response of the closed-loop system will approximate that of the deterministic one.

VI. DISCUSSION

We have outlined the philosophy, assumptions, formulation, and mathematical characterization of a design process for controlling a nonlinear uncertain system about a desired trajectory, through the use of the so-called LQG problem. This design process represents a relatively well-understood byproduct of modern control theory. Of course, successful control systems have been designed using alternate approaches. However, this design process is characterized by a clear-cut division of responsibilities between the modeling and the calculation aspects of the problem.

We outline in the following paragraphs, for the sake of completeness, the 13 key steps in the design process. All of these are carried out off-line; the on-line computational requirements are minimal. We have outlined the steps involved for the general nonstationary case. The corresponding steps for the stationary steady-state type of design are given parenthetically.

Part A: Deterministic Modeling

Step 1: Determine deterministic model of actuators and plant; this yields

$$\dot{x}(t) = f(x(t), u(t)).$$

Step 2: Determine deterministic model of plant and sensors; this yields

$$y(t) = g(x(t)).$$

Step 3: Determine ideal input-state-output response

- $u_0(t)$ (or u_0): ideal input
 - $x_0(t)$ (or x_0): ideal state response
 - $y_0(t)$ (or y_0): ideal output response
- for all $t \in [t_0, T]$ [or $t \in [t_0, \infty)$].

Part B: Stochastic Modeling

Step 4: Model uncertainty in initial plant state

- select mean: $\bar{x}_0 = E\{x(t_0)\}$
- select covariance: $\Sigma_0 = \text{cov}[x(t_0); x(t_0)]$.

Step 5: Model input uncertainty

- select covariance: $\Sigma(t)\delta(t - \tau) = \text{cov}[\xi(t); \xi(\tau)]$ (or stationary $\Sigma(t) = \Sigma$).

Step 6: Model sensor uncertainty

- select covariance: $\Theta(t)\delta(t - \tau) = \text{cov}[\theta(t); \theta(\tau)]$ (or stationary $\Theta(t) = \Theta$).

Part C: Linearization Modeling

Step 7: Establish matrices $A_0(t)$ (or A_0); $B_0(t)$ (or B_0); and $C_0(t)$ (or C_0) from information in Steps 1, 2, and 3

$$A_0(t) = \left. \frac{\partial f}{\partial x(t)} \right|_0, B_0(t) = \left. \frac{\partial f}{\partial u(t)} \right|_0, C_0(t) = \left. \frac{\partial g}{\partial x(t)} \right|_0.$$

Step 8: Depending on degree of nonlinearity, select

weighting matrices $Q_0(t)$ (or Q_0); $R_0(t)$ (or R_0); F_0 with due consideration of the values of Σ_0 ; $\Sigma(t)$ (or Σ); and $\Theta(t)$ (or Θ).

Part D: Control Problem Calculations

Step 9: Using matrices established in Steps 7 and 8, solve backward in time the control Riccati equation

$$\begin{aligned} \frac{d}{dt} K_0(t) = & -K_0(t)A_0(t) - A_0'(t)K_0(t) - Q_0(t) \\ & + K_0(t)B_0(t)R_0^{-1}(t)B_0'(t)K_0(t); K_0(T) = F_0. \end{aligned}$$

(In the time-invariant case find the positive definite solution \hat{K}_0 of algebraic Riccati equation $0 = -\hat{K}_0 A_0 - A_0' \hat{K}_0 - Q_0 + \hat{K}_0 B_0 R_0^{-1} B_0' \hat{K}_0$.)

Step 10: Using $K_0(t)$ (or \hat{K}_0) found in Step 9 and appropriate matrices established in Steps 7 and 8, determine the control gain matrix

$$G_0(t) = R_0^{-1}(t)B_0'(t)K_0(t).$$

(In the time-invariant case, $\hat{G}_0 = R_0^{-1}B_0' \hat{K}_0$.)

Part E: Filtering Problem Calculations

Step 11: Using matrices established in Steps 4, 5, and 6 solve forward in time the filter Riccati equation

$$\begin{aligned} \frac{d}{dt} \Sigma_0(t) = & A_0(t)\Sigma_0(t) + \Sigma_0(t)A_0'(t) + \Sigma(t) \\ & - \Sigma_0(t)C_0'(t)\Theta^{-1}(t)C_0(t)\Sigma_0(t); \Sigma_0(t_0) = \Sigma_0. \end{aligned}$$

(In the stationary case find positive definite solution $\hat{\Sigma}_0$ of the algebraic Riccati equation $0 = A_0 \hat{\Sigma}_0 + \hat{\Sigma}_0 A_0' + \Sigma - \hat{\Sigma}_0 C_0' \Theta^{-1} C_0 \hat{\Sigma}_0$.)

Step 12: Using $\Sigma_0(t)$ (or $\hat{\Sigma}_0$) and matrices established in Steps 5 and 7, determine the filter gain matrix

$$H_0(t) = \Sigma_0(t)C_0'(t)\Theta^{-1}(t).$$

(In the stationary case compute $\hat{H}_0 = \hat{\Sigma}_0 C_0' \Theta^{-1}$.)

Part F: Construction of Linearized Dynamic Compensator

Step 13: In the nonstationary case, design (on a digital computer) a data-processing algorithm that accepts an r vector $\delta z(t)$ and generates an m vector $\delta u(t)$ according to

$$\begin{aligned} \frac{d}{dt} \delta \hat{x}(t) = & [A_0(t) - B_0(t)G_0(t) - H_0(t)C_0(t)] \delta \hat{x}(t) \\ & + H_0(t)\delta z(t); \delta \hat{x}(t_0) = \bar{x}_0 - x_0(t_0); \delta u(t) = -G_0(t) \delta \hat{x}(t). \end{aligned}$$

(In the stationary case design, using an analog or digital computer, design a time-invariant system that generates $\delta u(t)$ from $\delta z(t)$ according to

$$\begin{aligned} \frac{d}{dt} \delta \hat{x}(t) = & [A_0 - B_0 \hat{G}_0 - \hat{H}_0 C_0] \delta \hat{x}(t) + \hat{H}_0 \delta z(t); \delta \hat{x}(t_0) \\ & = \bar{x}_0 - x_0(t_0) \\ \delta u(t) = & -\hat{G}_0 \delta \hat{x}(t). \end{aligned}$$

This completes the off-line computations. The system is now ready to run in real time as indicated in Fig. 14.

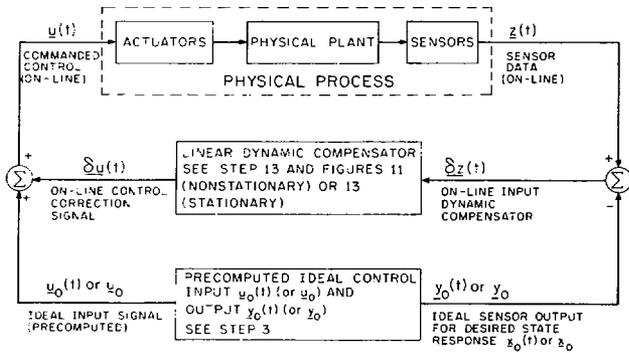


Fig. 14. Structure of overall system that can be used to control uncertain nonlinear system by combining precomputed signals with on-line measurements generated by linear dynamic compensator. Underlined symbols indicate boldface.

The different signals that are required are keyed to the outlined steps.

It is interesting to note that the conceptual block diagram of Fig. 14 masks the key role of the actual plant state and desired state response, as well as their deviations that underlie the whole design process, since it is essential for all the linearizations that are necessary in the design process.

The step-by-step development should convince the reader of the crucial importance of the modeling issue in this design process. The ability of the engineer to translate physical quantities into their mathematical counterparts in Steps 1, 2, 3, 4, 5, 6, and 8 is absolutely essential. Once the modeling has been carried out, the remaining steps 7, 9, 10, 11, 12, and 13 are mechanical. The availability of a variety of computational techniques for the solution of the differential and algebraic Riccati equations, using present-day digital computers, removes much of the "dogwork" associated with the design process.

Conclusions

It is the author's opinion that this shift in emphasis towards the modeling of the physical aspects of the problem represents a healthy direction when it is coupled with the routine computer calculations that fix both the structure and the parameters of the compensator. The major portion of the engineers time can now be focused upon understanding the real-world aspects of the system that he must control. Thus, this design process is indeed conducive to "creative engineering"—the art of making approximations and still obtaining something that works. Both the stochastic aspects of the problem, and the tightly coupled feedback structures that result, are ideally suited for this task, because they reduce the sensitivity of the final design to both modeling errors and uncertainty effects.

ACKNOWLEDGMENT

The author wishes to thank the referees for their constructive comments and suggestions; and L. A. Gould, R. W. Brockett, S. K. Mitter, J. Mendel, J. Peschon, S. R. McReynolds, and E. Tse for their helpful remarks.

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Good, Bad, or Optimal?

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Abstract—Optimal control with a quadratic performance index has been suggested as a solution to the problem of regulating an industrial plant in the vicinity of a steady state. It is shown that such control is usually not feasible, and if feasible can have serious defects.

I. INTRODUCTION

LET a controllable system be given with

$$\dot{x} = Ax + Bu; \quad x(0) = c \quad (1)$$

where A , B are, respectively, $n \times n$ and $n \times m$, with $m \leq n$, and let feedback be applied according to

$$u = -Cx. \quad (2)$$

If

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \quad (3)$$

Manuscript received June 17, 1971. Paper recommended by S. K. Mitter, Associate Guest Editor.

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it is known [1] that there exists a unique choice C_0 for C which minimizes J for all c , provided that Q , R satisfy the following conditions: 1) R is a symmetric positive definite matrix; 2) $Q = L^T L$, with (A, L) observable. Define the optimal transfer function matrix by

$$G_0(s) = C_0(sI_n - A)^{-1}B. \quad (4)$$

The closed-loop system (1), (2) is optimal in the special sense just defined, namely, that given A , B , Q , R , the choice $C = C_0$ minimizes J . The word optimal carries with it the suggestion that the system has desirable properties in general, but of course this need not be the case. In particular, it will be shown below that the optimal system has properties which cannot usually be achieved in practice and may have properties which are highly undesirable.

II. PHASE ADVANCE

It is known [1] that C_0 is given by

$$C_0 = R^{-1}B^T P \quad (5)$$

where P is a symmetric positive definite matrix satisfying the usual matrix Riccati equation. It also follows [2] from (4) that

$$\lim_{s \rightarrow \infty} sG_0(s) = C_0 B \quad (6)$$